Dr Oliver Mathematics
Histograms, Frequency Polygons, Mode from a Histogram, Cumulative Frequency Diagrams, and Histograms Without A Histogram

There is evident variation in the syllabus and presentation of statistical diagrams. The variation arises in textbooks, in syllabus documentation, and on examination papers. Statistical techniques can appear can appear as a set of discrete skills but, in fact, from an integrated field of study.

A note regarding the labelling of axes and scale. As an ex-teacher once said to me, "Remember that *values* are plotted and not *measurements*; consequently, neither the gradient nor the area have units. (The gradient may represent velocity in metres per second, for example, but is itself dimensionless.) Hence, the only meaningful labelling involves dividing the measurement by the relevant units to obtain a value; e.g., length/cm, *not* length in cm, *nor* length (cm), etc."

Now, a lot of the above makes sense, especially the part of gradient nor the area have units. But we will ignore that.

Part I

Histograms

1 Examples

Example 1: Continuous data
Draw a histogram to represent these data.

Mass (x kg)	Frequency
$0 \leqslant x < 6$	12
$6 \leqslant x < 10$	7
$10 \leqslant x < 12$	11
$12 \leqslant x < 16$	8

Solution 1
First, we need to add two columns to be table: 'Width' and 'Frequency Density.'

Mass (x kg)	Frequency	Width	Frequency Density
$0 \leqslant x < 6$	12		
$6 \leqslant x < 10$	7		
$10 \leqslant x < 12$	11		
$12 \leqslant x < 16$	8		

Table 1: 'Width' and 'Frequency Density' added

We will start with the $0 \leqslant x < 6$: how wide is that? Subtract:

$$6 - 0 = 6.$$

Mass (x kg)	Frequency	Width	Frequency Density
$0 \leqslant x < 6$	12	$6 - 0 = 6$	
$6 \leqslant x < 10$	7	$10 - 6 = 4$	
$10 \leqslant x < 12$	11	$12 - 10 = 2$	
$12 \leqslant x < 16$	8	$16 - 12 = 4$	

Table 2: 'Width' added

What is 'Frequency Density'? Usually,

$$\text{frequency density} \propto \frac{\text{frequency}}{\text{width}}.$$

Now, it could be that

$$\text{frequency density} = \frac{2 \times \text{frequency}}{\text{width}}$$

or

$$\text{frequency density} = \frac{0.637 \times \text{frequency}}{\text{width}},$$

and so on. But we will (unless circumstances choose otherwise) stick with the classic:

$$\boxed{\text{frequency density} = \frac{\text{frequency}}{\text{width}}.}$$

For example, $0 \leqslant x < 6$:

$$\text{Frequency density} = \frac{12}{6} = 2.$$

Mass (x kg)	Frequency	Width	Frequency Density
$0 \leqslant x < 6$	12	6	$\frac{12}{6} = 2$
$6 \leqslant x < 10$	7	4	$\frac{7}{4} = 1.75$
$10 \leqslant x < 12$	11	2	$\frac{11}{2} = 5.5$
$12 \leqslant x < 16$	8	4	$\frac{8}{4} = 2$

Table 3: 'Frequency Density' added

And, finally, draw a set of axes, with 'Mass' going to the right and 'Frequency Density' up.

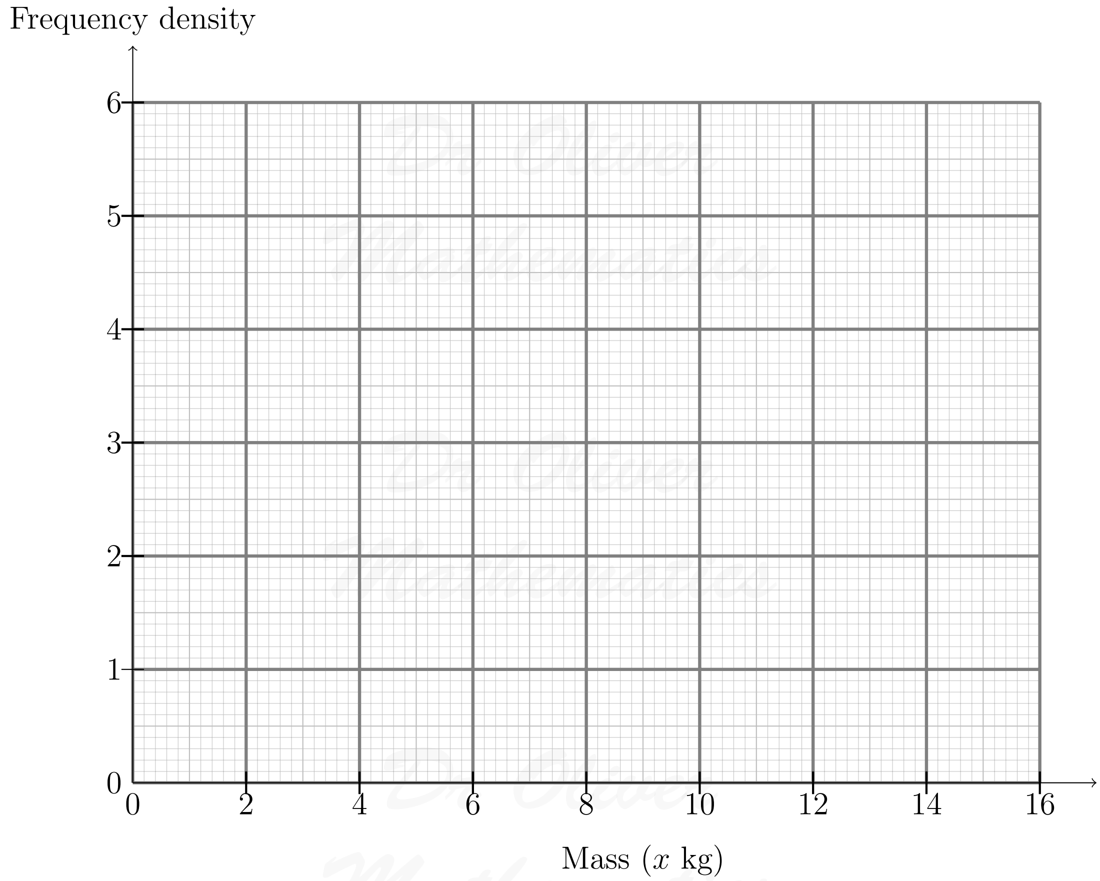


Figure 1: drawing the axes

And we complete the histogram:

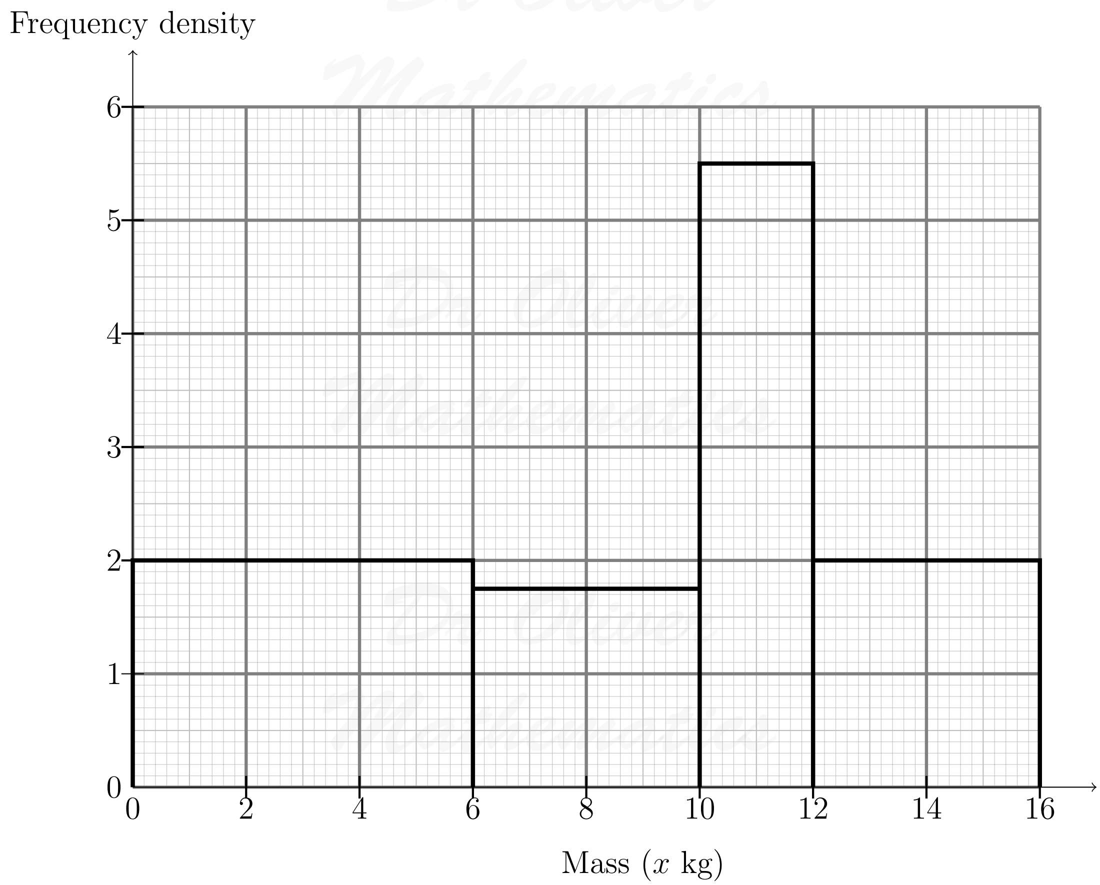


Figure 2: histogram completed

And that's it!

Example 2: Discrete data (whole numbers)

Marks	Frequency
$0-4$	4
$5-8$	12
$9-11$	5
$12-15$	6

Solution 2
$0-4$ and $5-8$ will leave a gap between 4 and 5.

Hmm.

Well, what if we widened the gap between $5-8$ into $4\frac{1}{2}-8\frac{1}{2}$?

That works: 5, 6, 7, and 8 (unlike $8-5=3$) and we count four items, just like

$$8\tfrac{1}{2}-4\tfrac{1}{2}=4.$$

What will happen to $0-4$? Simply do $-\frac{1}{2}-4\frac{1}{2}$ and count: 0, 1, 2, 3, and 4 and

$$4\tfrac{1}{2}-(-\tfrac{1}{2})=5.$$

In short, we expand all the intervals by $\frac{1}{2}$ from the left and from the right.

Marks	Frequency	Marks	Width	Frequency Density
$0-4$	4	$-\frac{1}{2}\leqslant \text{marks} < 4\frac{1}{2}$		
$5-8$	12	$4\frac{1}{2}\leqslant \text{marks} < 8\frac{1}{2}$		
$9-11$	5	$8\frac{1}{2}\leqslant \text{marks} < 11\frac{1}{2}$		
$12-15$	6	$11\frac{1}{2}\leqslant \text{marks} < 15\frac{1}{2}$		

Table 4: new ‘Width’ added

Complete the table:

Marks	Frequency	Marks	Width	Frequency Density
$0-4$	4	$-\frac{1}{2}\leqslant \text{marks} < 4\frac{1}{2}$	5	$\frac{4}{5}=0.8$
$5-8$	12	$4\frac{1}{2}\leqslant \text{marks} < 8\frac{1}{2}$	4	$\frac{12}{4}=3$
$9-11$	5	$8\frac{1}{2}\leqslant \text{marks} < 11\frac{1}{2}$	3	$\frac{5}{3}=1\frac{2}{3}$
$12-15$	6	$11\frac{1}{2}\leqslant \text{marks} < 15\frac{1}{2}$	4	$\frac{6}{4}=1.5$

Table 5: table completed

Complete the histogram:

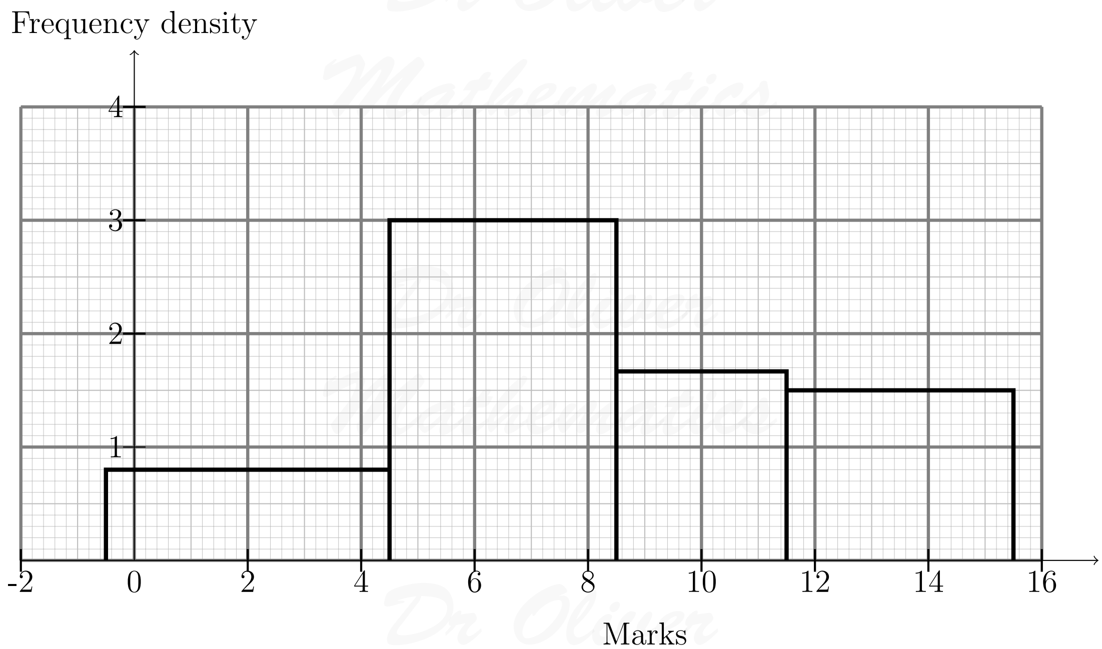


Figure 3: histogram completed

or even this:

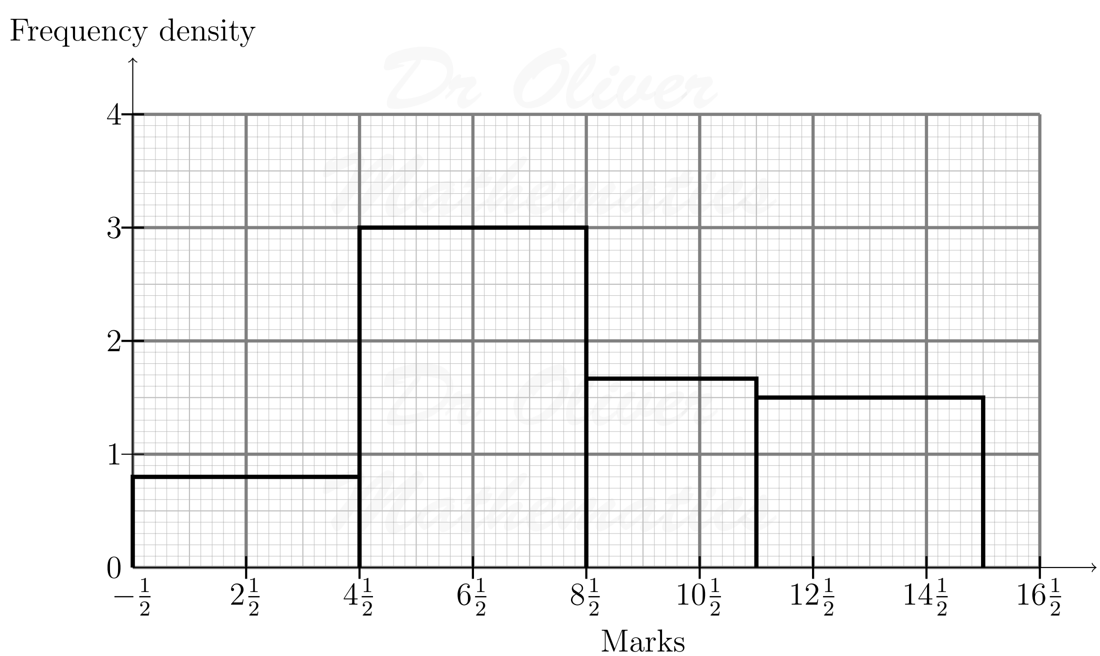


Figure 4: shifted along by $\frac{1}{2}$

Example 3: Discrete data (fractional values)
Draw a histogram to represent these data.

Shoe Size	Frequency
$2 - 2\frac{1}{2}$	4
$3 - 4$	3
$4\frac{1}{2} - 6$	5
$6\frac{1}{2} - 8$	4

Solution 3
Where do $2 - 2\frac{1}{2}$ and $3 - 4$ meet? That's right: at

$$\frac{2\frac{1}{2} + 3}{2} = 2\frac{3}{4}$$

so we expand all the intervals by $\frac{1}{4}$ from the left and from the right.

Shoe size	Frequency	Shoe Size	Width	Frequency Density
$2 - 2\frac{1}{2}$	4	$1\frac{3}{4} \leqslant \text{s/s} < 2\frac{3}{4}$	1	$\frac{4}{1} = 4$
$3 - 4$	3	$2\frac{3}{4} \leqslant \text{s/s} < 4\frac{1}{4}$	1.5	$\frac{3}{1.5} = 2$
$4\frac{1}{2} - 6$	5	$4\frac{1}{4} \leqslant \text{s/s} < 6\frac{1}{4}$	2	$\frac{5}{2} = 2.5$
$6\frac{1}{2} - 8$	4	$6\frac{1}{4} \leqslant \text{s/s} < 8\frac{1}{4}$	2	$\frac{4}{2} = 2$

Table 6: table completed

Complete the histogram:

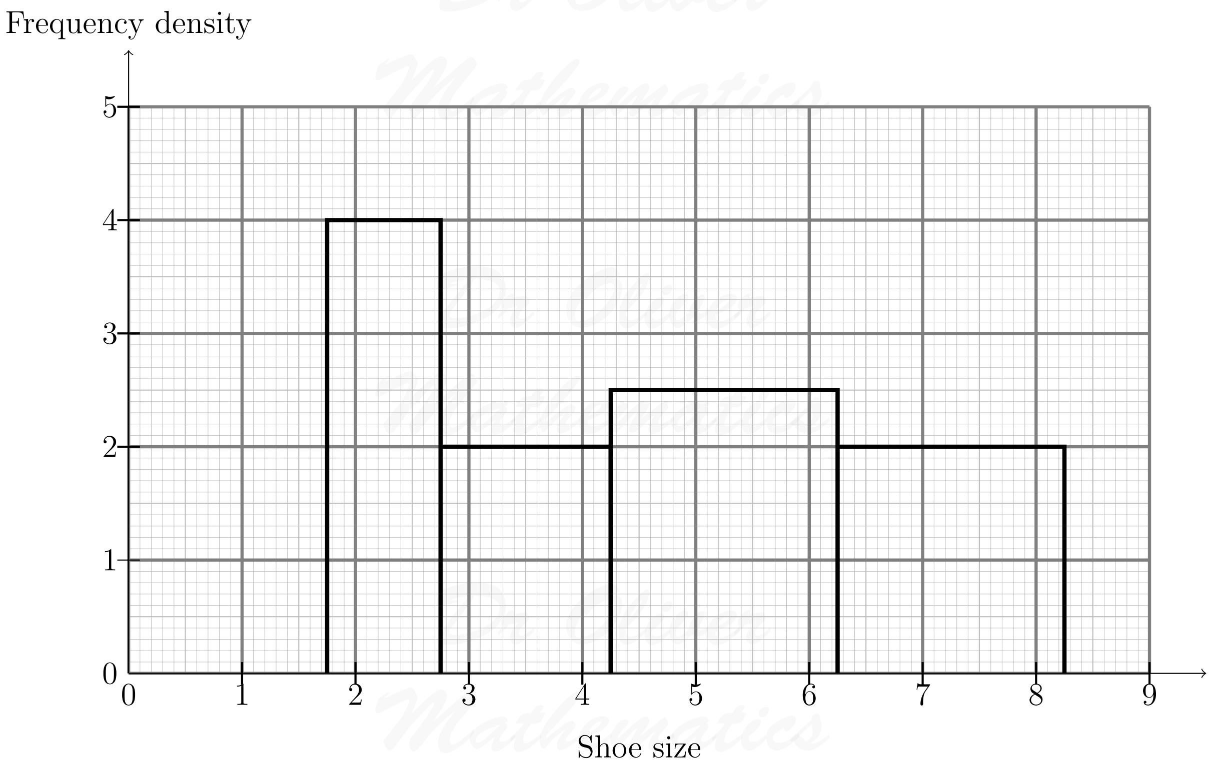


Figure 5: histogram completed

or even this:

