Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2009 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. The angle θ is greater than 90° and less than 360° and $\cos\theta = \frac{2}{3}$. (3)

   Find the exact value of $\tan\theta$.

Solution

The angle obviously lies in $270^\circ < \theta < 360^\circ$ (why?). The adjacent is 2, the hypotenuse is 3, which means the

$$\begin{aligned} \text{opposite} &= \sqrt{3^2 - 2^2} \\ &= \sqrt{5}. \end{aligned}$$

Finally,

$$\tan\theta = \underline{\underline{-\tfrac{\sqrt{5}}{2}}}.$$

2. Find the equation of the normal to the curve (5)

$$y = x^3 + 5x - 7$$

at the point $(1, -1)$.

Solution

$$y = x^3 + 5x - 7 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 5.$$

Now,

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 8$$
$$\Rightarrow m_{\text{normal}} = -\tfrac{1}{8}.$$

Finally,

$$y + 1 = -\tfrac{1}{8}(x-1) \Rightarrow y + 1 = -\tfrac{1}{8}x + \tfrac{1}{8}$$
$$\Rightarrow \underline{\underline{y = -\tfrac{1}{8}x - \tfrac{7}{8}}}.$$

3. A is the point $(1, 5)$ and C is the point $(3, p)$.

(a) Find the equation of the line through A which is parallel to the line (2)

$$2x + 5y = 7.$$

Solution

$$2x + 5y = 2(1) + 5(5)$$
$$= 27;$$

hence, the equation of the line is

$$\underline{\underline{2x + 5y = 27}}.$$

This line also passes through the point C.

(b) Find the value of p. (2)

Solution

$$2(3) + 5p = 27 \Rightarrow 5p = 21$$
$$\Rightarrow \underline{\underline{p = 4\tfrac{1}{5}}}.$$

4. AB is a diameter of a circle, where A is $(1, 1)$ and B is $(5, 3)$.

Find

(a) the exact length of AB, (2)

Solution

$$\begin{aligned} AB &= \sqrt{(5-1)^2 + (3-1)^2} \\ &= \sqrt{16+4} \\ &= \sqrt{20} \\ &= \underline{\underline{2\sqrt{5}}}. \end{aligned}$$

(b) the coordinates of the midpoint of AB, (1)

Solution

$$\left(\frac{1+5}{2}, \frac{1+3}{2}\right) = \underline{\underline{(3, 2)}}.$$

(c) the equation of the circle. (3)

Solution

$$(x-3)^2 + (y-2)^2 = (\sqrt{5})^2 \Rightarrow \underline{\underline{(x-3)^2 + (y-2)^2 = 5}}.$$

5. Parcels slide down a ramp. Due to resistance, the deceleration is 0.25 ms^{-2}.

One parcel is given an initial velocity of 2 ms^{-1}.

(a) Find the distance travelled before the parcel comes to rest. (3)

Solution

$s = ?$, $u = 2$, $v = 0$, $a = -0.25$, and $t = ?$: use $v^2 = u^2 + 2as$:

$$\begin{aligned} 0 = 2^2 + 2 \times (-0.25) \times s &\Rightarrow \tfrac{1}{2}s = 4 \\ &\Rightarrow \underline{\underline{s = 8 \text{ m}}}. \end{aligned}$$

A second parcel is given an initial velocity of 3 ms^{-1} and takes 4 seconds to reach the bottom of the ramp.

(b) Find the length of the ramp. (3)

Solution

$s = ?$, $u = 3$, $v = 0$, $a = -0.25$, and $t = 4$: use $s = ut + \frac{1}{2}at^2$:

$$\begin{aligned} s &= (3 \times 4) + [\tfrac{1}{2} \times (-0.25) \times 4^2] \\ &= 12 - 2 \\ &= \underline{\underline{10 \text{ m}}}. \end{aligned}$$

6. The gradient function of a curve is given by (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 4x + 3x^2.$$

Find the equation of the curve given that it passes through the point $(2, 6)$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 4x + 3x^2 \Rightarrow y = x - 2x^2 + x^3 + c,$$

