Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2005 November Paper 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Find the set of values of x for which (3)

$$(x-6)^2 > x.$$

Solution

Well,

$\times$	x	-6
x	x^2	$-6x$
-6	$-6x$	$+36$

and so

$$\begin{aligned}(x-6)^2 > x &\Rightarrow x^2 - 12x + 36 > x \\ &\Rightarrow x^2 - 13x + 36 > 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -13 \\ \text{multiply to:} & +36\end{array}\right\} -9,\ -4$$

e.g.,

$$\Rightarrow (x-9)(x-4) > 0.$$

We need a 'table of signs':

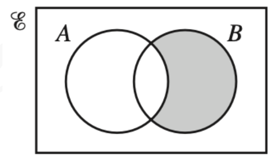
	$x < 4$	$x = 4$	$4 < x < 9$	$x = 9$	$x > 9$
$x-4$	$-$	0	$+$	$+$	$+$
$x-9$	$-$	$-$	$-$	0	$+$
$(x-9)(x-4)$	$+$	0	$-$	0	$+$

and so

$$(x-9)(x-4) > 0 \Rightarrow \underline{\underline{x < 4 \text{ or } x > 9}}.$$

2. (a) For each of the Venn diagrams above,

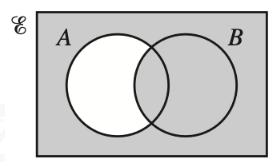
   (i) express the shaded region in set notation: (1)



Solution

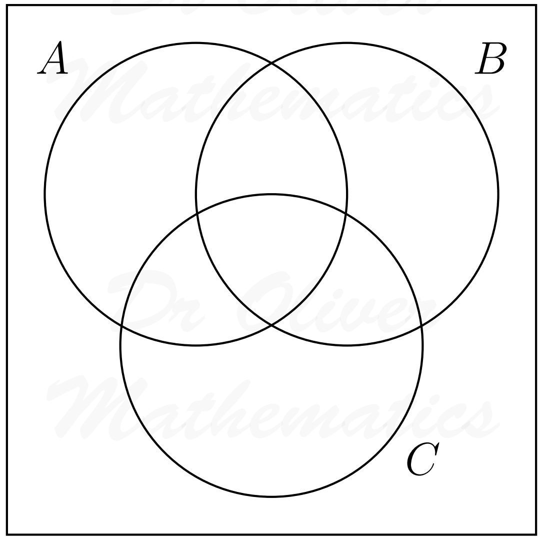
$\underline{\underline{A' \cap B}}$.

   (ii) express the shaded region in set notation: (1)



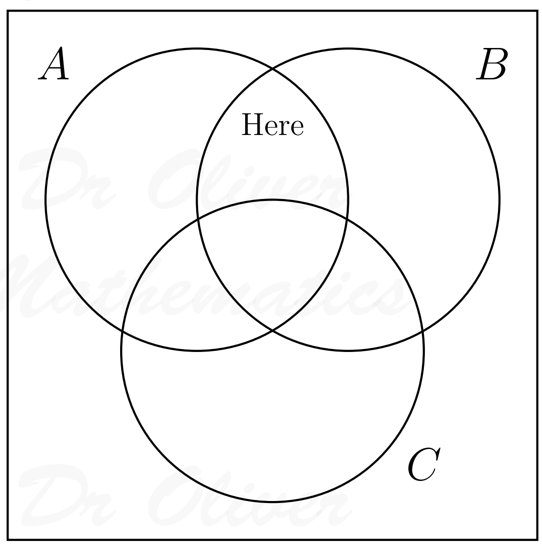
Solution

$\underline{\underline{A' \cup B}}$.