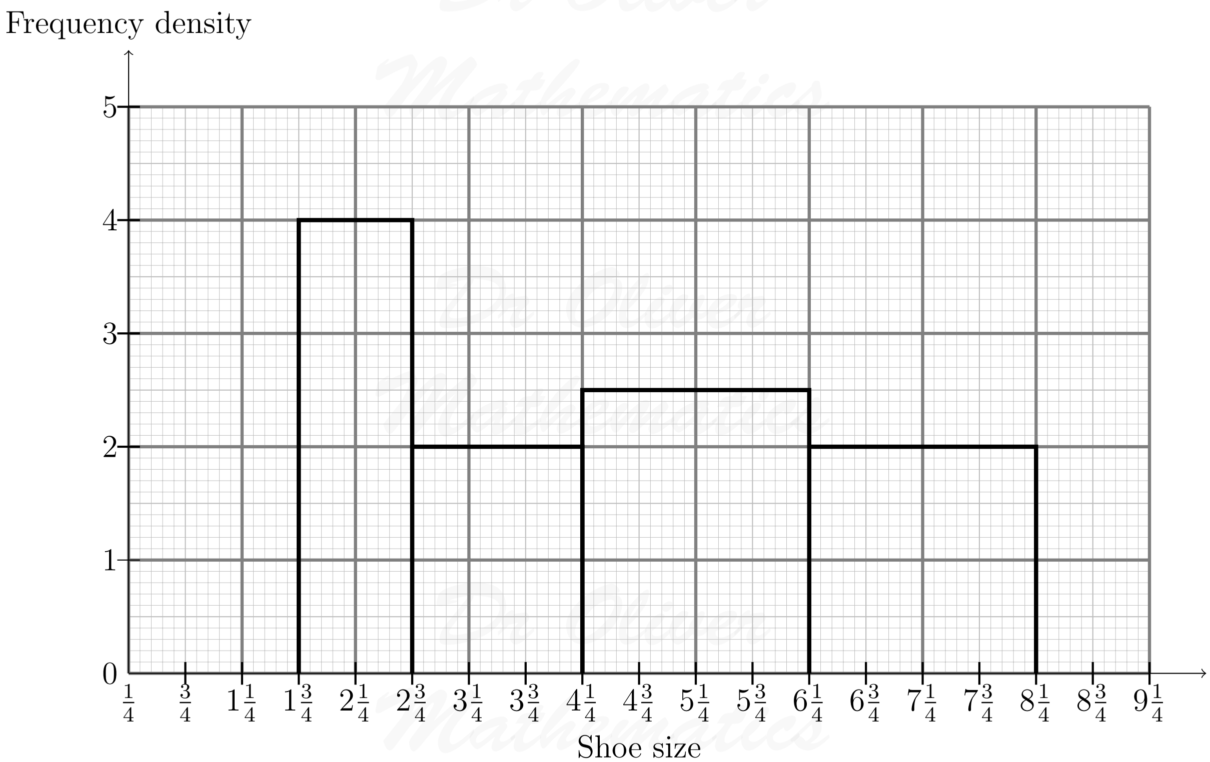


Figure 6: histogram completed

Example 4: Age (continuous – time – but represented by a discrete number)
Draw a histogram to represent these data.

Age (years)	Frequency
0 – 5	10
6 – 8	12
9 – 17	18
18 – 20	7

Solution 4
Because someone who is 5 years, 364 days, 59 minutes, and 59 seconds is *still* 5 years old.

Complete the table:

Age (years)	Frequency	Age (years)	Width	Frequency Density
$0-5$	10	$0 < x$ years < 6	6	$\frac{10}{6} = 1\frac{2}{3}$
$6-8$	12	$6 \leqslant x$ years < 9	3	$\frac{12}{3} = 4$
$9-17$	18	$9 \leqslant x$ years < 18	9	$\frac{18}{9} = 2$
$18-20$	7	$18 \leqslant x$ years < 21	3	$\frac{7}{3} = 2\frac{1}{3}$

Table 7: table completed

Complete the histogram:

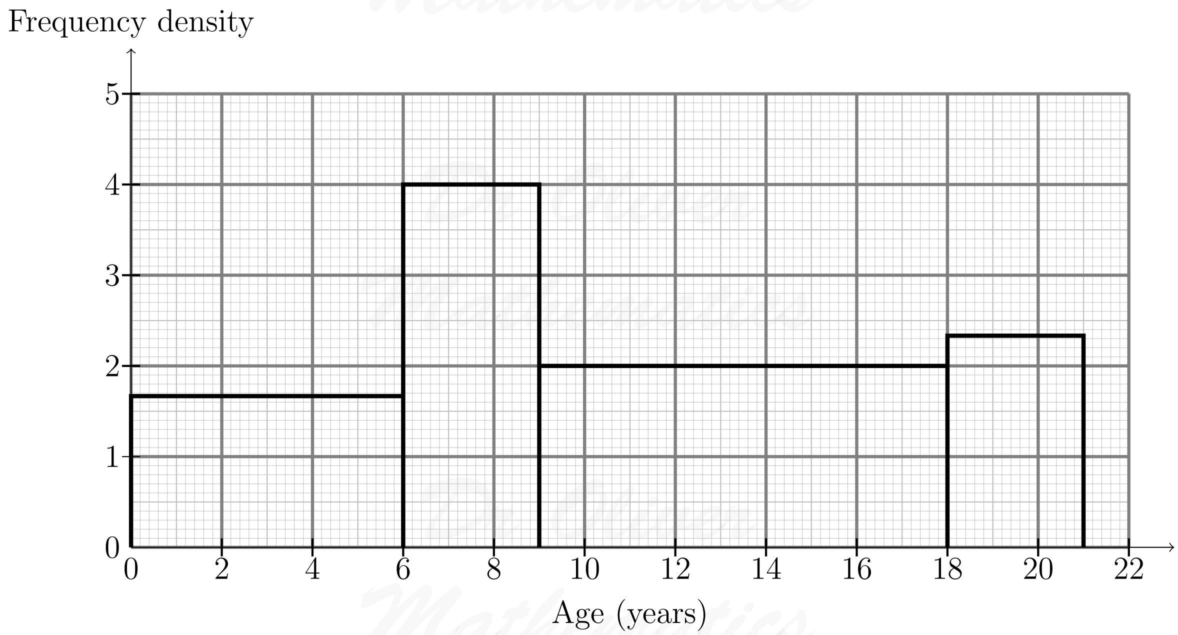
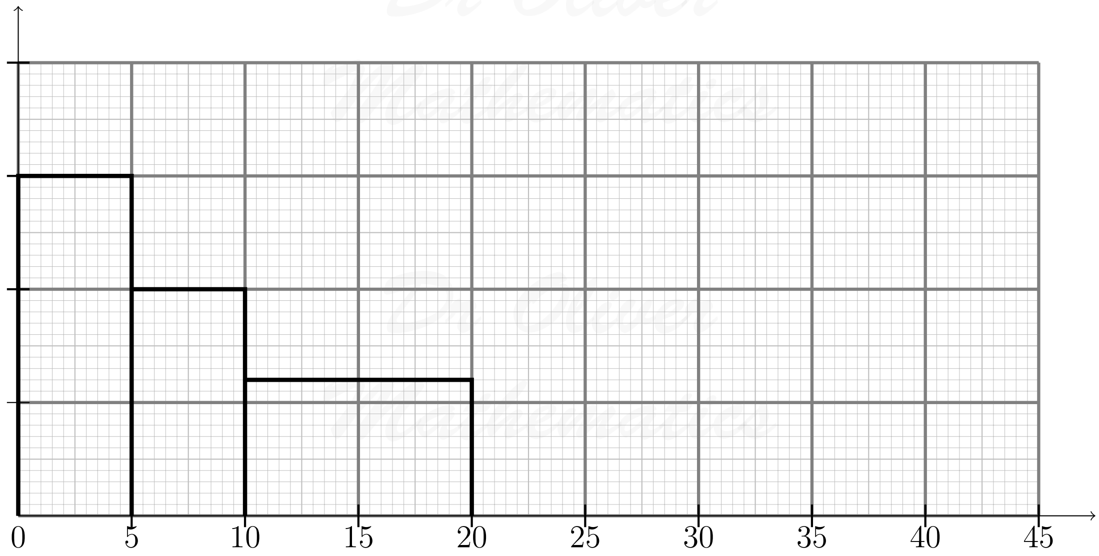


Figure 7: histogram completed

What happen if we were only given part of the histogram?

Example 5: Continuous data
The incomplete table and histogram show, some of the times taken by a number of students to complete a mathematical puzzle.

Time (t minutes)	Frequency
$0 \leqslant t < 5$	15
$5 \leqslant t < 10$	
$10 \leqslant t < 20$	
$20 \leqslant t < 40$	7



(a) Use the information in the table to complete the histogram.

(b) Use the information in the histogram to complete the table.

Solution 5
Well, we complete what we can of the table:

Time (t minutes)	Frequency	Width	Frequency Density
$0 \leqslant t < 5$	15	5	$\frac{15}{5} = 3$
$5 \leqslant t < 10$		5	2
$10 \leqslant t < 20$		10	1.2
$20 \leqslant t < 40$	7	20	$\frac{7}{20} = 0.35$

Table 8: we complete what we can of the table

$\underline{0 \leqslant t < 5}$: its frequency density is 3 so we write that in – plus the fact that the vertical direction is 'Frequency Density.'

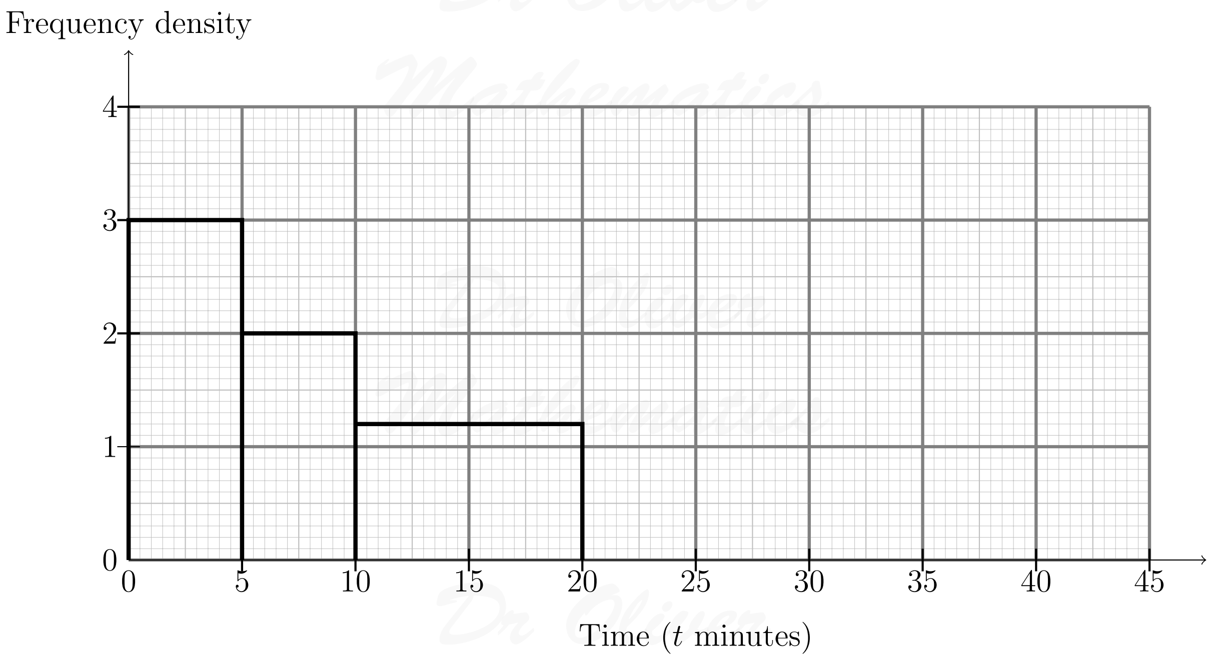


Figure 8: $0 \leqslant t < 5$: its frequency density is 3

And, all that remains, is to complete the histogram.

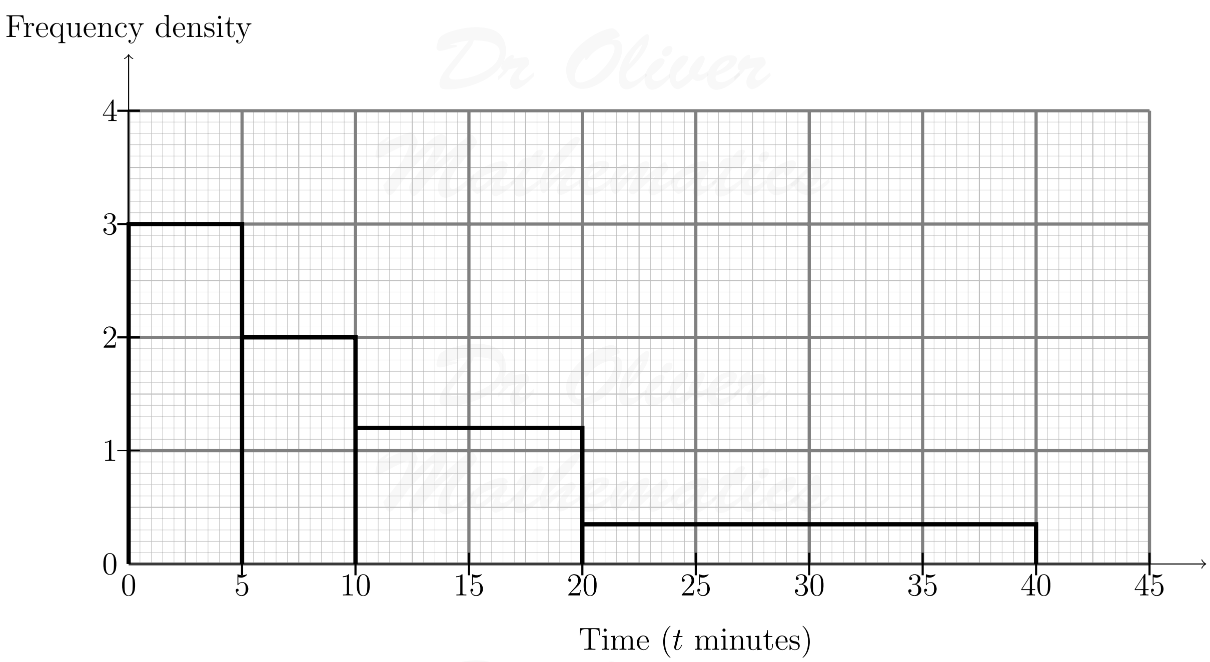


Figure 9: histogram completed

(b) Now, we want the frequencies of $5 \leqslant t < 10$ and $10 \leqslant t < 20$. Recall,

$$\text{frequency density} = \frac{\text{frequency}}{\text{width}} \Rightarrow \text{frequency} = \text{width} \times \text{frequency density}.$$

$\underline{5 \leqslant t < 10}$:

$$\text{Frequency} = 5 \times 2 = 10.$$

$\underline{10 \leqslant t < 20}$:

$$\text{Frequency} = 10 \times 1.2 = 12.$$

Finally, we complete the table.

Time (t minutes)	Frequency	Width	Frequency Density
$0 \leqslant t < 5$	15	5	$\frac{15}{5} = 3$
$5 \leqslant t < 10$	$\underline{\underline{10}}$	5	$\frac{10}{5} = 2$
$10 \leqslant t < 20$	$\underline{\underline{12}}$	10	$\frac{12}{10} = 1.2$
$20 \leqslant t < 40$	7	20	$\frac{7}{20} = 0.35$

Table 9: table completed

2 Problems

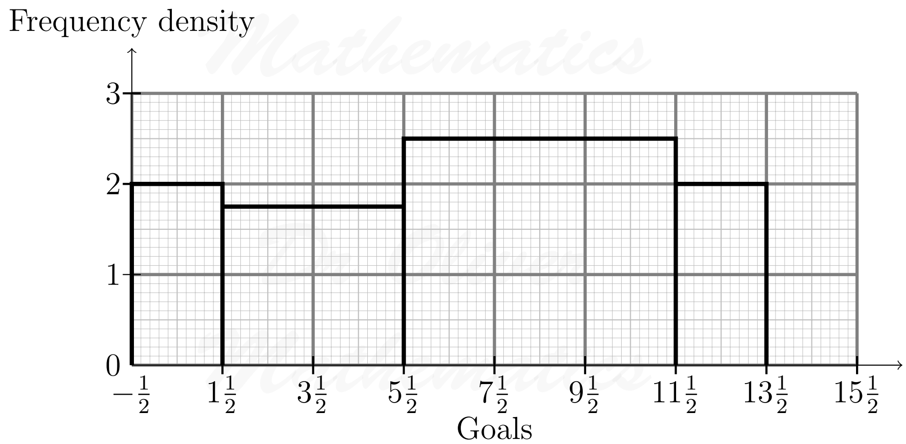
Here are a few problems for you to try.

1. Draw a histogram to represent these data.

Goals	Frequency
$0 - 1$	4
$2 - 5$	7
$6 - 11$	15
$12 - 13$	4

Solution

Goals	Frequency	Marks	Width	Frequency Density
$0 - 1$	4	$-\frac{1}{2} - 1\frac{1}{2}$	2	$\frac{4}{2} = 2$
$2 - 5$	7	$1\frac{1}{2} - 5\frac{1}{2}$	4	$\frac{7}{4} = 1.75$
$6 - 11$	15	$5\frac{1}{2} - 11\frac{1}{2}$	6	$\frac{15}{6} = 2.5$
$12 - 13$	4	$11\frac{1}{2} - 13\frac{1}{2}$	2	$\frac{4}{2} = 2$

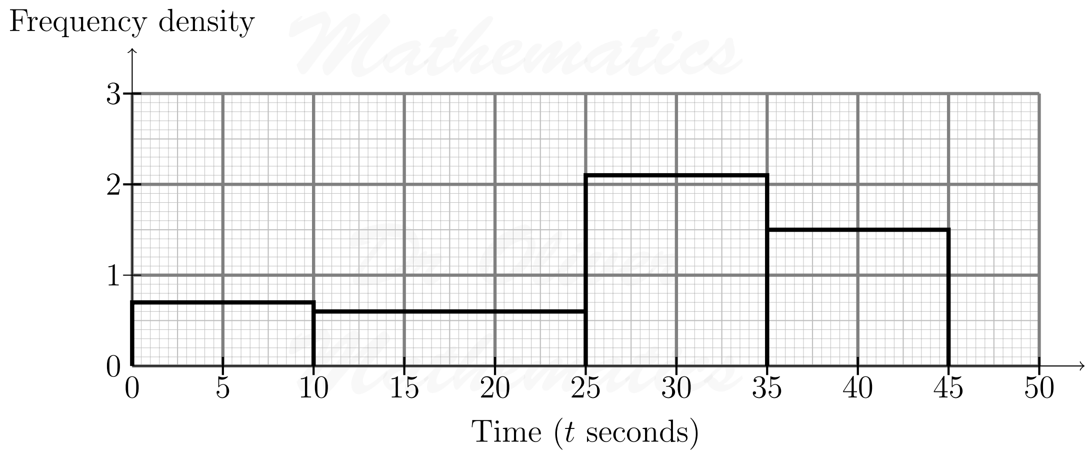


2. Draw a histogram to represent these data.

Time (t seconds)	Frequency
$0 \leqslant t < 10$	7
$10 \leqslant t < 25$	9
$25 \leqslant t < 35$	21
$35 \leqslant t < 45$	15

Solution

Time (t seconds)	Frequency	Width	Frequency Density
$0 \leqslant t < 10$	7	10	$\frac{7}{10} = 0.7$
$10 \leqslant t < 25$	9	15	$\frac{9}{15} = 0.6$
$25 \leqslant t < 35$	21	10	$\frac{21}{10} = 2.1$
$35 \leqslant t < 45$	15	10	$\frac{15}{10} = 1.5$