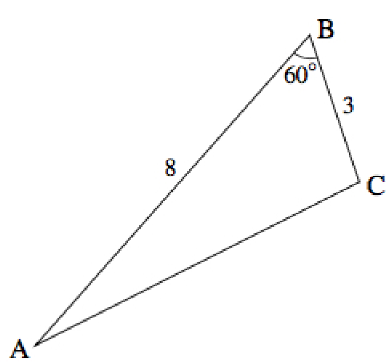
for some constant c. Now, $(2, 6)$ lies on the curve:

$$\begin{aligned} 2 - (2 \times 2^2) + 2^3 + c = 6 &\Rightarrow 2 - 8 + 8 + c = 6 \\ &\Rightarrow c = 4; \end{aligned}$$

hence, the equation is

$$\underline{\underline{y = 4 + x - 2x^2 + x^3}}.$$

7. The course of a cross-country race is in the shape of a triangle ABC.



Not to scale

$AB = 8$ km, $BC = 3$ km, and angle $ABC = 60^\circ$.

(a) Calculate the distance AC and hence the total length of the course. (4)

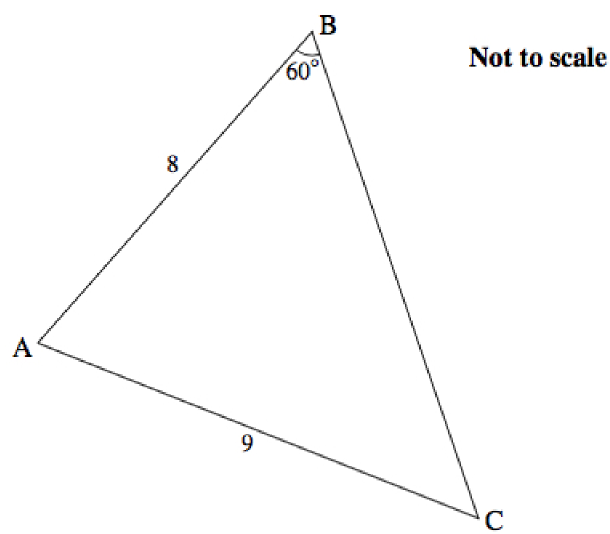
Solution

$$\begin{aligned} AC &= \sqrt{8^2 + 3^2 - 2 \times 8 \times 3 \times \cos 60^\circ} \\ &= \sqrt{49} \\ &= \underline{\underline{7\text{ km}}} \end{aligned}$$

and the total length of the course is

$$8 + 3 + 7 = \underline{\underline{18\text{ km}}}.$$

The organisers extend the course so that $AC = 9$ km.



(b) Calculate the angle BCA. (3)

Solution

$$\frac{\sin\angle BCA}{AB} = \frac{\sin\angle ABC}{AC} \Rightarrow \frac{\sin\angle BCA}{8} = \frac{\sin\angle 60^\circ}{9}$$
$$\Rightarrow \sin\angle BCA = \frac{8\sin\angle 60^\circ}{9}$$
$$\Rightarrow \sin\angle BCA = \tfrac{4\sqrt{3}}{9}$$
$$\Rightarrow \angle BCA = 50.335\,964\,64 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\angle BCA = 50.3^\circ \text{ (3 sf)}}}.$$

8. Calculate the x-coordinates of the points of intersection of the line (5)

$$y = 2x + 11$$

and the curve

$$y = x^2 - x + 5.$$

Give your answers correct to 2 decimal places.

Solution

$$x^2 - x + 5 = 2x + 11 \Rightarrow x^2 - 3x - 6 = 0$$
$$\Rightarrow x = \frac{3 \pm \sqrt{(-3)^2 - 4 \times 1 \times (-6)}}{2 \times 1}$$
$$\Rightarrow x = \frac{3 \pm \sqrt{33}}{2}$$
$$\Rightarrow x = -1.372\,281\,323 \text{ or } 4.372\,281\,323 \text{ (FCD)}.$$

Now,

$$x = -1.372\ldots \Rightarrow y = 8.255\ldots$$

and

$$x = 4.372\ldots \Rightarrow y = 19.744\ldots.$$

Finally, the answers are

$$\underline{\underline{(-1.37, 8.26) \text{ or } (4.37, 19.74) \text{ (2 dp)}}}.$$

9. A car accelerates from rest. At time t seconds, its acceleration is given by

$$a = (4 - 0.2t) \text{ ms}^{-2}$$

until $t = 20$.

