Dr Oliver Mathematics
Applied Mathematics: Binomial Theorem

The total number of marks available is 23.
You must write down all the stages in your working.

1. Expand and simplify (3)

$$\left(2a - \frac{3}{a}\right)^4.$$

Solution

$$\begin{aligned} & \left(2a - \frac{3}{a}\right)^4 \\ = \quad & (2a)^4 + \binom{4}{1}(2a)^3\left(-\frac{3}{a}\right) + \binom{4}{2}(2a)^2\left(-\frac{3}{a}\right)^2 + \binom{4}{3}(2a)\left(-\frac{3}{a}\right)^3 + \left(-\frac{3}{a}\right)^4 \\ & = \underline{\underline{16a^4 - 96a^2 + 216 - \frac{216}{a^2} + \frac{81}{a^4}}}. \end{aligned}$$

2. Use the binomial theorem to expand (4)

$$\left(x^3 - \frac{2}{x}\right)^4$$

and simplify your answer.

Solution

$$\begin{aligned} & \left(x^3 - \frac{2}{x}\right)^4 \\ = \quad & (x^3)^4 + \binom{4}{1}(x^3)^3\left(-\frac{2}{x}\right) + \binom{4}{2}(x^3)^2\left(-\frac{2}{x}\right)^2 + \binom{4}{3}(x^3)\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4 \\ = \quad & \underline{\underline{x^{12} - 8x^8 + 24x^4 - 32 + \frac{16}{x^4}}}. \end{aligned}$$

3. Obtain the binomial expansion of (4)

$$\left(b - \frac{2}{b}\right)^5$$

and simplify the expression.

Solution

$$\begin{aligned} & \left(b - \frac{2}{b}\right)^5 \\ = \; & b^5 + \binom{5}{1} b^4 \left(-\frac{2}{b}\right) + \binom{5}{2} b^3 \left(-\frac{2}{b}\right)^2 + \binom{5}{3} b^2 \left(-\frac{2}{b}\right)^3 + \binom{5}{4} b \left(-\frac{2}{b}\right)^4 + \left(-\frac{2}{b}\right)^5 \\ = \; & \underline{\underline{b^5 - 10b^3 + 40b - \frac{80}{b} + \frac{80}{b^3} - \frac{32}{b^5}}}. \end{aligned}$$

4. Find the term in a^6 in the binomial expansion of (4)

$$\left(\frac{1}{a} + 3a\right)^{10}.$$

Solution

The general term is

$$\begin{aligned} \binom{10}{r} \left(\frac{1}{a}\right)^r (3a)^{10-r} &= \binom{10}{r} (a^{-r})(3^{10-r})(a^{10-r}) \\ &= \binom{10}{r} (3^{10-r})(a^{10-2r}). \end{aligned}$$

Now,

$$\begin{aligned} 10 - 2r = 6 &\Rightarrow 2r = 4 \\ &\Rightarrow r = 2 \end{aligned}$$

and the term in a^6 is

$$\binom{10}{2}(3^8) = \underline{\underline{295\,245}}.$$

5. (a) Write down and simplify the general term in the expansion of (3)

$$(x^2 + 3x)^8.$$

Solution

The general term is

$$\begin{aligned}\binom{8}{r}(x^2)^r(3x)^{8-r} &= \binom{8}{r}(x^{2r})(3^{8-r}x^{8-r}) \\ &= \underline{\underline{\binom{8}{r}3^{8-r}x^{8+r}}}.\end{aligned}$$

(b) Hence, or otherwise, obtain the coefficient of x^{13}. (2)

Solution

Now,

$$8 + r = 13 \Rightarrow r = 5$$

and the coefficient of x^{13} is

$$\binom{8}{5}3^3 = \underline{\underline{1\,512}}.$$

6. Write down and simplify the binomial expansion of (3)

$$(\mathrm{e}^x + 2)^4.$$

Solution

You ought to know the binomial sequence is 1 4 6 4 1:

$$\begin{aligned}(\mathrm{e}^x + 2)^4 &= (\mathrm{e}^x)^4 + 4(\mathrm{e}^x)^3(2) + 6(\mathrm{e}^x)^2(2)^2 + 4(\mathrm{e}^x)(2)^3 + (2)^4 \\ &= \underline{\underline{\mathrm{e}^{4x} + 8\mathrm{e}^{3x} + 24\mathrm{e}^{2x} + 32\mathrm{e}^x + 16}}.\end{aligned}$$