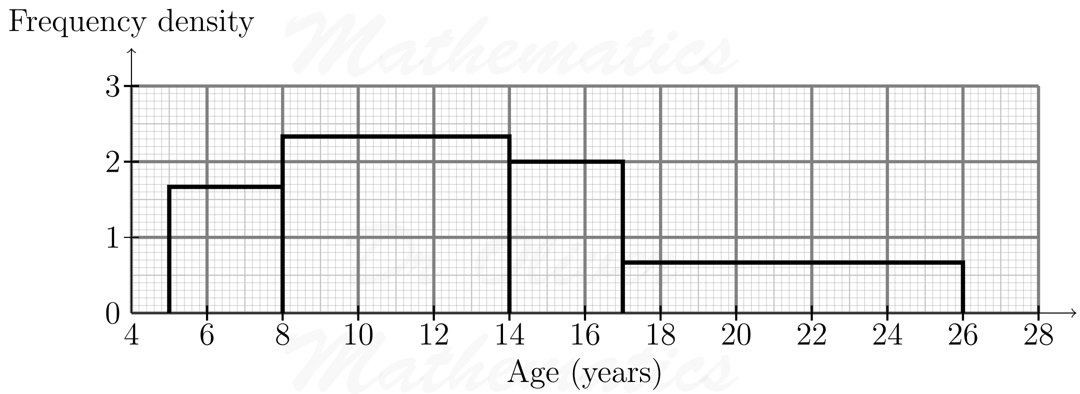


3. Draw a histogram to represent these data.

Age (years)	Frequency
5 – 7	5
8 – 13	14
14 – 16	6
17 – 25	6

Solution

Age (years)	Frequency	Age (years)	Width	Frequency Density
5 – 7	5	$5 \leqslant x$ years < 8	3	$\frac{5}{3} = 1\frac{2}{3}$
8 – 13	14	$8 \leqslant x$ years < 14	6	$\frac{14}{6} = 2\frac{1}{3}$
14 – 16	6	$14 \leqslant x$ years < 17	3	$\frac{6}{3} = 2$
17 – 25	6	$17 \leqslant x$ years < 26	9	$\frac{6}{9} = \frac{2}{3}$

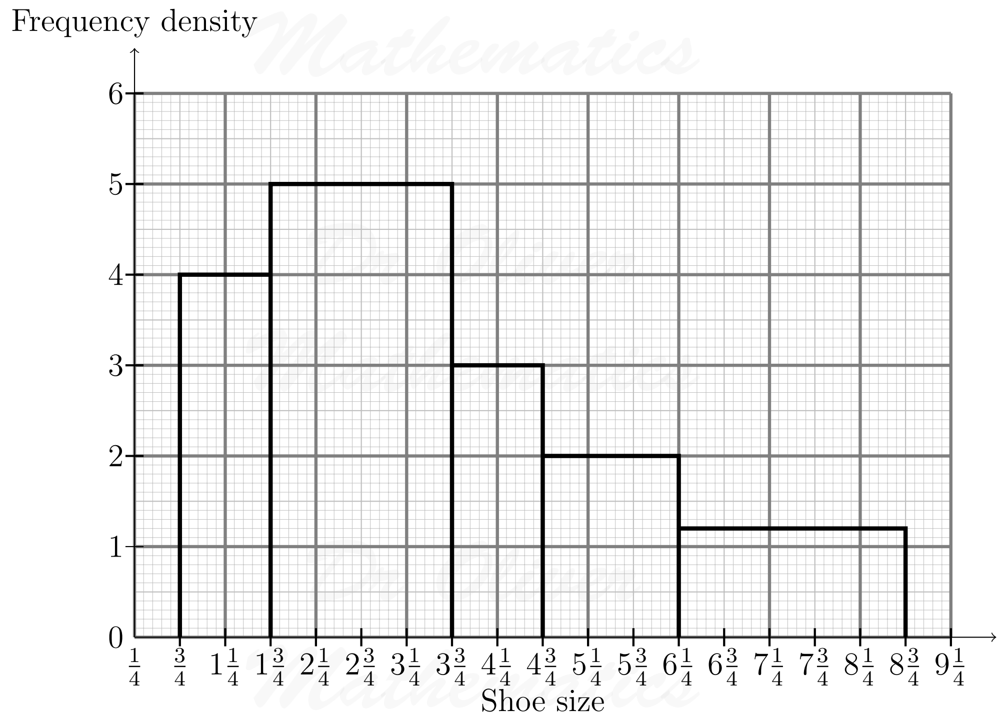


4. Draw a histogram to represent these data.

Shoe Size	Frequency
$1-1\frac{1}{2}$	4
$2-3\frac{1}{2}$	10
$4-4\frac{1}{2}$	3
$5-6$	3
$6\frac{1}{2}-8\frac{1}{2}$	3

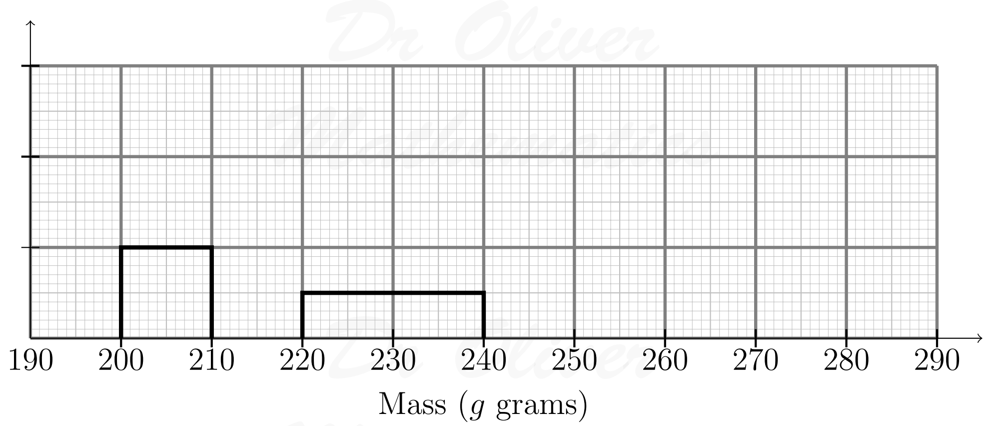
Solution

Shoe Size	Frequency	Shoe Size	Width	Frequency Density
$1-1\frac{1}{2}$	4	$\frac{3}{4} \leqslant \text{s/s} < 1\frac{3}{4}$	1	$\frac{4}{1}=4$
$2-3\frac{1}{2}$	10	$1\frac{3}{4} \leqslant \text{s/s} < 3\frac{3}{4}$	2	$\frac{10}{2}=5$
$4-4\frac{1}{2}$	3	$3\frac{3}{4} \leqslant \text{s/s} < 4\frac{3}{4}$	1	$\frac{3}{1}=3$
$5-6$	3	$4\frac{3}{4} \leqslant \text{s/s} < 6\frac{1}{4}$	$1\frac{1}{2}$	$\frac{3}{1\frac{1}{2}}=2$
$6\frac{1}{2}-8\frac{1}{2}$	3	$6\frac{1}{4} \leqslant \text{s/s} < 8\frac{3}{4}$	$2\frac{1}{2}$	$\frac{3}{2\frac{1}{2}}=1\frac{1}{5}$



5. The mass g grams of grapefruit are summarised in the table and in the partially completed histogram.

Mass (g grams)	Frequency
$200 \leqslant g < 210$	10
$210 \leqslant g < 220$	20
$220 \leqslant g < 240$	
$240 \leqslant g < 280$	60



(a) Use the information in the table to complete the histogram.

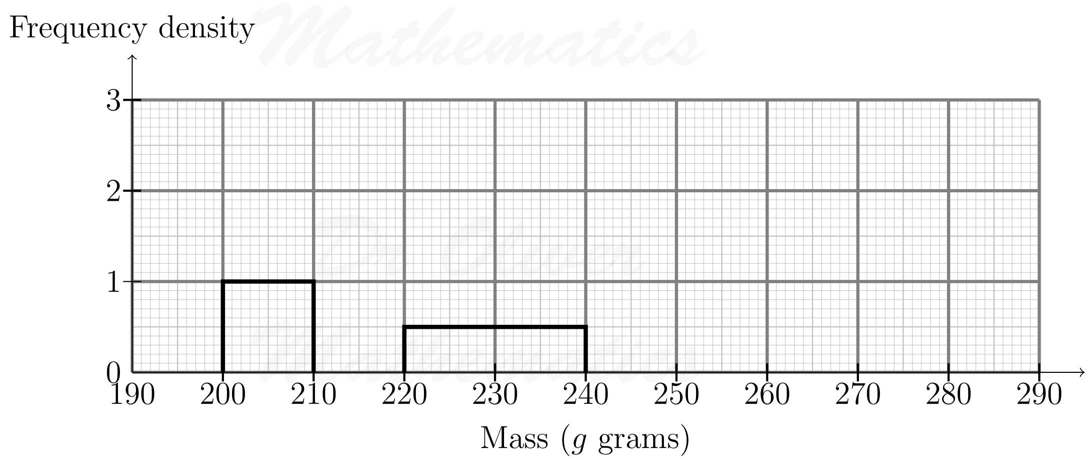
Solution

Well, we complete what we can of the table:

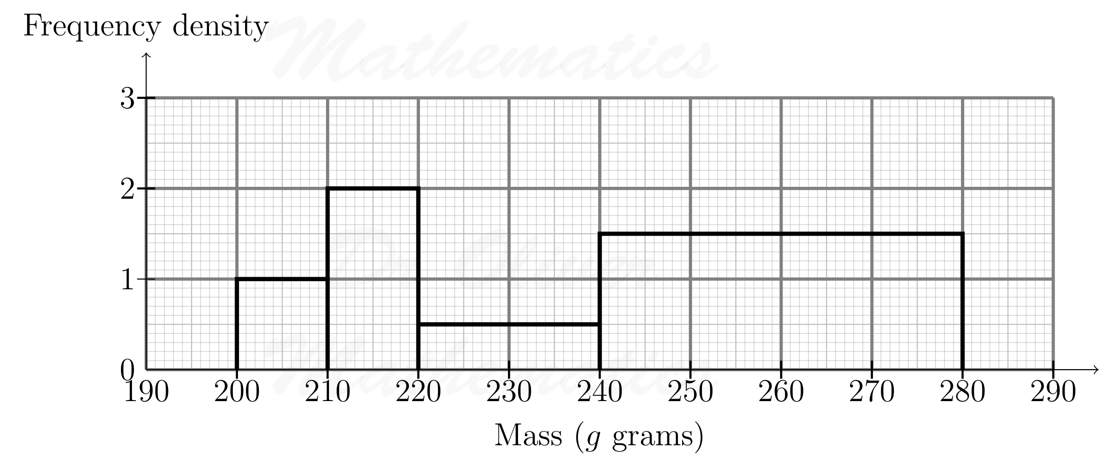
Mass (g grams)	Frequency	Width	Frequency Density
$200 \leqslant g < 210$	10	10	$\frac{10}{10} = 1$
$210 \leqslant g < 220$	20	10	$\frac{20}{10} = 2$
$220 \leqslant g < 240$		20	
$240 \leqslant g < 280$	60	40	$\frac{60}{40} = 1.5$

Table 10: we complete what we can of the table

$\underline{200 \leqslant g < 210}$: its frequency density is 1 so we write that in – plus the fact that the vertical direction is 'Frequency Density.'



And, all that remains, is to complete the histogram.



(b) Use the information in the histogram to complete the table

Solution

$$\text{Frequency} = 20 \times 0.5 = 10.$$

Mass (g grams)	Frequency	Width	Frequency Density
$200 \leqslant g < 210$	10	10	$\frac{10}{10} = 1$
$210 \leqslant g < 220$	20	10	$\frac{20}{10} = 2$
$220 \leqslant g < 240$	$\underline{\underline{10}}$	20	$\frac{10}{20} = 0.5$
$240 \leqslant g < 280$	60	40	$\frac{60}{40} = 1.5$

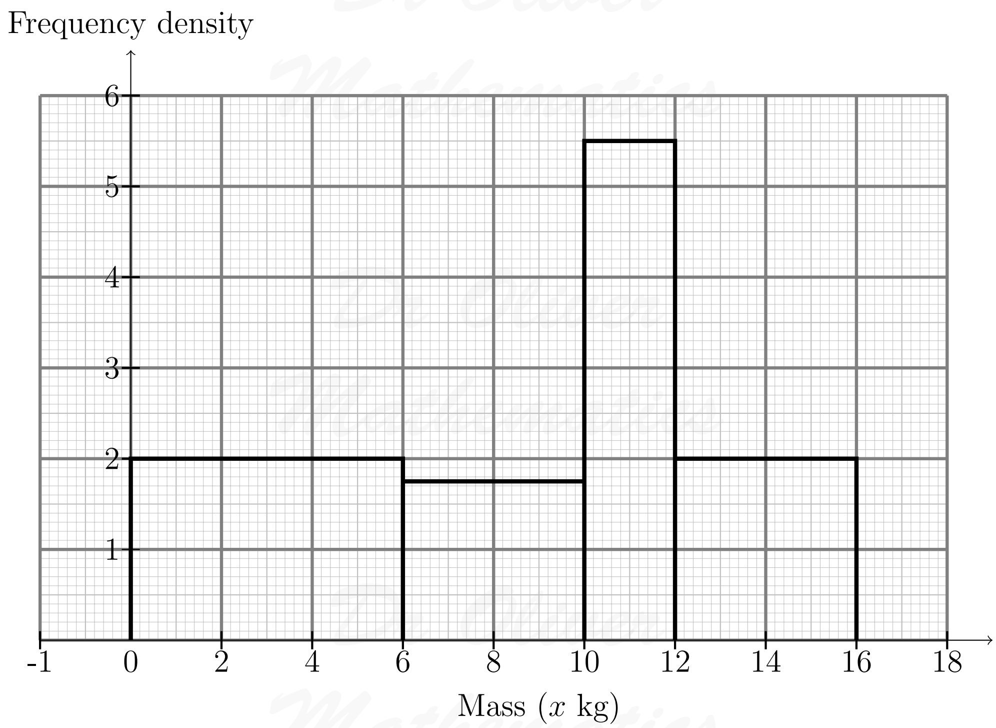
Part II

Frequency Polygons and Histograms

3 Examples

Example 6

Draw the frequency polygon associated with the following histogram.



Solution 6
Isn't it just that?

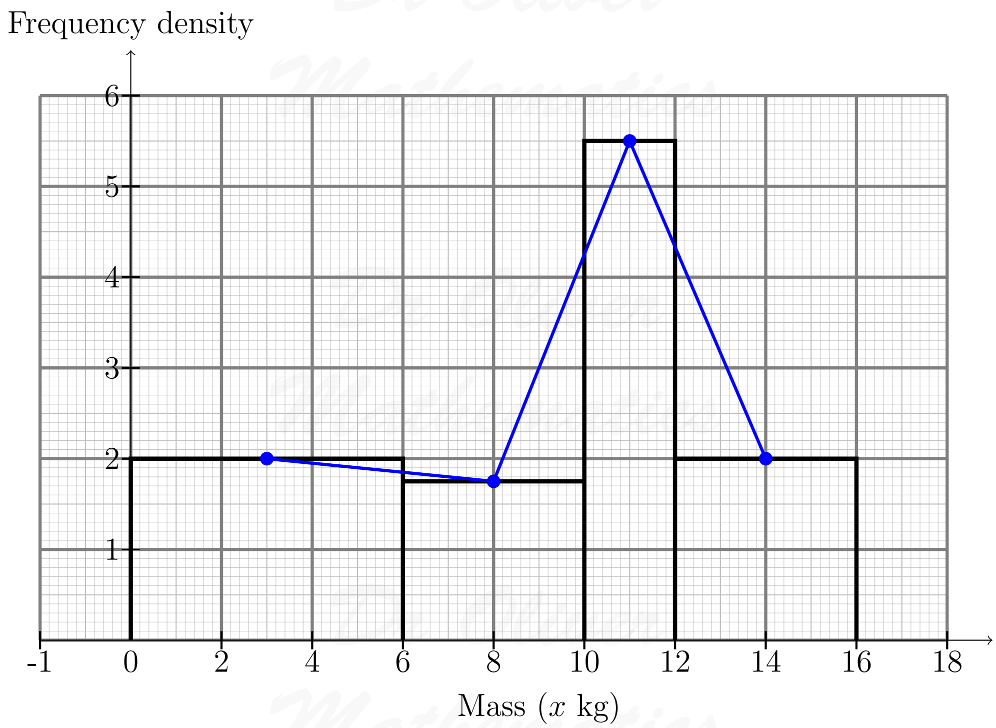


Figure 10: a frequency polygon?

Well, no.

First, ‘polygon’ suggests a closed shape but the frequency polygon is ‘open’. Second, the rationale behind joining the polygon to the x-axis is the area enclosed by the polygon represents the total frequency.

To conserve the area property, you must modify the procedure as follows:
(a) take the width of your narrowest bar to be the arbitrary interval. In this example, we have widths of 6, 4, 2, and 4 so we take 2.
(b) Plot mid-points on each each of those intervals (and this means you will end up with more than one point on any bar that is wider than the narrowest).
(c) Join up the points.
(d) And don’t forgot to do the point at the left-hand edge of the frequency polygon and the point at the right-hand edge of the frequency polygon.

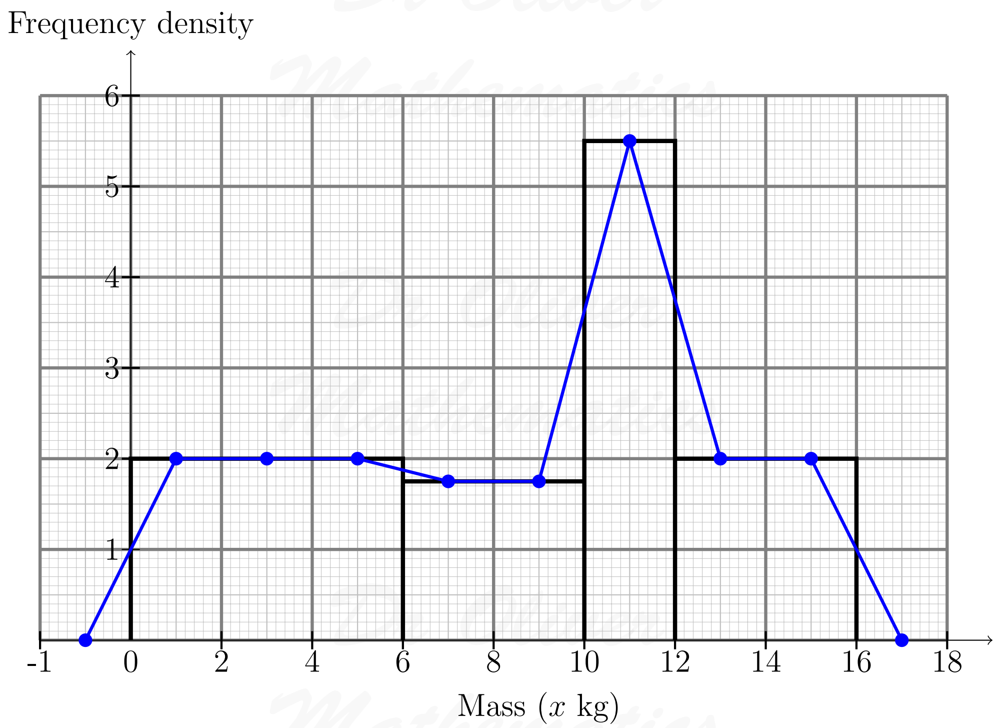
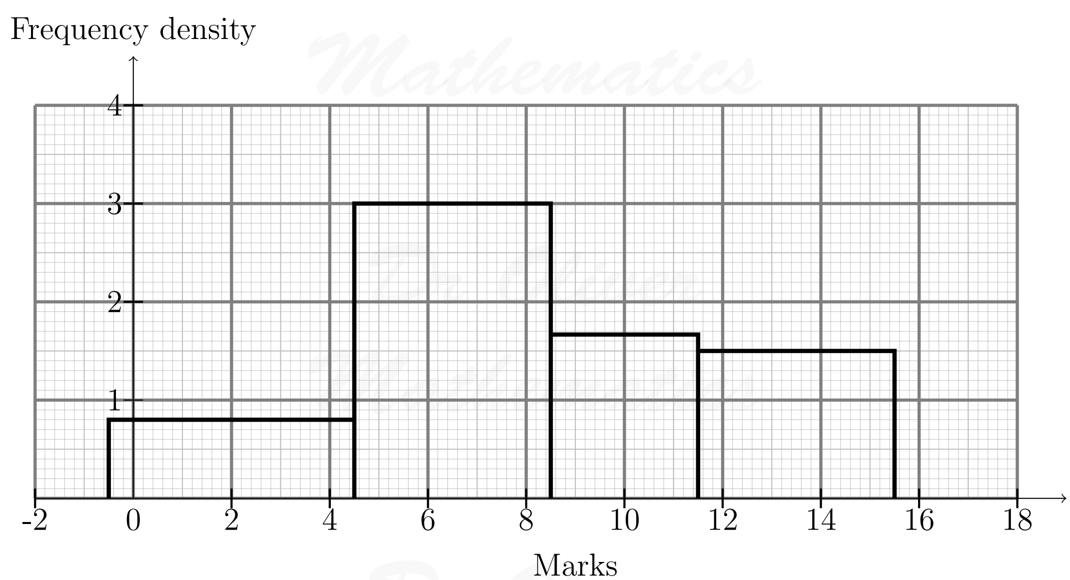