(a) Find the velocity after 5 seconds. (3)

Solution

$$a = (4 - 0.2t) \Rightarrow v = (c + 4t - 0.1t^2)\ \text{ms}^{-1}$$

for some constant c. Now,

$$v = 0 \Rightarrow c + 0 - 0 = 0 \Rightarrow c = 0$$

which means

$$v = (4t - 0.1t^2)\ \text{ms}^{-1}.$$

Finally,

$$\begin{aligned} t = 5 &\Rightarrow v = 4(5) - 0.1(5^2) \\ &\Rightarrow \underline{\underline{v = 17\tfrac{1}{2}\ \text{ms}^{-1}}}. \end{aligned}$$

(b) What is happening to the velocity at $t = 20$? (1)

Solution

$$t = 20 \Rightarrow a = 0$$

and, hence, the car is going at its $\underline{\underline{\text{maximum velocity}}}$.

(c) Find the distance travelled in the first 20 seconds. (3)

Solution

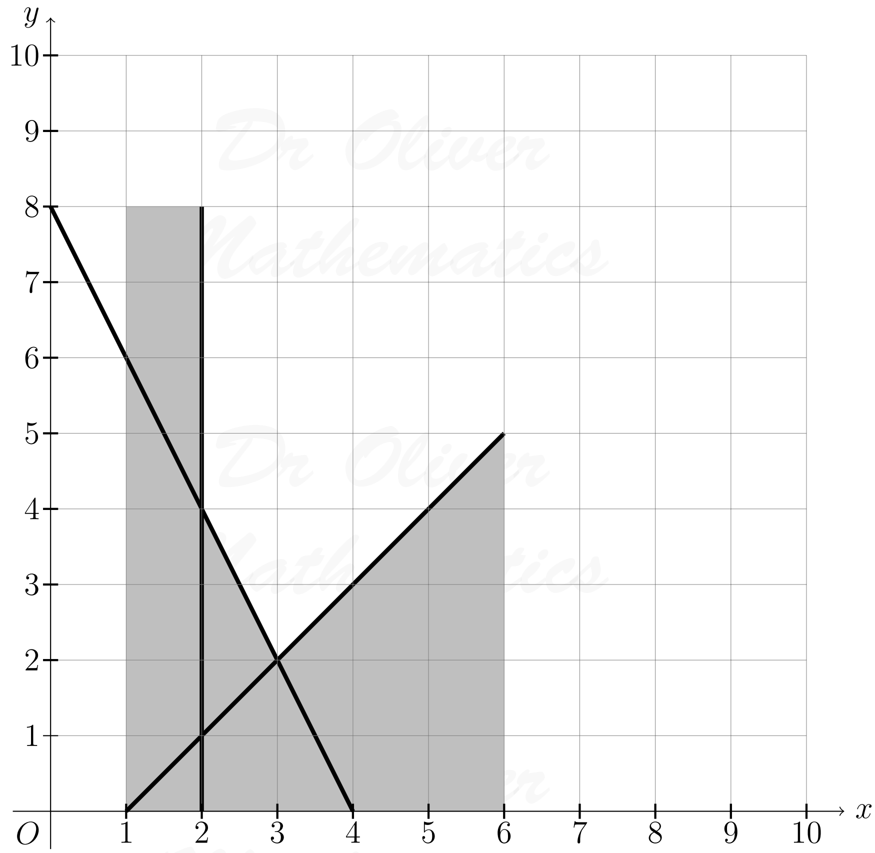
$$\begin{aligned} \text{Distance travelled} &= \int_0^{20} (4t - 0.1t^2)\,\mathrm{d}t \\ &= \left[2t^2 - \tfrac{1}{30}t^3\right]_{t=0}^{20} \\ &= \left(800 - 266\tfrac{2}{3}\right) - (0 - 0) \\ &= 533\tfrac{1}{3}\ \text{(exact value)} \\ &= \underline{\underline{533\ \text{m (3 sf)}}}. \end{aligned}$$

10. (a) Illustrate on one graph the following three inequalities. (4)

$$\begin{aligned} y &\geqslant x - 1 \\ x &\geqslant 2 \\ 2x + y &\geqslant 8. \end{aligned}$$

Draw suitable boundaries and shade areas that are **excluded**.