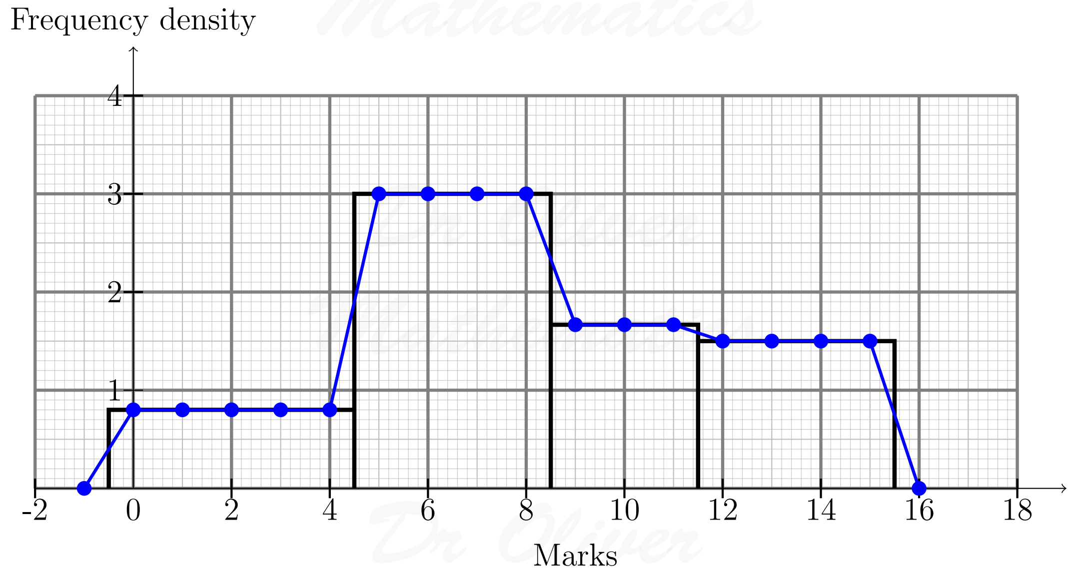
Figure 11: a frequency polygon

Example 7
Draw the frequency polygon associated with the following histogram.

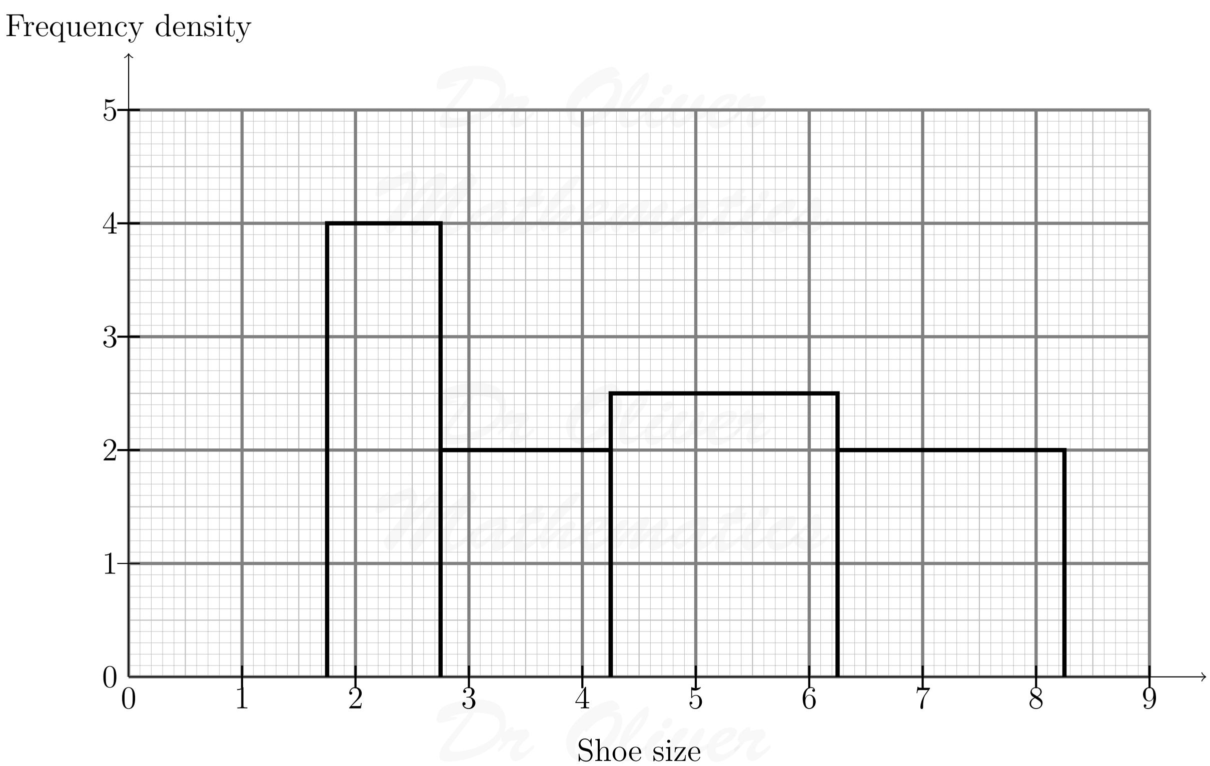


Solution 7

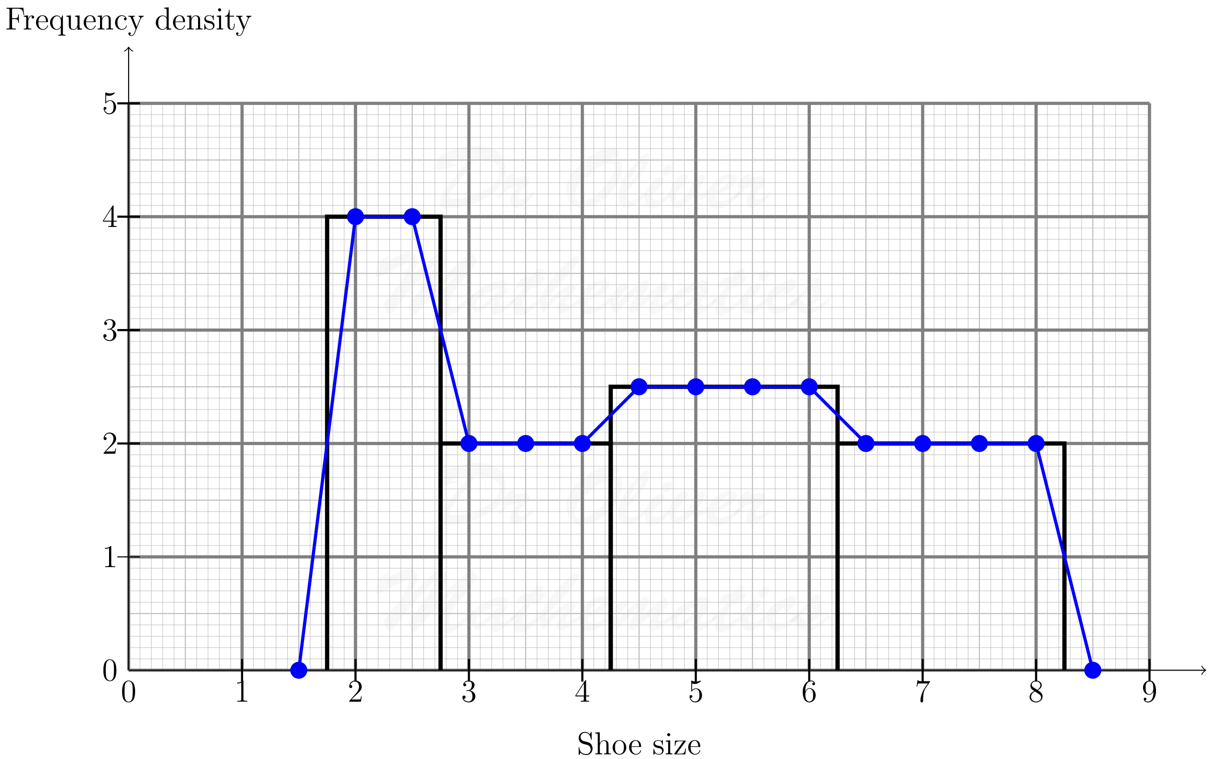
The widths are 5, 4, 3, and 4 and so we pick 1.



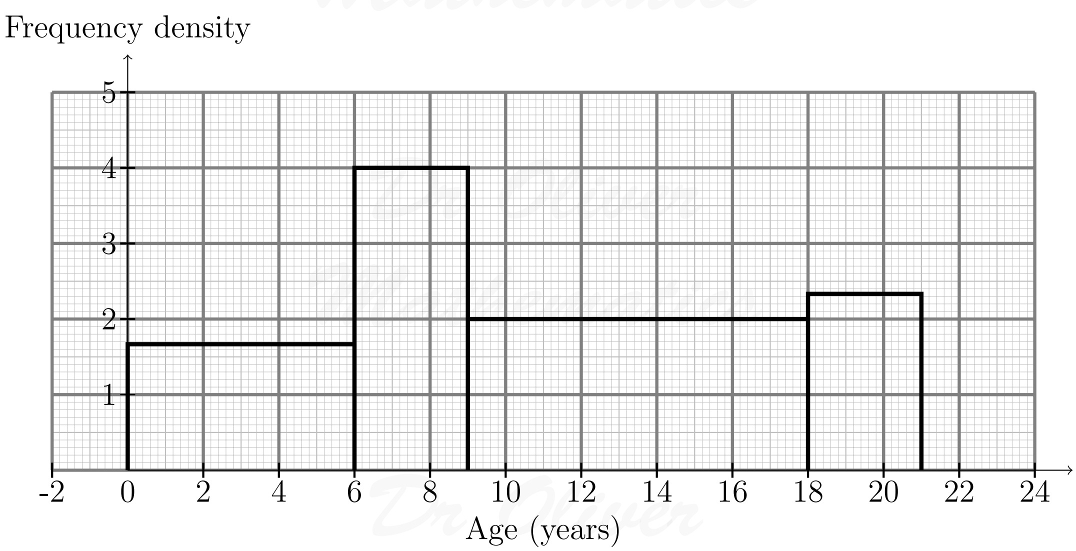
Example 8
Draw the frequency polygon associated with the following histogram.



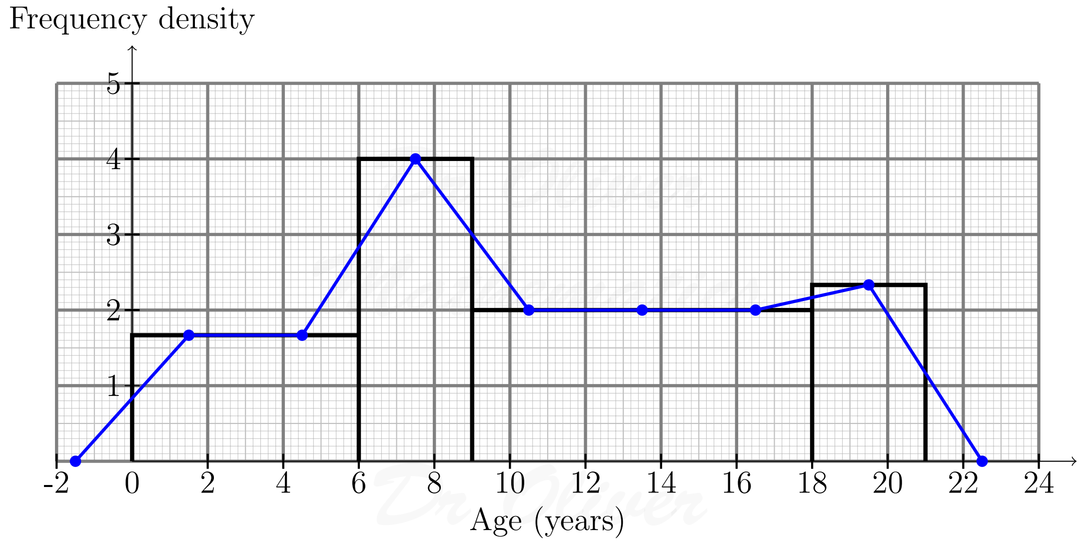
Solution 8
The widths are 1, 1.5, 2, and 2 and so we pick 0.5.



Example 9
Draw the frequency polygon associated with the following histogram.



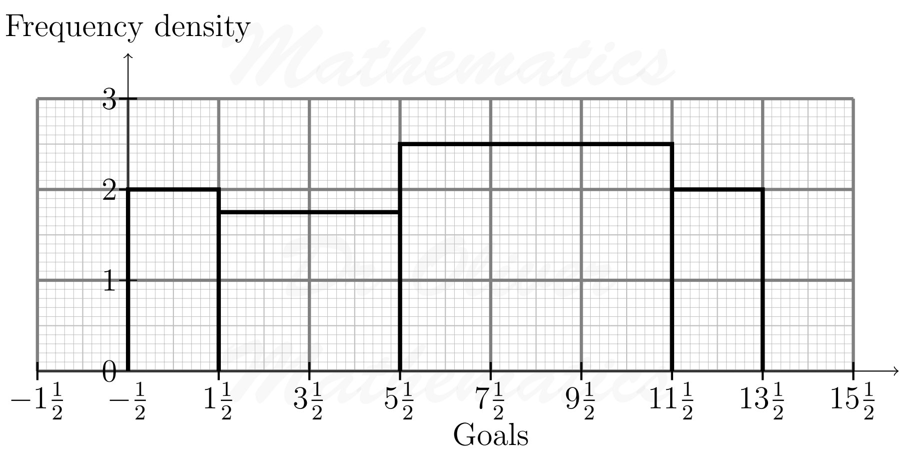
Solution 9
The widths are 6, 3, 9, and 3 and so we pick 3.



4 Problems

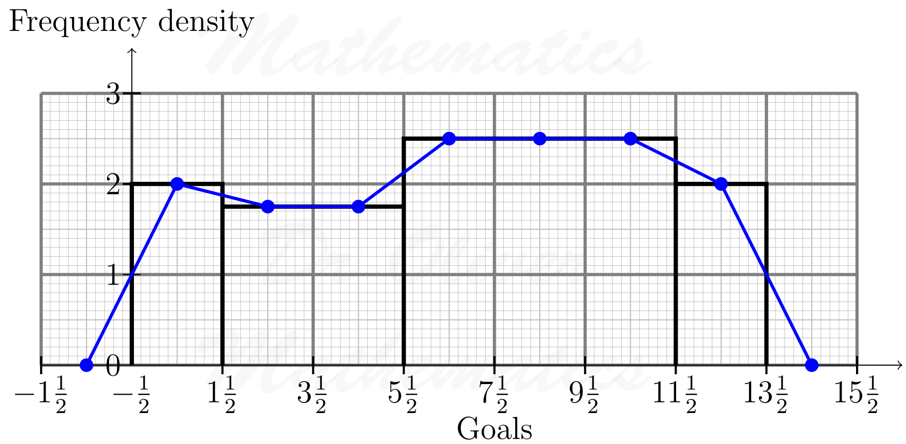
Here are a few problems for you to try.

6. Draw the frequency polygon associated with the following histogram.

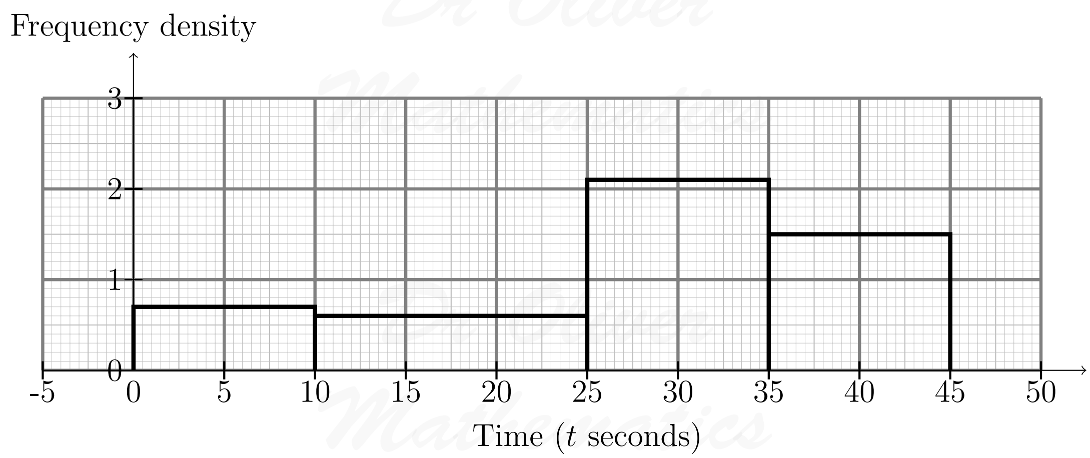


Solution

The widths are 2, 4, 6, and 2 so we pick 2.

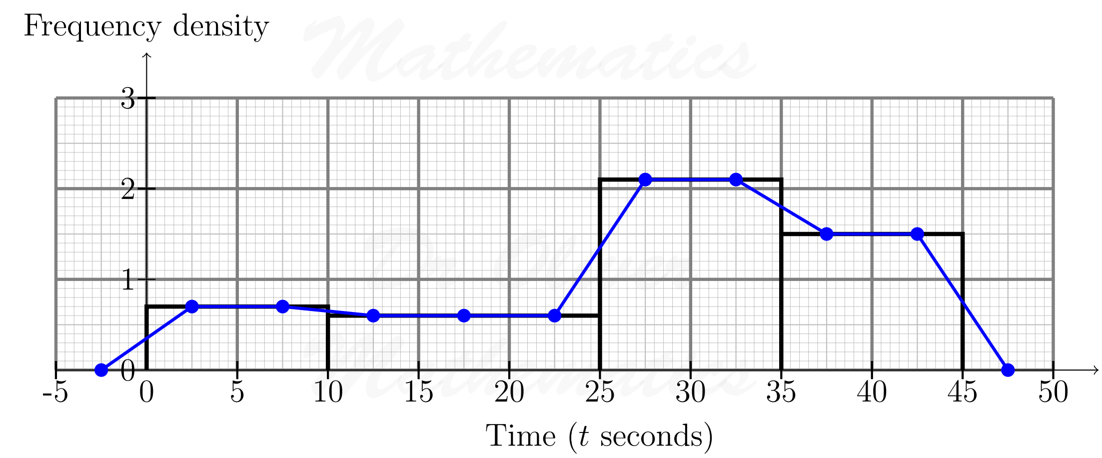


7. Draw the frequency polygon associated with the following histogram.

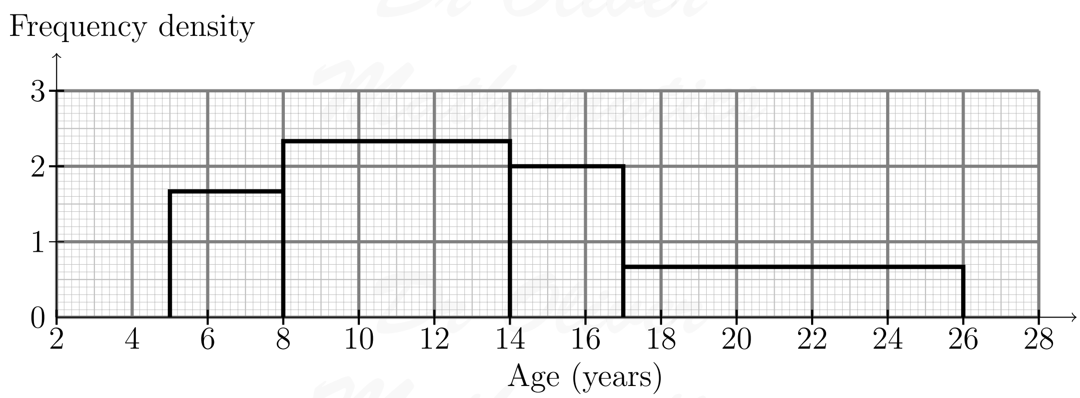


Solution

The widths are 10, 15, 10, and 10 so we pick 5.