Solution



(b) Write down the minimum value of y in this region. (1)

Solution

$\underline{\underline{y = 2}}$.

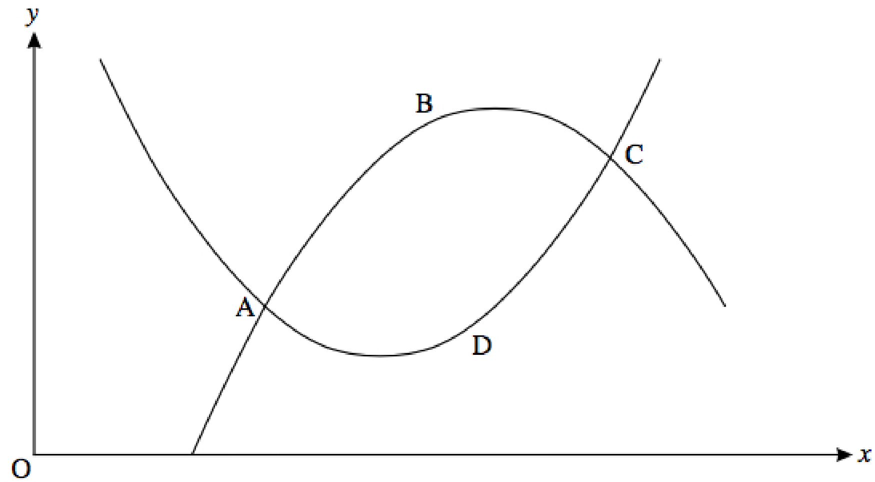
Section B

11. The shape $ABCD$ below represents a leaf.
The curve ABC has equation

$$y = -x^2 + 8x - 9.$$

The curve ADC has equation

$$y = x^2 - 6x + 11.$$



(a) Find algebraically the coordinates of A and C, the points where the curves intersect. (5)

Solution

$$x^2 - 6x + 11 = -x^2 + 8x - 9 \Rightarrow 2x^2 - 14x + 20 = 0$$
$$\Rightarrow 2(x^2 - 7x + 10) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -7 \\ \text{multiply to:} & +10\end{array}\right\} -5,\ -2$$

$$\Rightarrow 2(x-2)(x-5) = 0$$
$$\Rightarrow x - 2 = 0 \text{ or } x - 5 = 0$$
$$\Rightarrow x = 2 \text{ or } x = 5.$$

When

$$x = 2 \Rightarrow y = 3 \text{ and } x = 5 \Rightarrow y = 6.$$

Hence,

$$\underline{\underline{A(2,3) \text{ and } C(5,6)}}.$$

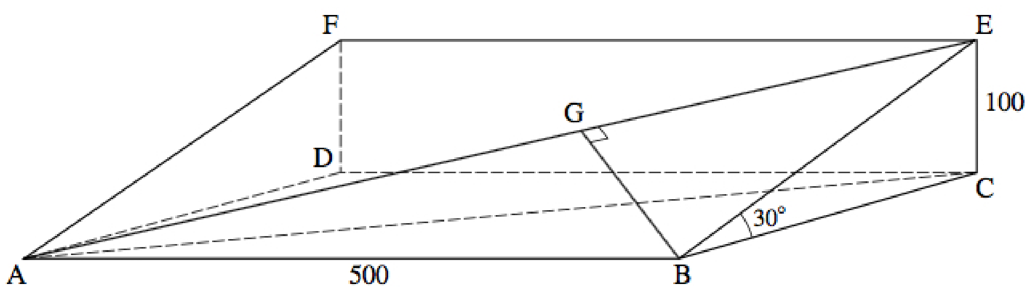
(b) Find the area of the leaf. (7)

Solution

$$
\begin{aligned}
\text{Area} &= \int_2^5 \left[(-x^2+8x-9)-(x^2-6x+11)\right] \mathrm{d}x \\
&= \int_2^5 (-2x^2+14x-20)\,\mathrm{d}x \\
&= \left[-\tfrac{2}{3}x^3+7x^2-20x\right]_{x=2}^5 \\
&= \left(-83\tfrac{1}{3}+175-100\right)-\left(-5\tfrac{1}{3}+28-40\right) \\
&= \underline{\underline{9}}.
\end{aligned}
$$

12. The diagram shows a rectangle $ABEF$ on a plane hillside which slopes at an angle of 30° to the horizontal. $ABCD$ is a horizontal rectangle. E and F are 100 m vertically above C and D respectively. $AB = DC = FE = 500$ m.