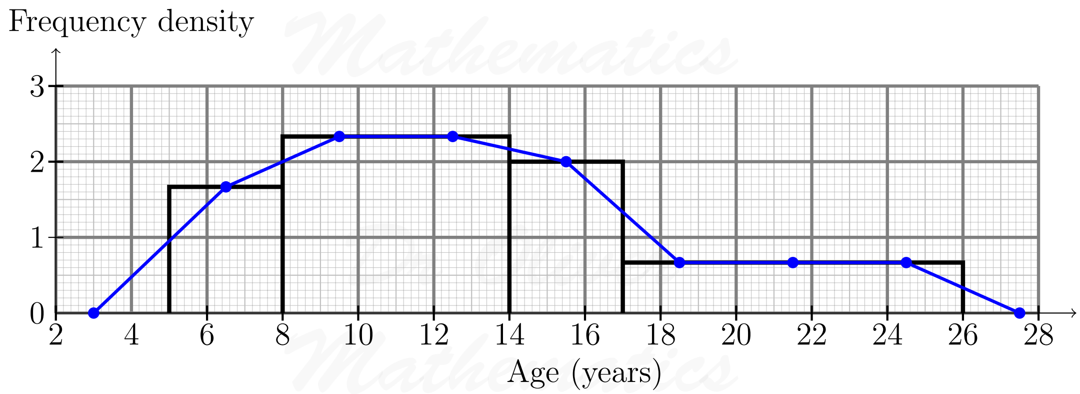


8. Draw the frequency polygon associated with the following histogram.

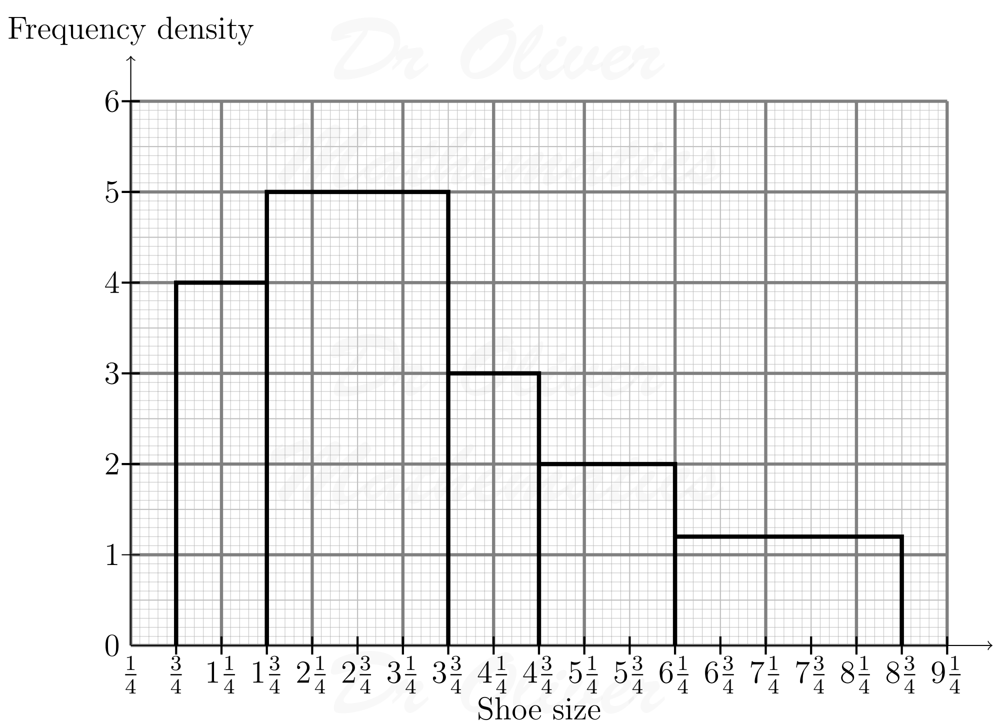


Solution

The widths are 3, 6, 3, and 9 and so we pick 3.

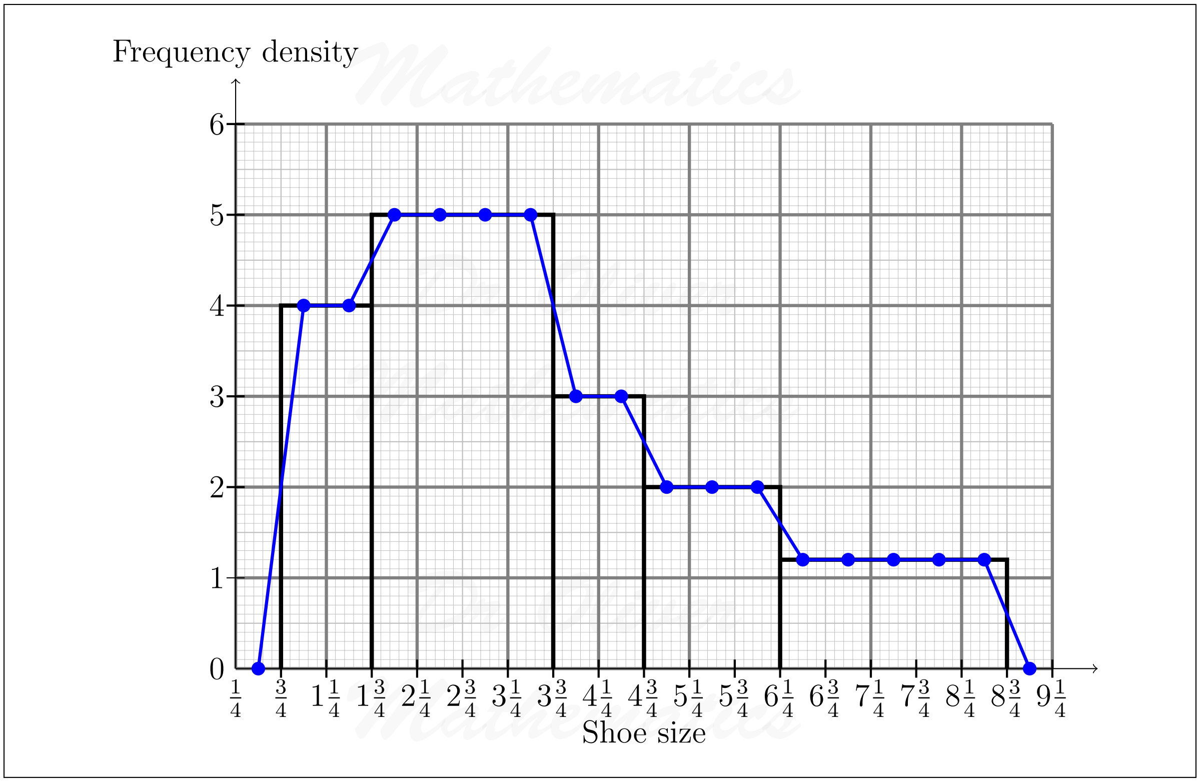


9. Draw the frequency polygon associated with the following histogram.



Solution

The widths are 1, 2, 1, $1\frac{1}{2}$, and $2\frac{1}{2}$ and so we pick $\frac{1}{2}$.



Part III

Mode from a Histogram

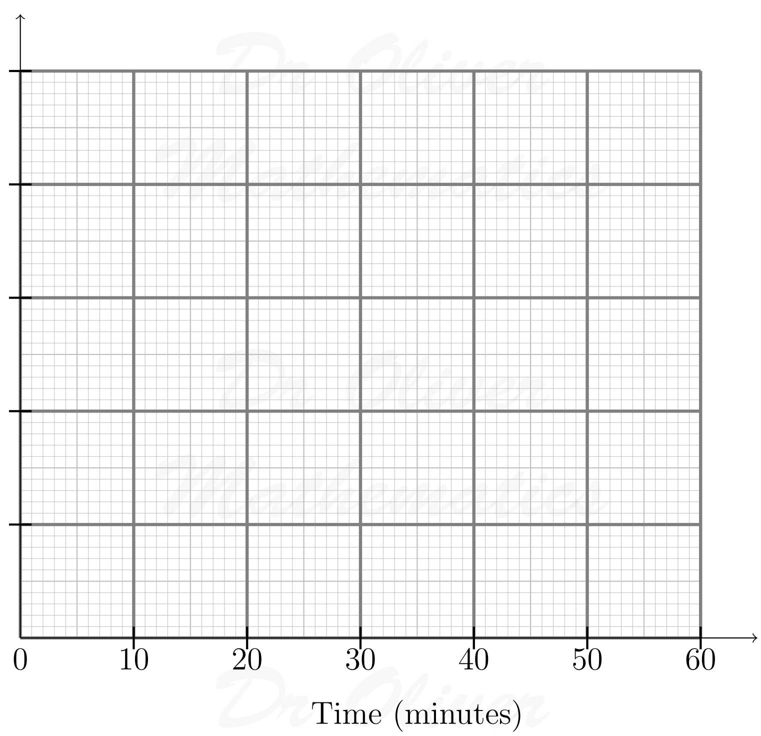
What is the mode of a histogram?
Do you mean, 'What is the modal class of a histogram?' Well, that's easy. All you do –
No. What is the mode of a histogram?
I'm sorry?
(Sigh.) The *actual* mode.
You're bonkers!
In the Southern Examination Group GCSE Statistics papers there was, on occasion, this type of question.
Which question?
(Very deep sigh.) Well, you could try the Southern Examination Group, GCSE Statistics, Summer 1997, Higher Tier, Paper 4, Question 2, for example.
Well, I'll go to the foot of our stairs! Can you list the question?
Certainly.

SEG, GCSE Statistics, Summer 1997, Higher Tier, Paper 4, Question 2

The waiting time at a doctors' surgery is given in the following table.

Time (minutes)	Frequency
$0 \leqslant t < 10$	13
$10 \leqslant t < 15$	22
$15 \leqslant t < 20$	18
$20 \leqslant t < 30$	15
$30 \leqslant t < 40$	9
$40 \leqslant t < 60$	8

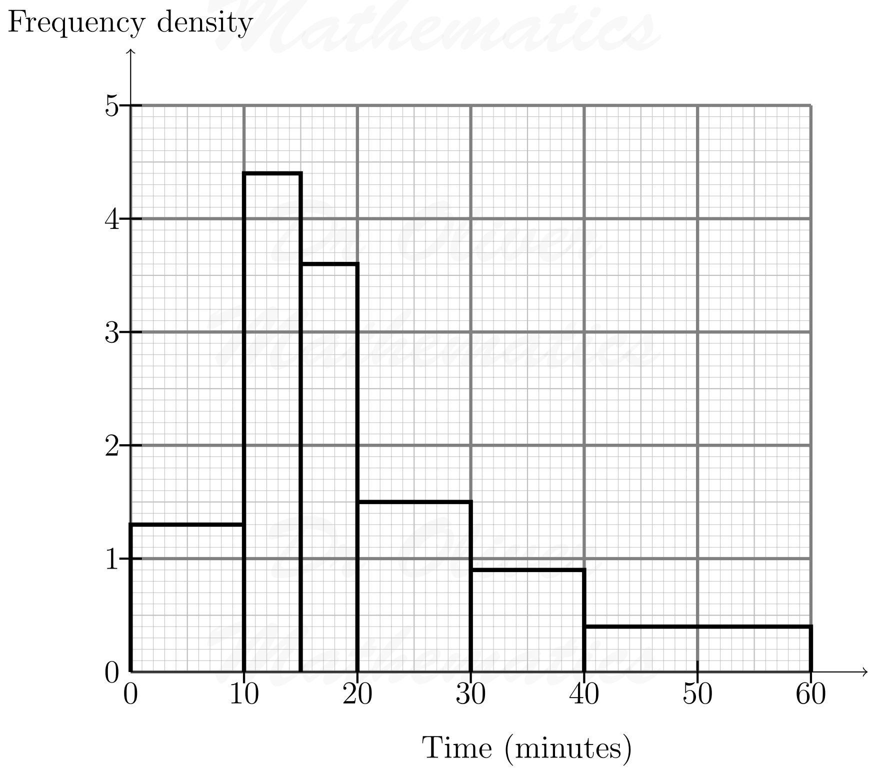
No patient waited for more than 60 minutes.
(a) (6 marks) Draw a histogram to represent these data.



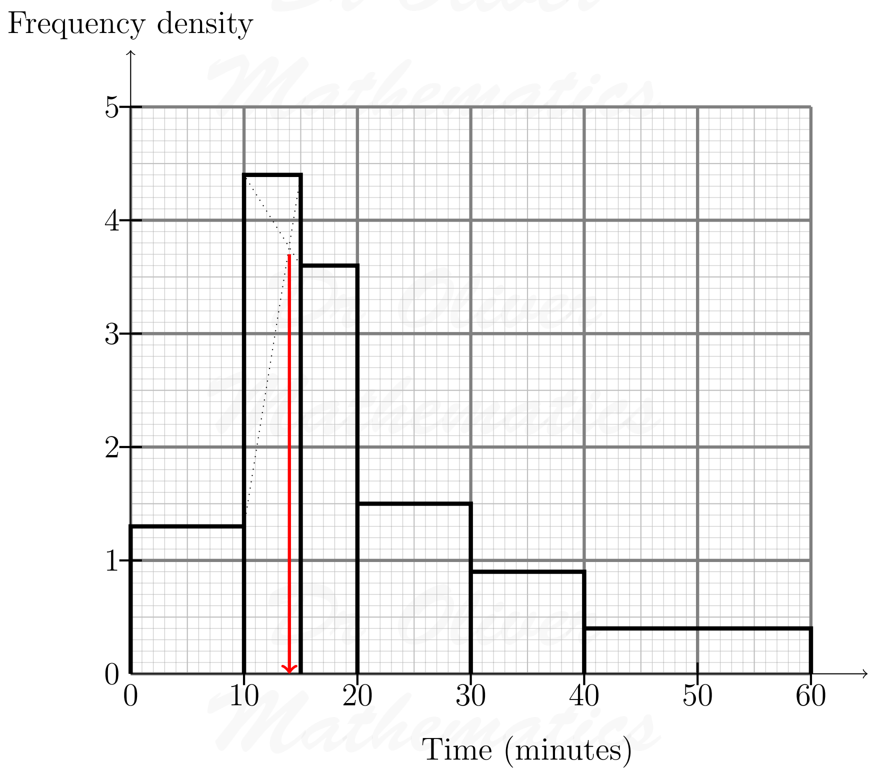
(b) (2 marks) Use the histogram to obtain an estimate of the mode.
(c) (2 marks) How many patients waited for a period of time greater than the mode?

Solution
(a) We begin by completing the table.

Time (minutes)	Frequency	Width	Frequency Density
$0 \leqslant t < 10$	13	10	$\frac{13}{10} = 1.3$
$10 \leqslant t < 15$	22	5	$\frac{22}{5} = 4.4$
$15 \leqslant t < 20$	18	5	$\frac{18}{5} = 3.6$
$20 \leqslant t < 30$	15	10	$\frac{15}{10} = 1.5$
$30 \leqslant t < 40$	9	10	$\frac{9}{10} = 0.9$
$40 \leqslant t < 60$	8	20	$\frac{8}{20} = 0.4$



(b) We find the mode of the histogram.



About $\underline{\underline{14}}$. (Actually, it is $13\frac{38}{39}$ but what's $\frac{1}{39}$ between friends?)

(c) There are

$$13 + 22 + 18 + 15 + 9 + 8 = 85$$

patients in total. Well,

$$\text{less than the mode} = 13 + \tfrac{4}{5} \times 22 = 30.6$$

which means

$$\text{greater than the mode} = 80 - 30.6 = 54.4;$$

so either $\underline{\underline{\text{54 or 55}}}$.

How did you do that?
Well, we construct the histogram.

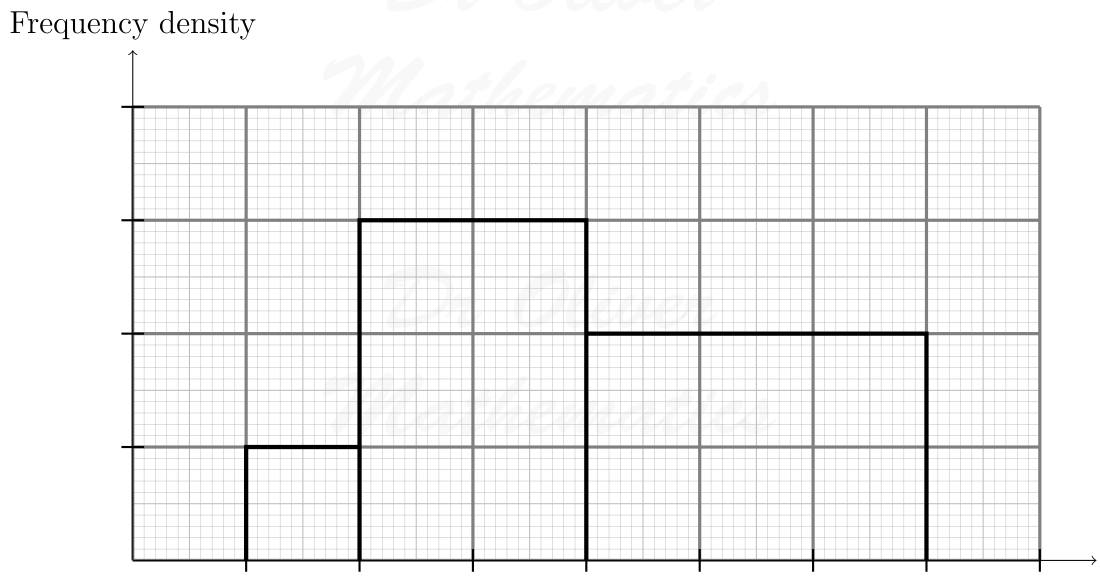


Figure 12: a sketch of the histogram

To find the mode, you find the bar with the maximum frequency density and proceed as follows:

(a) you draw from the right-hand edge the previous bar to the right-hand edge of the bar with the maximum frequency density,
(b) you draw from the left-hand edge the next bar to the left-hand edge of the bar with the maximum frequency density,
(c) where they interest, you read-off the value.

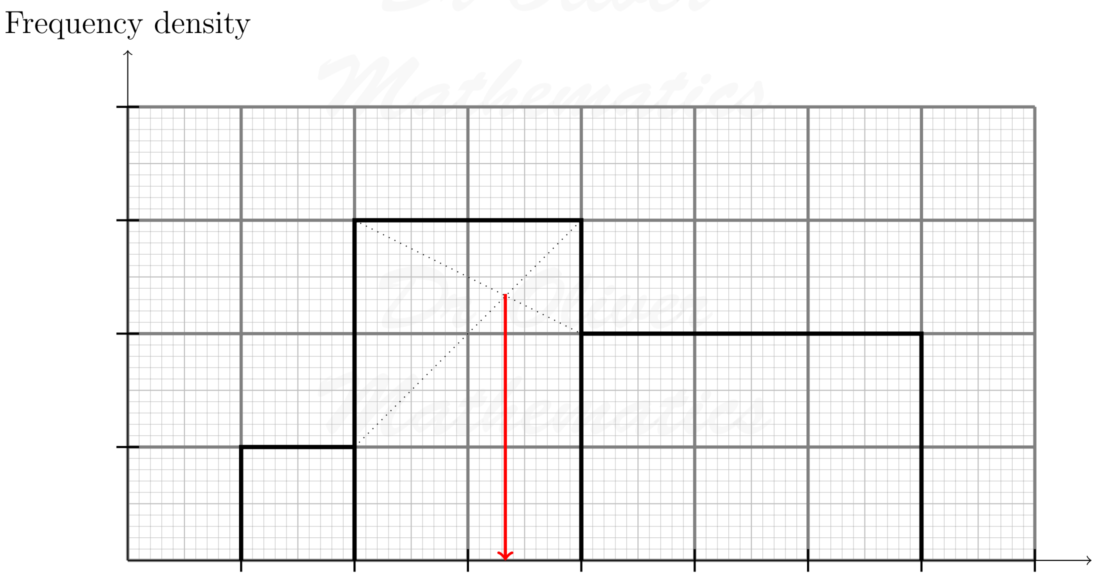


Figure 13: the ‘mode’ of the histogram

But what happens if . . .

(a) there is a mode either on the left-hand edge or the right-hand edge of the page,

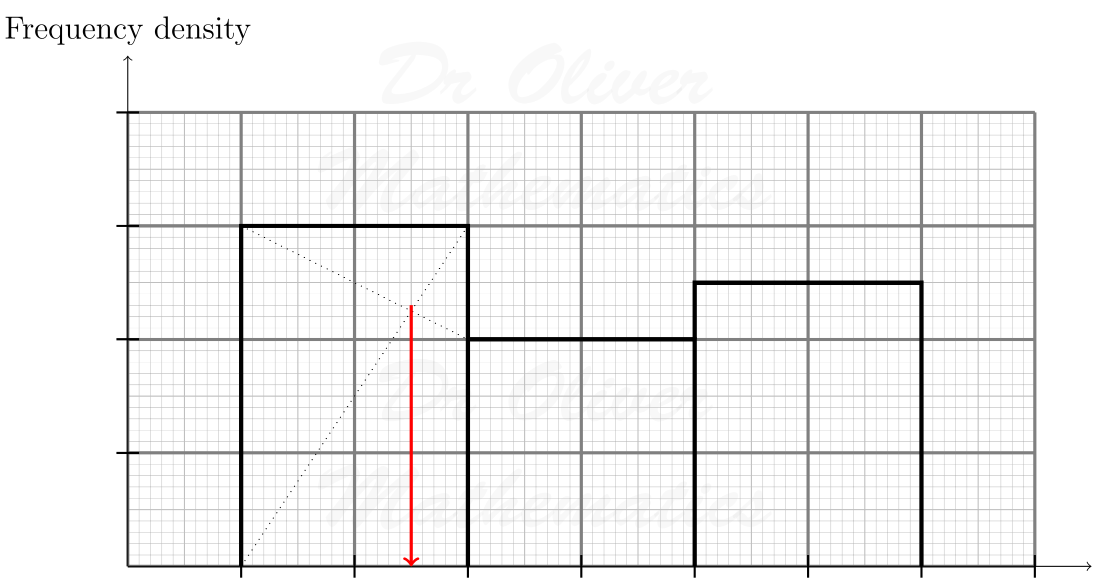


Figure 14: one group is absent or not recorded

(b) there are two or more equal highest frequency densities,

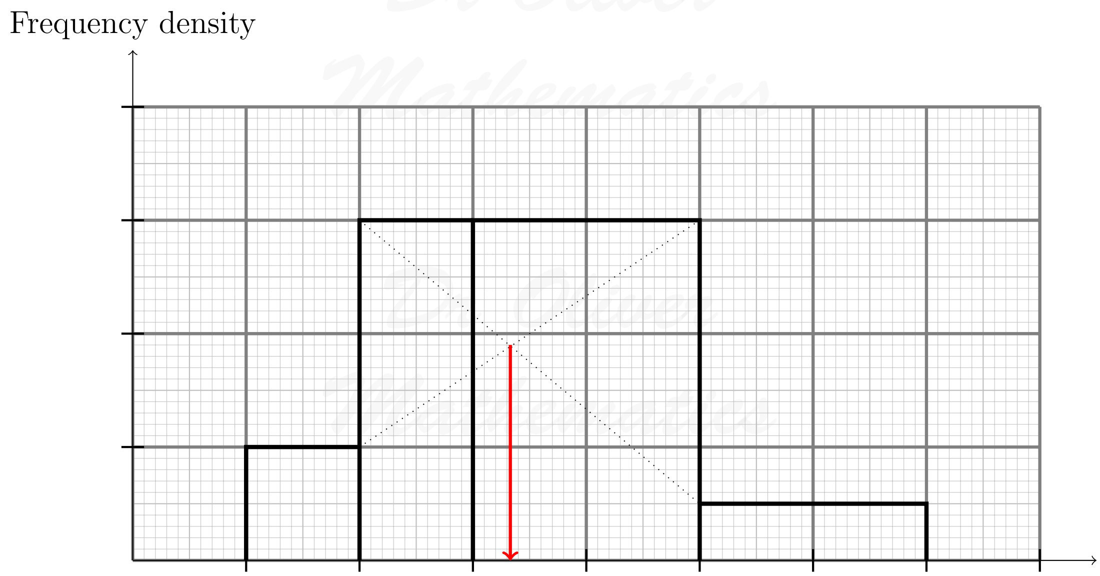


Figure 15: just join them together

(c) there are two or more modes.

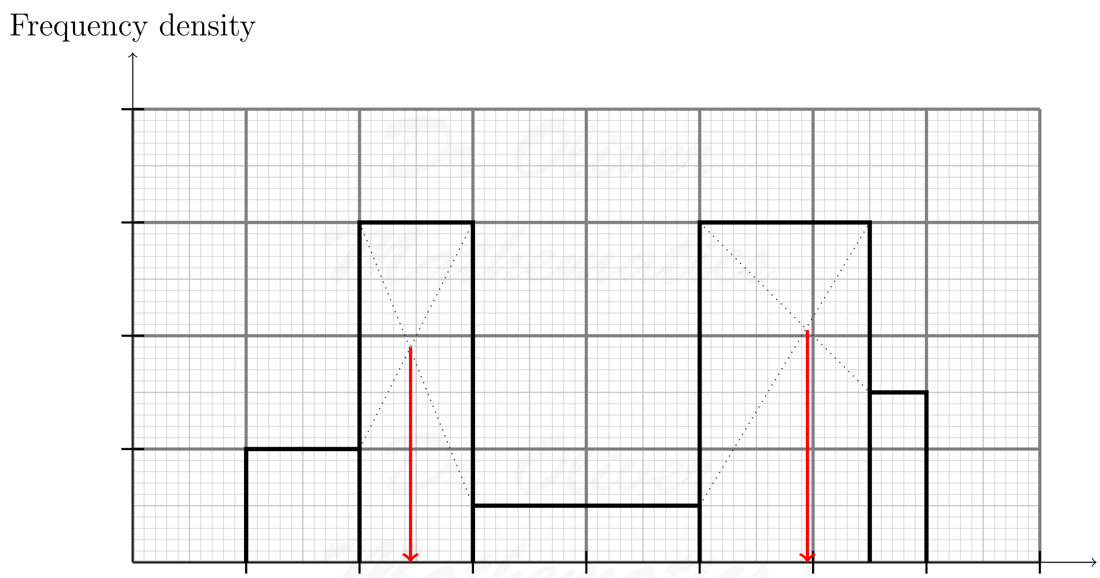


Figure 16: two 'modes'!

Algebraically, how do we go about finding the intersection?

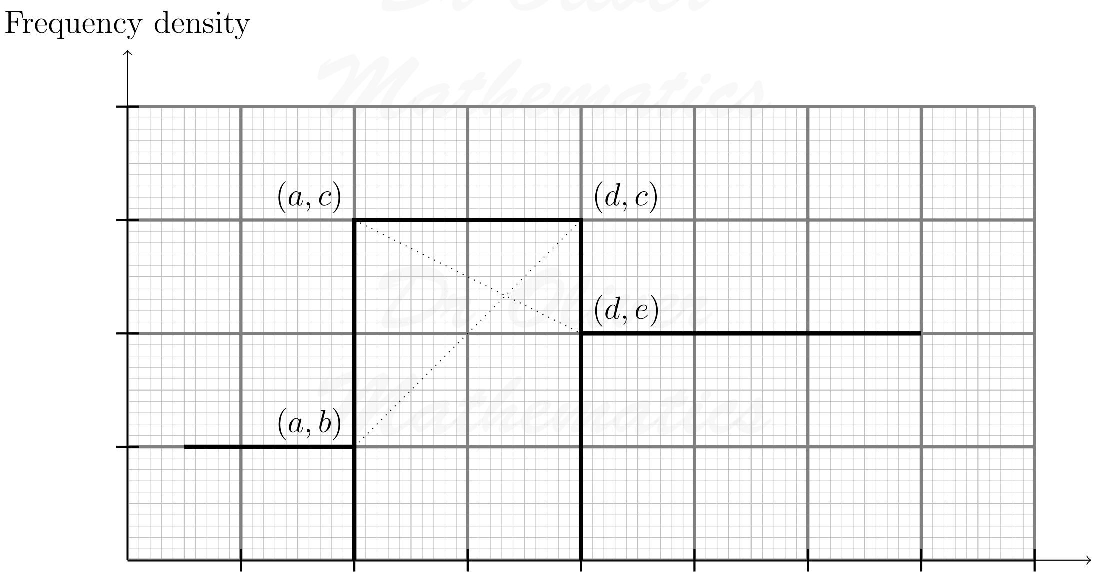


Figure 17: finding the intersection

<u>$(a, c) - (d, e)$</u>:

$$\text{Gradient} = \frac{e-c}{d-a}$$

and the equation is

$$y = \left(\frac{e-c}{d-a}\right)x + f$$

for some constant f. Now, it goes through, say, (a, c):

$$\begin{aligned}
c &= \left(\frac{e-c}{d-a}\right)a + f \\
\Rightarrow \quad f &= c - \left(\frac{e-c}{d-a}\right)a \\
\Rightarrow \quad f &= c - \left(\frac{ae-ac}{d-a}\right) \\
\Rightarrow \quad f &= \frac{c(d-a)-(ae-ac)}{d-a} \\
\Rightarrow \quad f &= \frac{cd-ac-ae+ac}{d-a} \\
\Rightarrow \quad f &= \frac{cd-ae}{d-a}
\end{aligned}$$

and the equation is

$$y = \left(\frac{e-c}{d-a}\right)x + \left(\frac{cd-ae}{d-a}\right). \quad (1)$$

$\underline{(a,b)-(d,c)}$:

$$\text{Gradient} = \frac{c-b}{d-a}$$

and the equation is

$$y = \left(\frac{c-b}{d-a}\right)x + g$$

for some constant g. Now, it goes through, say, (a,b):

$$\begin{aligned} b &= \left(\frac{c-b}{d-a}\right)a + g \\ \Rightarrow \quad g &= b - \left(\frac{c-b}{d-a}\right)a \\ \Rightarrow \quad g &= b - \left(\frac{ac-ab}{d-a}\right) \\ \Rightarrow \quad g &= \frac{b(d-a)-(ac-ab)}{d-a} \\ \Rightarrow \quad g &= \frac{bd-ab-ac+ab}{d-a} \\ \Rightarrow \quad g &= \frac{bd-ac}{d-a} \end{aligned}$$

and the equation is

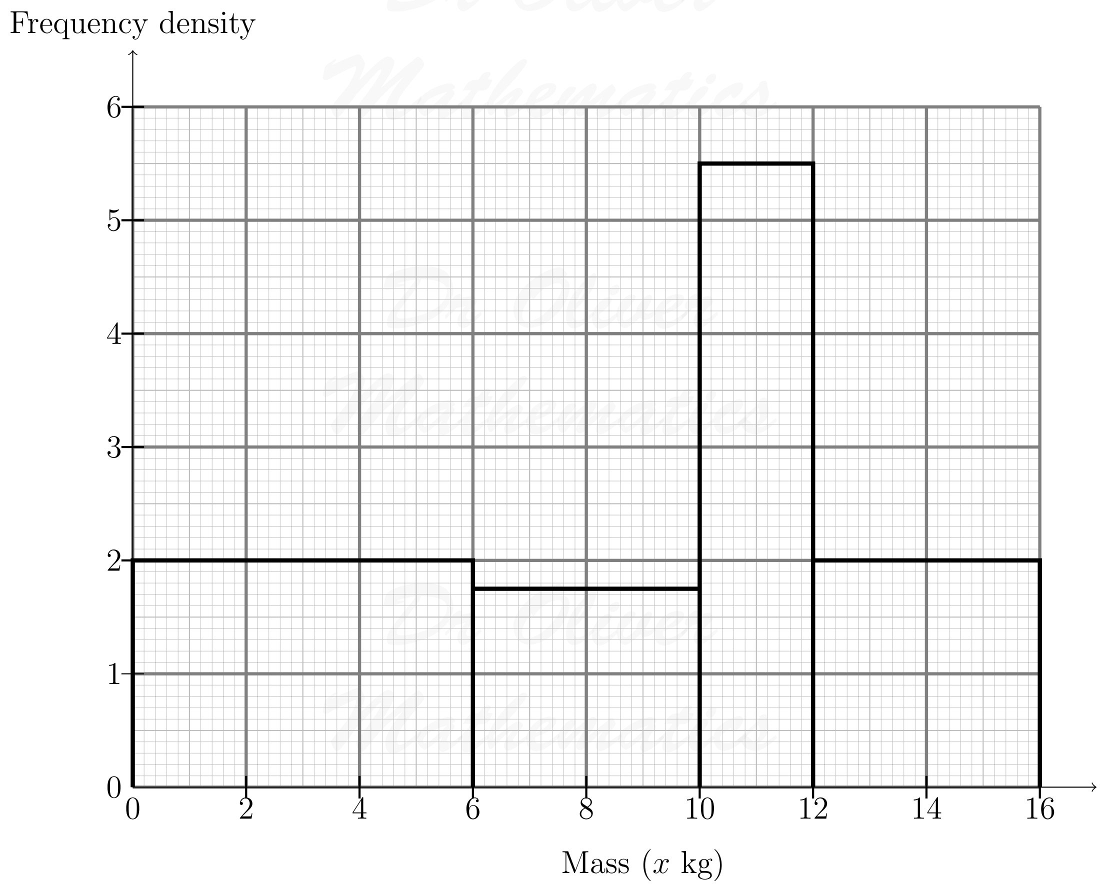
$$y = \left(\frac{c-b}{d-a}\right)x + \left(\frac{bd-ac}{d-a}\right). \quad (2)$$

Solve (1) and (2):

$$\begin{aligned} & \left(\frac{e-c}{d-a}\right)x + \left(\frac{cd-ae}{d-a}\right) = \left(\frac{c-b}{d-a}\right)x + \left(\frac{bd-ac}{d-a}\right) \\ \Rightarrow \quad & (e-c)x + (cd-ae) = (c-b)x + (bd-ac) \\ \Rightarrow \quad & (e-c)x - (c-b)x = (bd-ac) - (cd-ae) \\ \Rightarrow \quad & ex - cx - cx - bx = bd - ac - cd + ae \\ \Rightarrow \quad & x(b+e-2c) = bd - cd - ac + ae \\ \Rightarrow \quad & x(b+e-2c) = d(b-c) - a(c-e) \\ \Rightarrow \quad & \boxed{x = \frac{d(b-c)-a(c-e)}{b+e-2c}.} \end{aligned}$$

Of course, a number of rearrangements are possible but we will stick with that one.

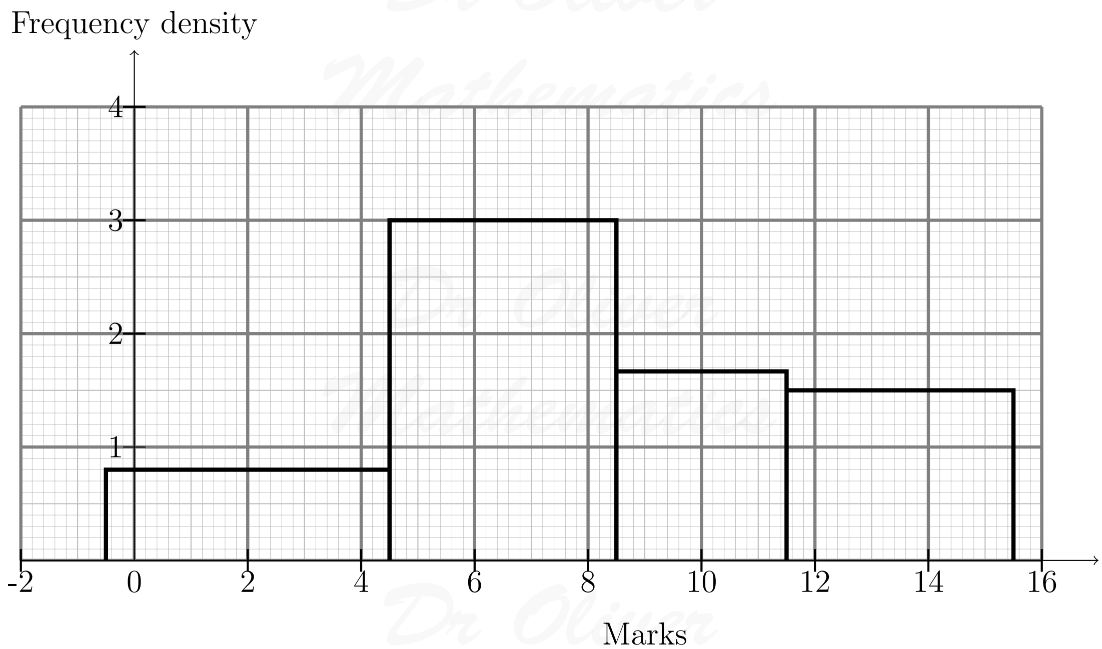
Example 10
Find the 'mode' that goes with the following histogram.



Solution 10
$a = 10$, $b = 1.75$, $c = 5.5$, $d = 12$, and $e = 2$:

$$\begin{aligned} x &= \frac{12(1.75 - 5.5) - 10(5.5 - 2)}{1.75 + 2 - 2 \times 5.5} \\ &= \underline{\underline{11\tfrac{1}{29}}}. \end{aligned}$$

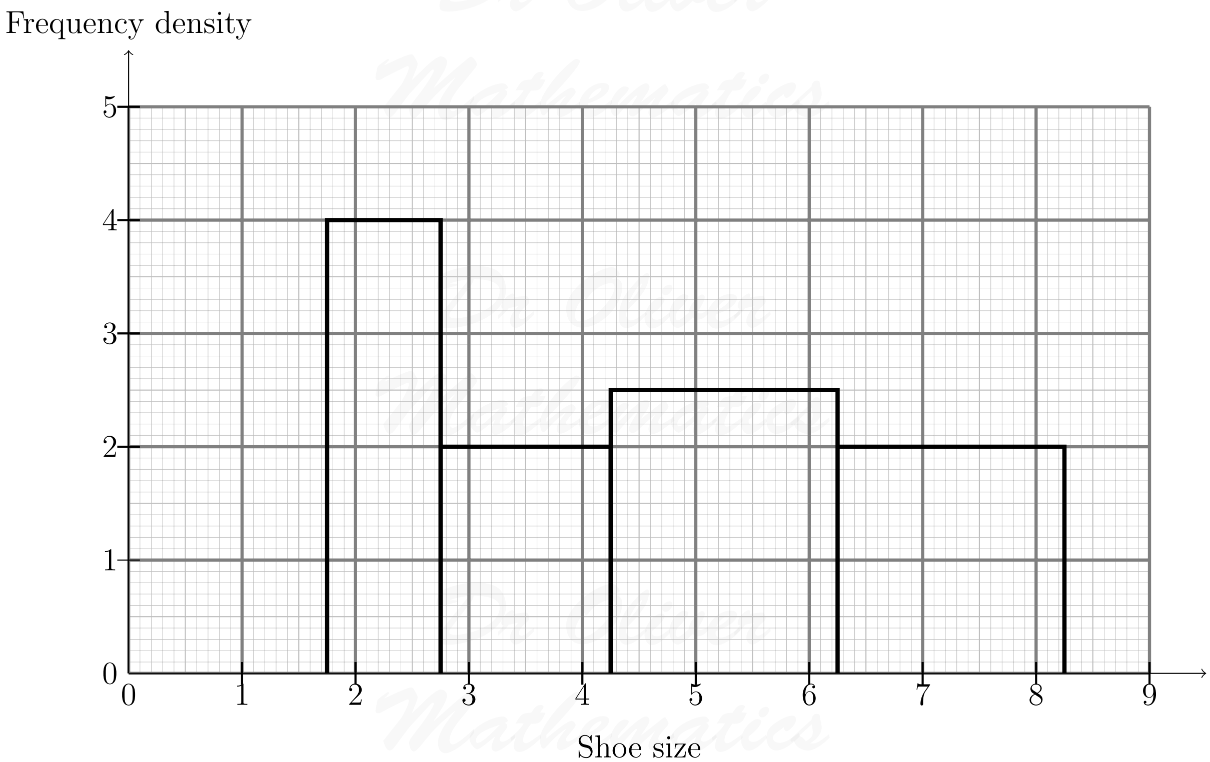
Example 11
Find the 'mode' that goes with the following histogram.