AE is a straight path.
From B there is a straight path which runs at right angles to AE, meeting it at G.



(a) Find the distance BE. (3)

Solution

$$
\frac{100}{BE} = \sin 30^\circ \Rightarrow BE = \frac{100}{\sin 30^\circ}
$$
$$
\Rightarrow \underline{\underline{BE = 200 \text{ m}}}.
$$

(b) Find the angle that the path AE makes with the horizontal. (4)

Solution

Let us call this angle x. First,

$$
\frac{100}{BC} = \tan 30^\circ \Rightarrow BC = \frac{100}{\tan 30^\circ}.
$$

Second,

$$\begin{aligned} AC &= \sqrt{AB^2 + BC^2} \\ &= \sqrt{500^2 + \left(\frac{100}{\tan 30^\circ}\right)^2} \\ &= \sqrt{280\,000}. \end{aligned}$$

Finally,

$$\begin{aligned} \tan x = \frac{CE}{AC} &\Rightarrow \tan x = \frac{100}{\sqrt{280\,000}} \\ &\Rightarrow x = 10.701\,674\,82 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 10.7^\circ \text{ (3 sf)}}}. \end{aligned}$$

(c) Find the area of the triangle ABE and hence find the length BG. (5)

Solution

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \times AB \times BE \\ &= \tfrac{1}{2} \times 500 \times 200 \\ &= \underline{\underline{50\,000 \text{ m}^2}} \end{aligned}$$

Now,

$$\begin{aligned} AE &= \sqrt{AB^2 + BC^2 + CE^2} \\ &= \sqrt{500^2 + \left(\frac{100}{\tan 30^\circ}\right)^2 + 100^2} \\ &= \sqrt{290\,000}. \end{aligned}$$

and

$$\begin{aligned} \tfrac{1}{2} \times AE \times BG = 50\,000 &\Rightarrow \tfrac{1}{2} \times \sqrt{290\,000} \times BG = 50\,000 \\ &\Rightarrow BG = 185.695\,338\,2 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{BG = 186 \text{ m (3 sf)}}}. \end{aligned}$$

13. In a supermarket chain there are a large number of employees, of whom 40% are male.

One employee is chosen to undergo training.

(a) What assumption is made if 0.4 is taken to be the probability that this employee is male? (1)

Solution

E.g., The selection is random.

6 employees are chosen at random to undergo training.

(b) (i) Show that (2)

$$P(\text{all 6 chosen are female}) = 0.046\,7,$$

correct to 4 decimal places.

Solution

$$\begin{aligned} P(\text{all 6 chosen are female}) &= (0.6)^6 \\ &= 0.046\,656 \text{ (exact value)} \\ &= 0.046\,7 \text{ (4 dp)}. \end{aligned}$$

Find the probability that

(ii) 3 are male and 3 are female, (4)

Solution

$$\begin{aligned} P(\text{3 are male and 3 are female}) &= \binom{6}{3}(0.4)^3(0.6)^3 \\ &= 0.276\,48 \text{ (exact value)} \\ &= 0.276\,5 \text{ (4 dp)}. \end{aligned}$$

(iii) there are more females than males chosen. (5)