Solution 11
$a = 4.5$, $b = 0.8$, $c = 3$, $d = 8.5$, and $e = 1\frac{2}{3}$:

$$x = \frac{8.5(0.8 - 3) - 4.5(3 - 1\frac{2}{3})}{1\frac{2}{3} + 0.8 - 2 \times 3}$$
$$= \underline{\underline{6\tfrac{105}{106}}}.$$

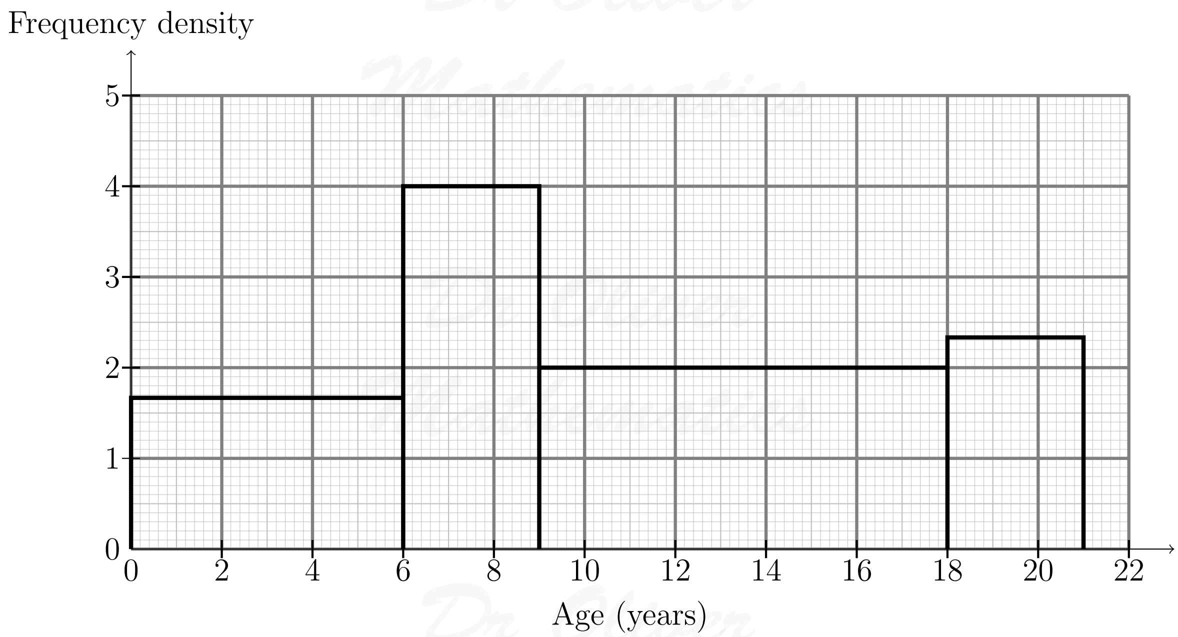
Example 12
Find the 'mode' that goes with the following histogram.



Solution 12
$a = 1.75$, $b = 0$, $c = 4$, $d = 2.75$, and $e = 2$:

$$x = \frac{2.75(0-4) - 1.75(4-2)}{2+0-2\times 4}$$
$$= \underline{\underline{2\tfrac{5}{12}}}.$$

Example 13
Find the 'mode' that goes with the following histogram.



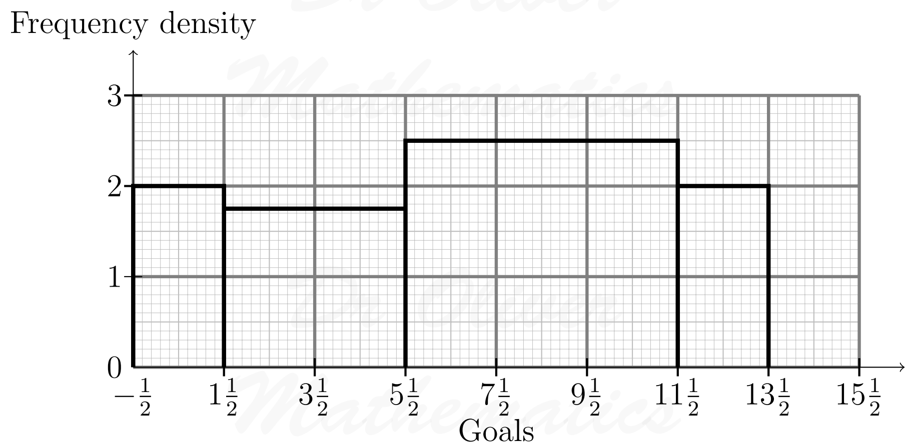
Solution 13
$a = 6$, $b = 1\frac{2}{3}$, $c = 4$, $d = 9$, and $e = 2$:

$$x = \frac{9(1\frac{2}{3} - 4) - 6(4 - 2)}{1\frac{2}{3} + 2 - 2 \times 4} = \underline{\underline{\underline{7\tfrac{8}{13}}}}.$$

5 Problems

Here are a few problems for you to try.

10. Find the 'mode' that goes with the following histogram.

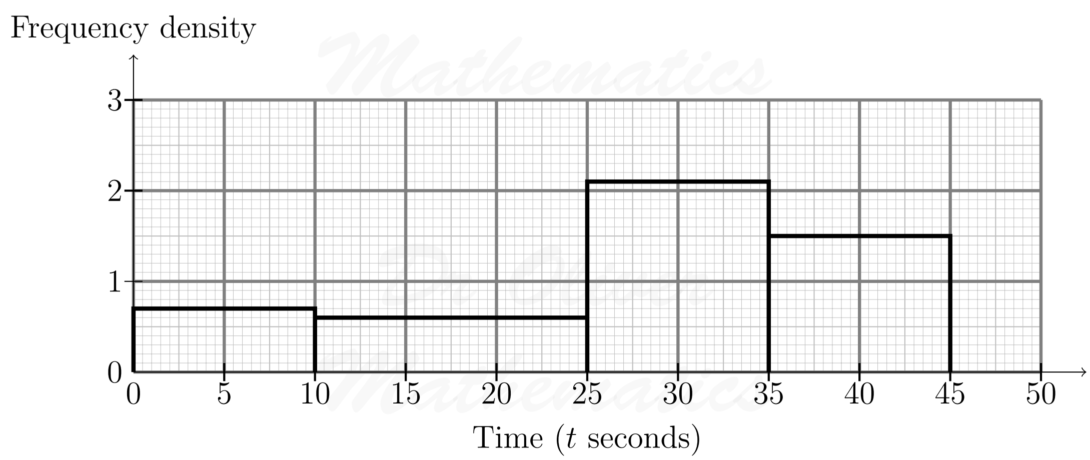


Solution

$a = 5.5$, $b = 1.75$, $c = 2.5$, $d = 11.5$, and $e = 2$:

$$\begin{aligned} x &= \frac{11.5(1.75 - 2.5) - 5.5(2.5 - 2)}{1.75 + 2 - 2 \times 2.5} \\ &= \underline{\underline{9\tfrac{1}{10}}}. \end{aligned}$$

11. Find the 'mode' that goes with the following histogram.

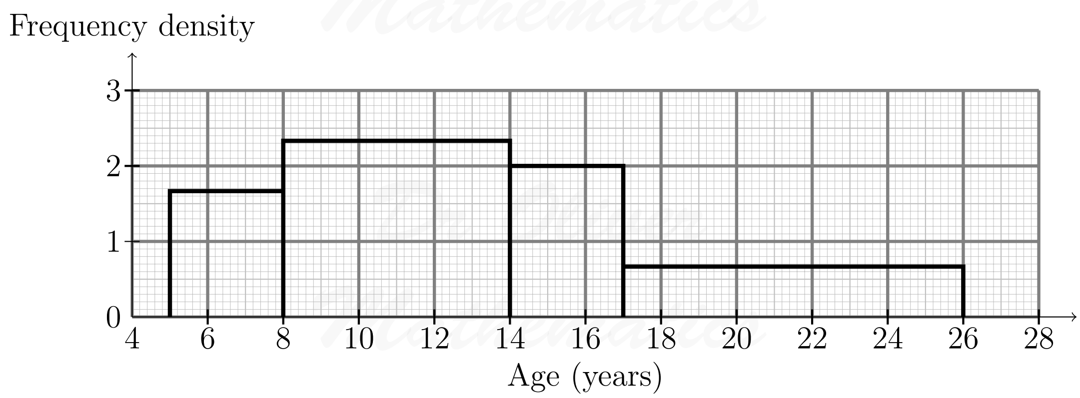


Solution

$a = 25$, $b = 0.6$, $c = 2.1$, $d = 35$, and $e = 1.5$:

$$x = \frac{35(0.6 - 2.1) - 25(2.1 - 1.5)}{0.6 + 1.5 - 2 \times 2.1}$$
$$= \underline{\underline{32\tfrac{1}{7}}}.$$

12. Find the 'mode' that goes with the following histogram.

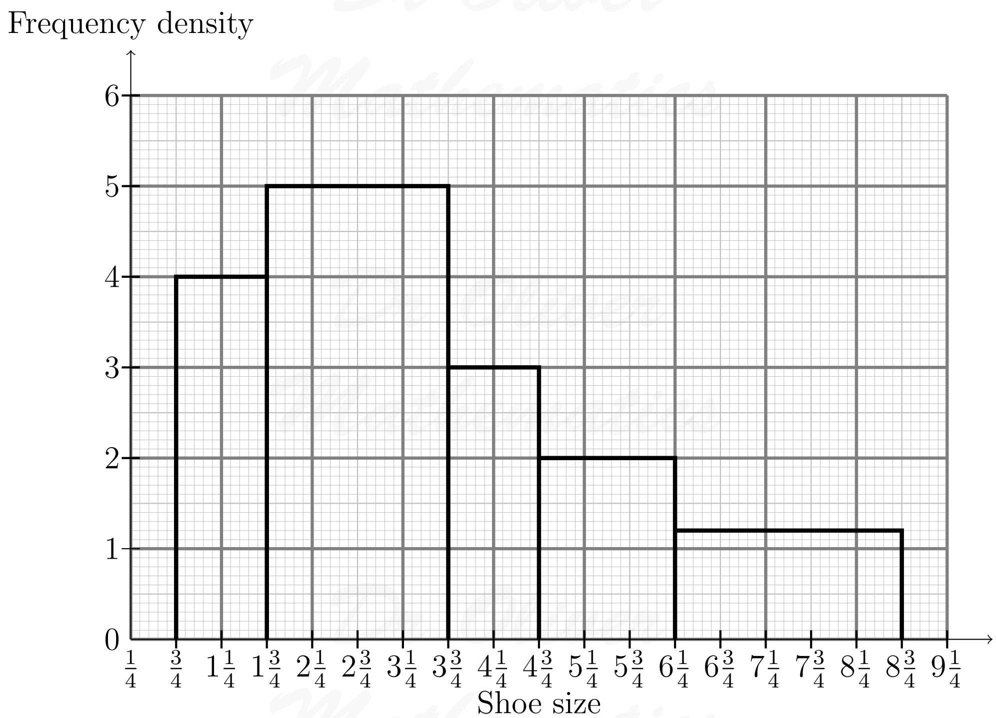


Solution

$a = 8$, $b = 1\frac{2}{3}$, $c = 2\frac{1}{3}$, $d = 14$, and $e = 2$:

$$x = \frac{14(1\frac{2}{3} - 2\frac{1}{3}) - 8(2\frac{1}{3} - 2)}{1\frac{2}{3} + 2 - 2 \times 2\frac{1}{3}}$$
$$= \underline{\underline{12}}.$$

13. Find the 'mode' that goes with the following histogram.



Solution

$a = 1.75$, $b = 4$, $c = 5$, $d = 3.75$, and $e = 3$:

$$x = \frac{3.75(4-5) - 1.75(5-3)}{4+3-2\times 5} = \underline{\underline{2\tfrac{5}{12}}}.$$

Part IV

Cumulative Frequency Diagrams

Example 14: Continuous data
Draw a cumulative frequency polygon or curve to represent these data.

Mass (x kg)	Frequency
$0 \leqslant x < 6$	12
$6 \leqslant x < 10$	7
$10 \leqslant x < 12$	11
$12 \leqslant x < 16$	8

Solution 14
First, you need to re-label the first axis as $0 \leqslant x$ (or whatever it is you have there). Second, you need to label the other axis as 'Cumulative Frequency.'

Mass (x kg)	Cumulative Frequency
$0 \leqslant x < 6$	12
$0 \leqslant x < 10$	
$0 \leqslant x < 12$	
$0 \leqslant x < 16$	

Table 11: 'Cumulative Frequency' labelled

What is $12 + 7 =$? That's correct: 19.

Mass (x kg)	Cumulative Frequency
$0 \leqslant x < 6$	12
$0 \leqslant x < 10$	$7 + 12 = 19$
$0 \leqslant x < 12$	
$0 \leqslant x < 16$	

Table 12: $7 + 12 = 19$

We complete the table:

Mass (x kg)	Cumulative Frequency
$0 \leqslant x < 6$	12
$0 \leqslant x < 10$	$7 + 12 = 19$
$0 \leqslant x < 12$	$19 + 11 = 30$
$0 \leqslant x < 16$	$8 + 30 = 38$

Table 13: table completed

Mark in the axes . . .

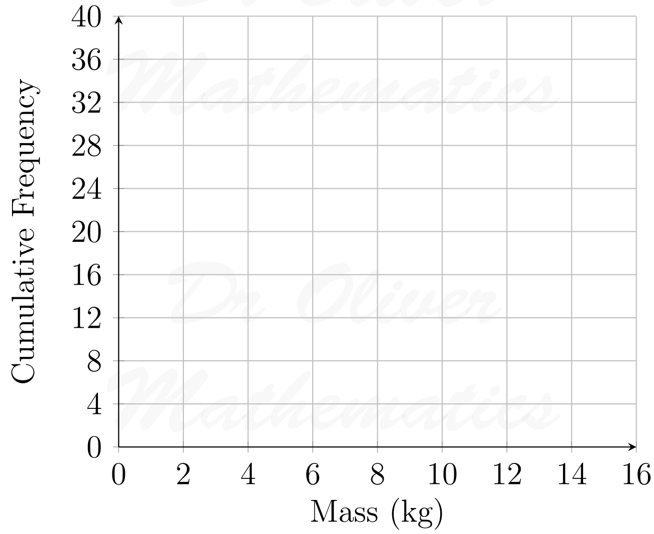


Figure 18: mark in the axes

. . . and complete the cumulative frequency polygon . . .

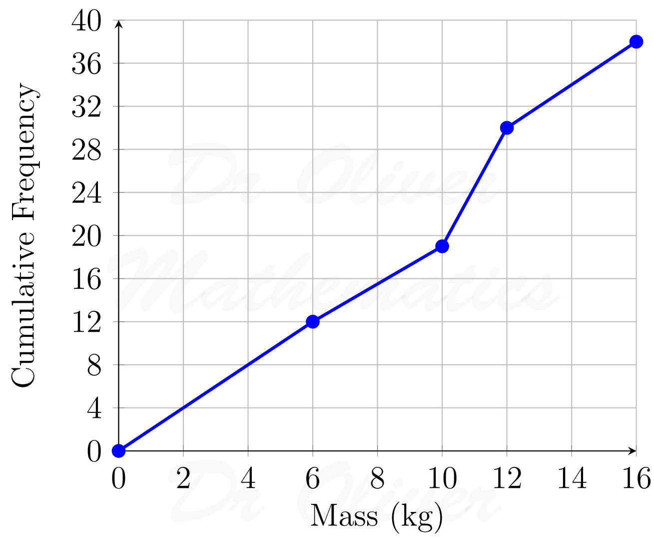


Figure 19: cumulative frequency polygon

. . . or the cumulative frequency curve.

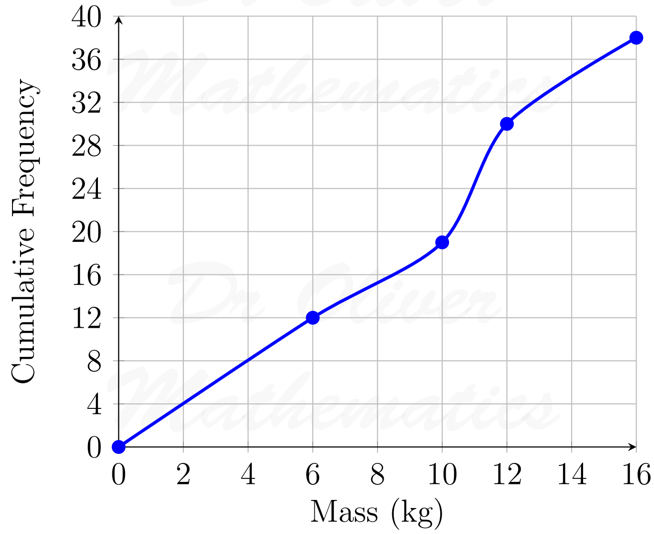


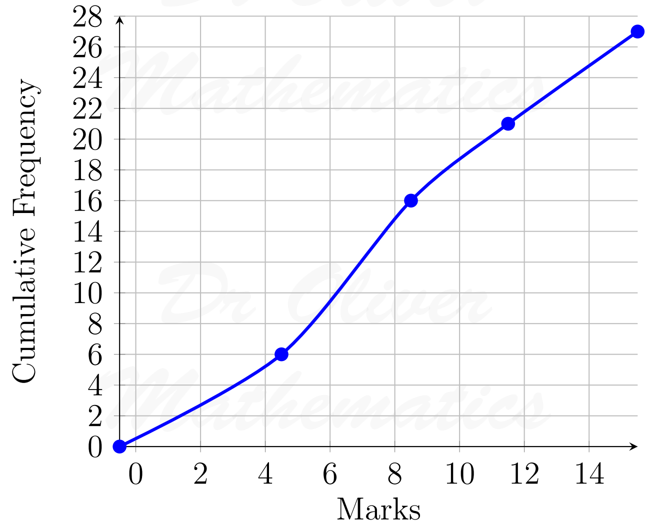
Figure 20: cumulative frequency curve – I prefer this

Example 15: Discrete data (whole numbers)
Draw a cumulative frequency polygon or curve to represent these data.

Marks	Frequency
0 – 4	4
5 – 8	12
9 – 11	5
12 – 15	6

Solution 15

Marks	Cumulative Frequency
$-\frac{1}{2} \leqslant \text{marks} < 4\frac{1}{2}$	4
$-\frac{1}{2} \leqslant \text{marks} < 8\frac{1}{2}$	$4 + 12 = 16$
$-\frac{1}{2} \leqslant \text{marks} < 11\frac{1}{2}$	$16 + 5 = 21$
$-\frac{1}{2} \leqslant \text{marks} < 15\frac{1}{2}$	$21 + 6 = 27$

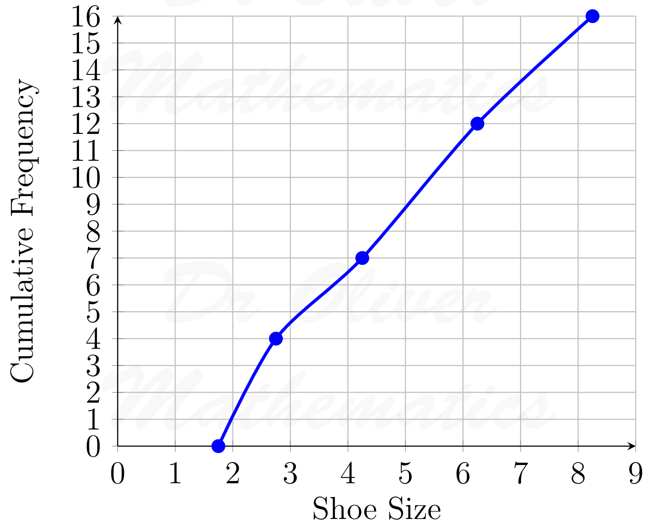


Example 16: Discrete data (fractional values)
Draw a cumulative frequency polygon or curve to represent these data.

Shoe Size	Frequency
$2 - 2\frac{1}{2}$	4
$3 - 4$	3
$4\frac{1}{2} - 6$	5
$6\frac{1}{2} - 8$	4

Solution 16

Shoe Size	Cumulative Frequency
$1\frac{3}{4} \leqslant \text{s/s} < 2\frac{3}{4}$	4
$1\frac{3}{4} \leqslant \text{s/s} < 4\frac{1}{4}$	$4 + 3 = 7$
$1\frac{1}{4} \leqslant \text{s/s} < 6\frac{1}{4}$	$7 + 5 = 12$
$1\frac{1}{4} \leqslant \text{s/s} < 8\frac{1}{4}$	$12 + 4 = 16$

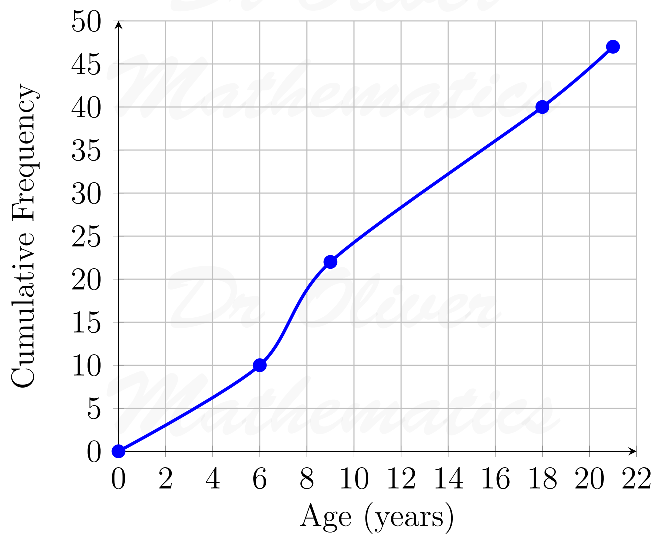


Example 17: Age (continuous – time – but represented by a discrete number)
Draw a cumulative frequency polygon or curve to represent these data.

Age (years)	Frequency
$0-5$	10
$6-8$	12
$9-17$	18
$18-20$	7

Solution 17

Age (years)	Cumulative Frequency
$0 < x$ years < 6	10
$0 \leqslant x$ years < 9	$10 + 12 = 22$
$0 \leqslant x$ years < 18	$22 + 18 = 40$
$0 \leqslant x$ years < 21	$40 + 7 = 47$



6 Problems

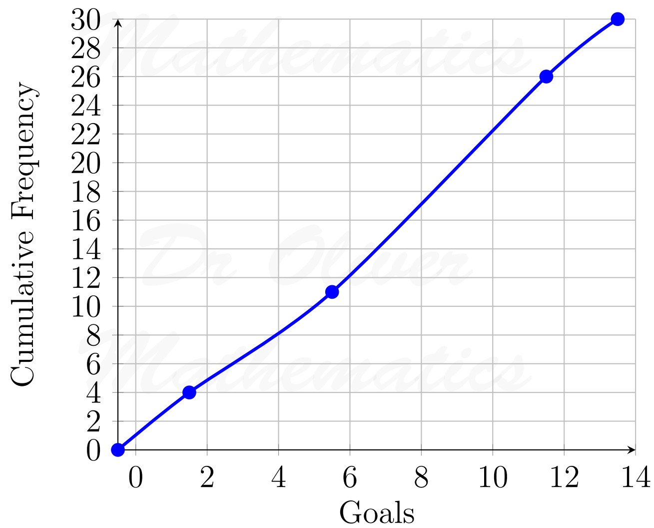
Here are a few problems for you to try.

14. Draw a cumulative frequency polygon or curve to represent these data.

Goals	Frequency
0 – 1	4
2 – 5	7
6 – 11	15
12 – 13	4

Solution

Goals	Cumulative Frequency
$-\frac{1}{2} \leqslant \text{goals} < 1\frac{1}{2}$	4
$-\frac{1}{2} \leqslant \text{goals} < 5\frac{1}{2}$	$7 + 4 = 11$
$-\frac{1}{2} \leqslant \text{goals} < 11\frac{1}{2}$	$11 + 15 = 26$
$-\frac{1}{2} \leqslant \text{goals} < 13\frac{1}{2}$	$26 + 4 = 30$

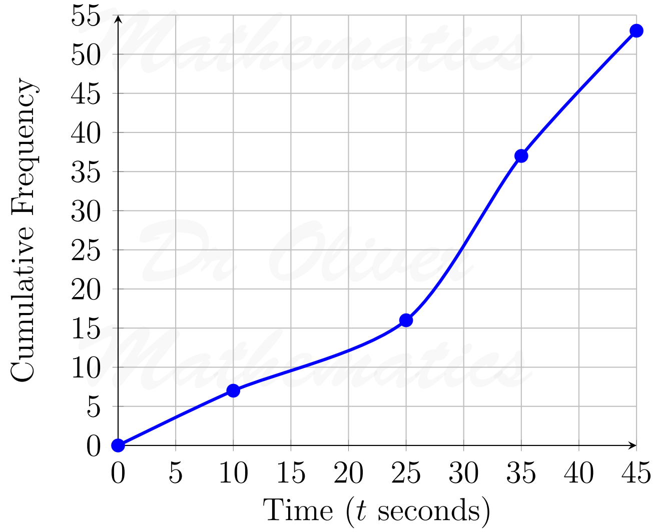


15. Draw a cumulative frequency polygon or curve to represent these data.

Time (t seconds)	Frequency
$0 \leqslant t < 10$	7
$10 \leqslant t < 25$	9
$25 \leqslant t < 35$	21
$35 \leqslant t < 45$	15

Solution

Time (t seconds)	Cumulative Frequency
$0 \leqslant t < 10$	7
$0 \leqslant t < 25$	$9 + 7 = 16$
$0 \leqslant t < 35$	$21 + 16 = 37$
$0 \leqslant t < 45$	$15 + 37 = 53$

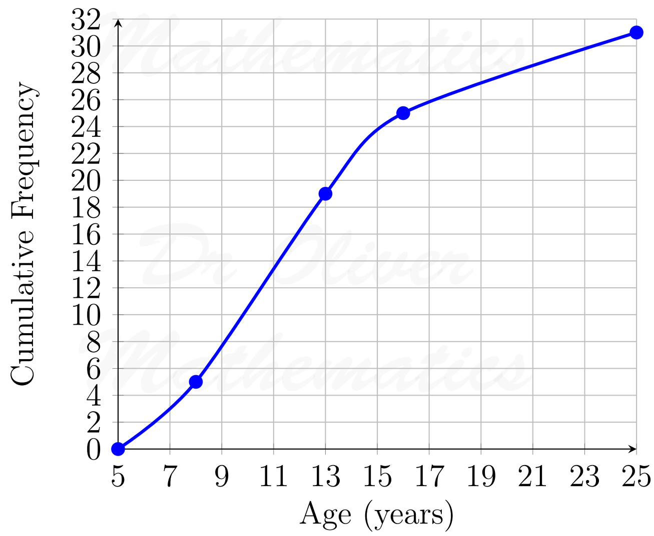


16. Draw a cumulative frequency polygon or curve to represent these data.

Age (years)	Frequency
$5 - 7$	5
$8 - 13$	14
$14 - 16$	6
$17 - 25$	6

Solution

Age (years)	Cumulative Frequency
$5 \leqslant \text{age} < 8$	5
$5 \leqslant \text{age} < 13$	$14 + 5 = 19$
$5 \leqslant \text{age} < 16$	$6 + 19 = 25$
$5 \leqslant \text{age} < 25$	$6 + 25 = 31$

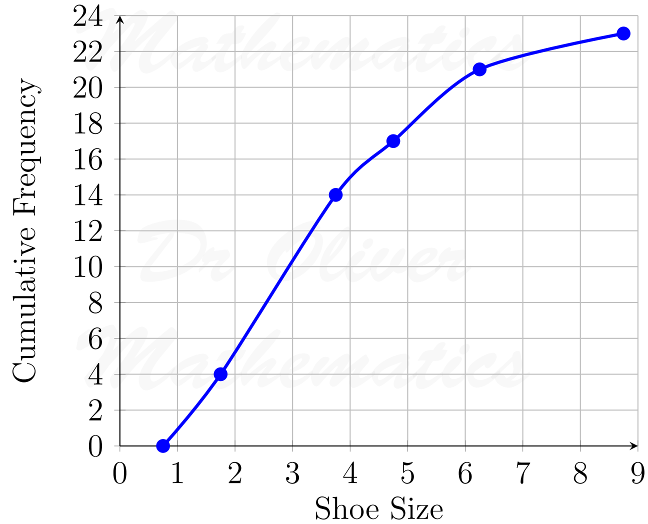


17. Draw a cumulative frequency polygon or curve to represent these data.

Shoe Size	Frequency
$1 - 1\frac{1}{2}$	4
$2 - 3\frac{1}{2}$	10
$4 - 4\frac{1}{2}$	3
$5 - 6$	3
$6\frac{1}{2} - 8\frac{1}{2}$	3

Solution

Shoe Size	Cumulative Frequency
$\frac{3}{4} \leqslant \text{s/s} < 1\frac{3}{4}$	4
$\frac{3}{4} \leqslant \text{s/s} < 3\frac{3}{4}$	$4 + 10 = 14$
$\frac{3}{4} \leqslant \text{s/s} < 4\frac{3}{4}$	$14 + 3 = 17$
$\frac{3}{4} \leqslant \text{s/s} < 6\frac{1}{4}$	$17 + 3 = 20$
$\frac{3}{4} \leqslant \text{s/s} < 8\frac{3}{4}$	$20 + 3 = 23$



Part V

Histograms Without a Histogram

Do you remember that

$$\text{frequency density} \propto \frac{\text{frequency}}{\text{width}}?$$

Example 18

Length (km)	Frequency	Frequency density
$0 \leqslant t < 20$	70	7
$20 \leqslant t < 30$	x	
$30 \leqslant t < 60$		y
$60 \leqslant t < z$	100	5

Find:
(i) the frequency density for the second row,
(ii) the frequency for the third row, and
(iii) the value of z.

Solution 18

What...!?
Calm down.
But...?!
Breathe. This is a very good question: I cribbed it from London Examinations, 1998, Paper 6, Question 14 and I have changed the numbers, etc.

So, where do we start? Well, the obvious thing is that there is no 'Width' column and let's insert a as the frequency density in the second line and b as the frequency in the third line.

Length (km)	Frequency	Width	Frequency density
$0 \leqslant t < 20$	70	20	7
$20 \leqslant t < 30$	x	10	a
$30 \leqslant t < 60$	b	30	y
$60 \leqslant t < z$	100	$z - 60$	5

Now, we do a ratio: $7 : \dfrac{70}{20}$ must be the ratio for all bars (why?).
<u>a</u>:

$$\begin{aligned} 7 : \frac{70}{20} = a : \frac{x}{10} &\Rightarrow \frac{7}{\frac{70}{20}} = \frac{a}{\frac{x}{10}} \\ &\Rightarrow 2 = \frac{10a}{x} \\ &\Rightarrow a = \tfrac{1}{5}x. \end{aligned}$$

<u>b</u>:

$$\begin{aligned} 7 : \frac{70}{20} = a : \frac{x}{10} &\Rightarrow \frac{7}{\frac{70}{20}} = \frac{y}{\frac{b}{30}} \\ &\Rightarrow 2 = \frac{30y}{b} \\ &\Rightarrow b = 15y. \end{aligned}$$

<u>z</u>:

$$\begin{aligned} 7 : \frac{70}{20} = 5 : \frac{100}{z - 60} &\Rightarrow \frac{7}{\frac{70}{20}} = \frac{5(z-60)}{100} \\ &\Rightarrow 40 = z - 60 \\ &\Rightarrow z = 100. \end{aligned}$$

Length (km)	Frequency	Frequency density
$0 \leqslant t < 20$	70	7
$20 \leqslant t < 30$	x	$\frac{1}{5}x$
$30 \leqslant t < 60$	$15y$	y
$60 \leqslant t < 100$	100	5

7 Problems

Here are a few problems for you to try.

18. In the incomplete table, find

Speed (ms^{-1})	Frequency	Frequency density
$v \leqslant t < 20$	10	3
$20 \leqslant t < 30$	x	
$30 \leqslant t < 45$	15	1.5
$45 \leqslant t < 55$		y
$55 \leqslant t < z$	28	1.05

(a) the value of v,

(b) the frequency density for the second row,

(c) the frequency for the fourth row, and

(d) the value of z.

Solution

Let's insert a as the frequency density in the second line and b as the frequency in the fourth line.

Speed (ms^{-1})	Frequency	Frequency density	Width
$v \leqslant t < 20$	10	3	$20 - v$
$20 \leqslant t < 30$	x	a	10
$30 \leqslant t < 45$	15	1.5	15
$45 \leqslant t < 55$	b	y	10
$55 \leqslant t < z$	28	1.05	$z - 55$

$\underline{v}$:

$$\begin{aligned}1.5 : \frac{15}{15} = 3 : \frac{10}{20-v} &\Rightarrow \frac{1.5}{\frac{15}{15}} = \frac{3}{\frac{10}{20-v}} \\ &\Rightarrow 1.5 = \tfrac{3}{10}(20-v) \\ &\Rightarrow 5 = 20 - v \\ &\Rightarrow \underline{\underline{v = 15}}.\end{aligned}$$

$\underline{x}$:

$$\begin{aligned}1.5 : \frac{15}{15} = a : \frac{x}{10} &\Rightarrow \frac{1.5}{\frac{15}{15}} = \frac{a}{\frac{x}{10}} \\ &\Rightarrow 1.5 = \frac{10a}{x} \\ &\Rightarrow \underline{\underline{a = 0.15x}}.\end{aligned}$$

$\underline{y}$:

$$\begin{aligned}1.5 : \frac{15}{15} = y : \frac{b}{10} &\Rightarrow \frac{1.5}{\frac{15}{15}} = \frac{y}{\frac{b}{10}} \\ &\Rightarrow 1.5 = \frac{10y}{b} \\ &\Rightarrow \underline{\underline{b = \tfrac{20}{3}y}}.\end{aligned}$$

$\underline{z}$:

$$\begin{aligned}1.5 : \frac{15}{15} = 1.05 : \frac{28}{z-55} &\Rightarrow \frac{1.5}{\frac{15}{15}} = \frac{1.05}{\frac{28}{z-55}} \\ &\Rightarrow 1.5 = \tfrac{3}{80}(z-55) \\ &\Rightarrow 40 = z - 55 \\ &\Rightarrow \underline{\underline{z = 95}}.\end{aligned}$$

We present the final table:

Speed (ms^{-1})	Frequency	Frequency density
$\underline{\underline{15}} \leqslant t < 20$	10	3
$20 \leqslant t < 30$	x	$\underline{\underline{0.15x}}$
$30 \leqslant t < 45$	15	1.5
$45 \leqslant t < 55$	$\underline{\underline{\tfrac{20}{3}y}}$	y
$55 \leqslant t < \underline{\underline{95}}$	28	1.05

19. In the following, find x.

Length (km)	Frequency	Frequency density
$10 \leqslant t < x$	18	6.75
$x \leqslant t < 30$	48	4.5

Solution

$$\begin{aligned}6.75 : \frac{18}{x-10} = 4.5 : \frac{48}{30-x} &\Rightarrow \frac{6.75}{\frac{18}{x-10}} = \frac{4.5}{\frac{48}{30-x}} \\ &\Rightarrow \tfrac{3}{8}(x-10) = \tfrac{3}{32}(30-x) \\ &\Rightarrow 4(x-10) = 30-x \\ &\Rightarrow 4x - 40 = 30 - x \\ &\Rightarrow 5x = 70 \\ &\Rightarrow \underline{\underline{x = 14}}.\end{aligned}$$

20. In the following, find x and y.

Mass (g)	Frequency	Frequency density
$16 < m \leqslant x$	4	1
$x < m \leqslant y$	30	5
$y < m \leqslant 40$	6	3

Solution

First two lines of the table:

$$\begin{aligned}1 : \frac{4}{x-16} = 5 : \frac{30}{y-x} &\Rightarrow \frac{1}{\frac{4}{x-16}} = \frac{5}{\frac{30}{y-x}} \\ &\Rightarrow \tfrac{1}{4}(x-16) = \tfrac{1}{6}(y-x) \\ &\Rightarrow 3(x-16) = 2(y-x) \\ &\Rightarrow 3x - 48 = 2y - 2x \\ &\Rightarrow 5x - 2y = 48 \quad (1)\end{aligned}$$

Final two lines of the table:

$$5 : \frac{30}{y-x} = 3 : \frac{6}{40-y} \Rightarrow \frac{5}{\frac{30}{y-x}} = \frac{3}{\frac{6}{40-y}}$$
$$\Rightarrow \tfrac{1}{6}(y-x) = \tfrac{1}{2}(40-y)$$
$$\Rightarrow y - x = 3(40-y)$$
$$\Rightarrow y - x = 120 - 3y$$
$$\Rightarrow -x + 4y = 120$$
$$\Rightarrow -5x + 20y = 600 \quad (2)$$

Add (1) + (2):

$$18y = 648 \Rightarrow \underline{\underline{y = 36}}$$
$$\Rightarrow 5x - 72 = 48$$
$$\Rightarrow 5x = 120$$
$$\Rightarrow \underline{\underline{x = 24}}.$$

21. In the following, find x, y, and z.

Length (m)	Frequency	Frequency density
$0 \leqslant t < x$	30	2.4
$x \leqslant t < y$	25	4
$y \leqslant t < z$	60	3.2
$z \leqslant t < 50$	20	0.8

Solution

x and y:

$$2.4 : \frac{30}{x} = 4 : \frac{25}{y-x} \Rightarrow \frac{2.4}{\frac{30}{x}} = \frac{4}{\frac{25}{y-x}}$$
$$\Rightarrow \tfrac{2}{25}x = \tfrac{4}{25}(y-x)$$
$$\Rightarrow x = 2(y-x)$$
$$\Rightarrow x = 2y - 2x$$
$$\Rightarrow 3x - 2y = 0 \quad (1)$$

$\underline{x \text{ and } z}$:

$$\begin{aligned} 2.4 : \frac{30}{x} = 0.8 : \frac{20}{50-z} &\Rightarrow \frac{2.4}{\frac{30}{x}} = \frac{0.8}{\frac{20}{50-z}} \\ &\Rightarrow \tfrac{2}{25}x = \tfrac{1}{25}(50-z) \\ &\Rightarrow 2x = 50 - z \\ &\Rightarrow 2x + z = 50 \quad (2) \end{aligned}$$

$\underline{y \text{ and } z}$:

$$\begin{aligned} 3.2 : \frac{60}{z-y} = 0.8 : \frac{20}{50-z} &\Rightarrow \frac{3.2}{\frac{60}{z-y}} = \frac{0.8}{\frac{20}{50-z}} \\ &\Rightarrow \tfrac{4}{75}(z-y) = \tfrac{1}{25}(50-z) \\ &\Rightarrow 4(z-y) = 3(50-z) \\ &\Rightarrow 4z - 4y = 150 - 3z \\ &\Rightarrow -4y + 7z = 150 \quad (3) \end{aligned}$$

Now, if you have one of those calculators that will solve 3×3 equations (such as the Casio fx-991EX, for example), use that. If you do not, then proceed as follows.

Now,

$$\begin{aligned} 3x - 2y = 0 &\Rightarrow 2y = 3x \\ &\Rightarrow y = \tfrac{3}{2}x \quad (4) \end{aligned}$$

and

$$-4(\tfrac{3}{2}x) + 7z = 150 \Rightarrow -6x + 7z = 150 \quad (5)$$

From (2),

$$2x + z = 50 \Rightarrow 6x + 3z = 150 \quad (6)$$

Add (5) + (6):

$$\begin{aligned} 10z = 300 &\Rightarrow \underline{\underline{z = 30}} \\ &\Rightarrow -6x + 210 = 150 \\ &\Rightarrow -6x = -60 \\ &\Rightarrow \underline{\underline{x = 10}} \\ &\Rightarrow \underline{\underline{y = 15}}. \end{aligned}$$

We finish with the table:

Length (m)	Frequency	Frequency density
$0 \leqslant t < \underline{\underline{10}}$	30	2.4
$10 \leqslant t < \underline{\underline{15}}$	25	4
$15 \leqslant t < \underline{\underline{30}}$	60	3.2
$\underline{\underline{30}} \leqslant t < 50$	20	0.8