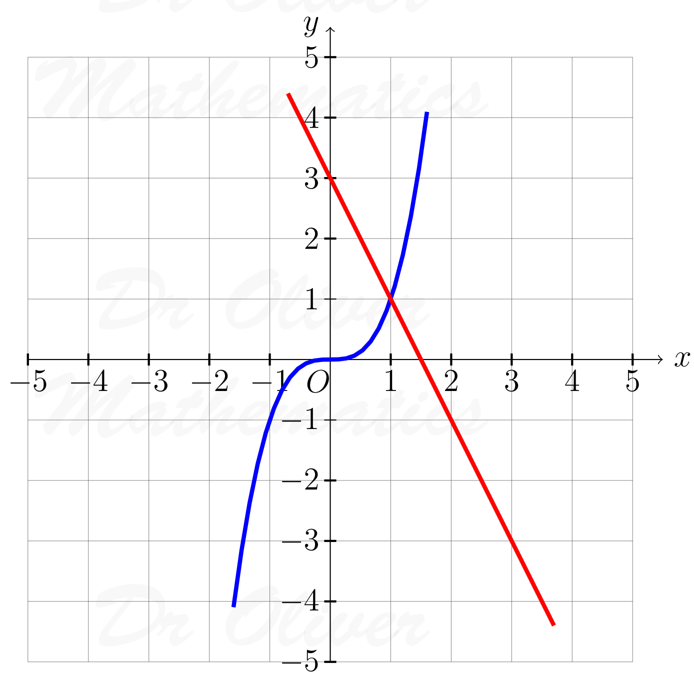
Solution

$$\begin{aligned} \text{P(more females than males)} &= \binom{6}{4}(0.4)^2(0.6)^4 + \binom{6}{5}(0.4)(0.6)^5 + (0.6)^6 \\ &= 0.311\,04 + 0.186\,624 + 0.046\,656 \\ &= 0.544\,32 \text{ (exact value)} \\ &= \underline{\underline{0.544\,3 \text{ (4 dp)}}}. \end{aligned}$$

14. (a) (i) On the same graph, draw sketches of the curve (2)

$$y = x^3 \text{ and the line } y = 3 - 2x.$$

Solution



(ii) Use your sketch to explain why the equation (1)

$$x^3 + 2x - 3 = 0$$

has only one root.

Solution

$$x^3 + 2x - 3 = 0 \Rightarrow x^3 = 3 - 2x$$

and the graphs only intersect at one point.

(b) (i) Show by differentiation that there are no stationary points on the curve (3)

$$y = x^3 + 3x - 4.$$

Solution

$$y = x^3 + 3x - 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 3 \geqslant 3$$

and, hence, there are no stationary points on the curve.

(ii) Hence explain why the equation (1)

$$x^3 + 3x - 4 = 0$$

has only one root.

Solution

E.g., the curve is always increasing it can only cross the x-axis in one point (which is the root).

(c) (i) Use the factor theorem to find an integer root of the equation (1)

$$x^3 + x - 10 = 0.$$

Solution

We will go through the factors of -10: ± 1, ± 2, ± 5, ± 10 until we find the root. Let

$$\mathrm{f}(x) = x^3 + x - 10.$$

Then

$$\begin{aligned}\mathrm{f}(1) &= 1 + 1 - 10 = -8 \\ \mathrm{f}(-1) &= (-1) + (-1) - 10 = -12 \\ \mathrm{f}(2) &= 8 + 2 - 10 = 0;\end{aligned}$$

hence, $x = 2$ is a factor.

(ii) Write the equation (2)

$$x^3 + x - 10 = 0$$

in the form

$$(x-a)(x^2+px+q)=0$$

where a, p, and q are values to be determined.

Solution

We use synthetic division:

2	1	0	1	−10
	↓	2	4	10
	1	2	5	0

Hence,

$$x^3+x-10=0 \Rightarrow \underline{\underline{(x-2)(x^2+2x+5)=0}}.$$

(iii) By considering the quadratic equation (1)

$$x^2+px+q=0$$

found in part (ii), show that the cubic equation

$$x^3+x-10=0$$

has only one root.

Solution

$$\begin{aligned} x^2+2x+5 &= (x^2+2x+1)+4 \\ &= (x+1)^2+4 \\ &\geqslant 4; \end{aligned}$$

hence, the cubic equation <u>has only one root</u>.

You are given that r and s are positive numbers.

(d) What do the results in parts (a), (b) and (c) suggest about the equation (1)

$$x^3+rx-s=0?$$

Solution

E.g., for all r and s, the equation will only have <u>one root</u>.