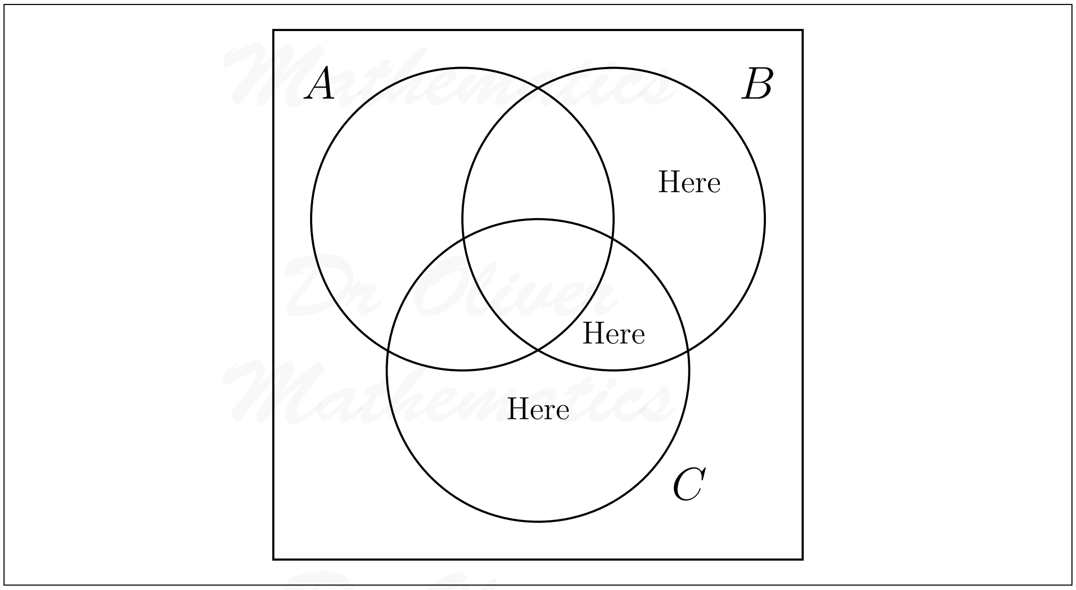
(b) (i) Copy the Venn diagram above and shade the region that represents $A \cap B \cap C'$. (1)

Solution



(ii) Copy the Venn diagram above and shade the region that represents $A' \cap (B \cup C)$. (1)

Solution



3. Find the values of the constant c for which the line (4)

$$2y = x + c$$

is a tangent to the curve

$$y = 2x + \frac{6}{x}.$$

Solution

Well,

$$\begin{aligned} y = 2x + \frac{6}{x} &\Rightarrow 2y = 4x + 12x^{-1} \\ &\Rightarrow x + c = 4x + 12x^{-1} \\ &\Rightarrow x^2 + cx = 4x^2 + 12 \\ &\Rightarrow 3x^2 - cx + 12 = 0. \end{aligned}$$

Now,

$$\begin{aligned} b^2 - 4ac &\Rightarrow (-c)^2 - 4(3)(12) = 0 \\ &\Rightarrow c^2 = 144 \\ &\Rightarrow \underline{\underline{c = \pm 12}}. \end{aligned}$$

4. A cuboid has a square base of side $(2-\sqrt{3})$ m and a volume $(2\sqrt{3}-3)$ m^3. (4)

Find the height of the cuboid in the form $(a+b\sqrt{3})$ m, where a and b are integers.

Solution

Now,

$\times$	2	$-\sqrt{3}$
2	2	$-2\sqrt{3}$
$-\sqrt{3}$	$-2\sqrt{3}$	$+3$

and so

$$\begin{aligned}\text{height} &= \frac{2\sqrt{3}-3}{(2-\sqrt{3})^2}\\ &= \frac{2\sqrt{3}-3}{7-4\sqrt{3}}\\ &= \frac{2\sqrt{3}-3}{7-4\sqrt{3}} \times \frac{7+4\sqrt{3}}{7+4\sqrt{3}}\end{aligned}$$

$\times$	$2\sqrt{3}$	-3
7	$14\sqrt{3}$	-21
$+4\sqrt{3}$	$+24$	$-12\sqrt{3}$

$\times$	7	$+4\sqrt{3}$
7	49	$+28\sqrt{3}$
$-4\sqrt{3}$	$-28\sqrt{3}$	-48

$$\begin{aligned}&= \frac{3+2\sqrt{3}}{1}\\ &= \underline{\underline{3+2\sqrt{3}}}.\end{aligned}$$

5. The diagram, which is not drawn to scale, shows a horizontal rectangular surface. (6)

One corner of the surface is taken as the origin O and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the edges of the surface.

A fly, F, starts at the point with position vector $(\mathbf{i} + 12\mathbf{j})$ cm and crawls across the surface with a velocity of $(3\mathbf{i} + 2\mathbf{j})$ cm s^{-1}.

At the instant that the fly starts crawling, a spider, S, at the point with position vector $(85\mathbf{i} + 5\mathbf{j})$ cm, sets off across the surface with a velocity of $(-5\mathbf{i} + k\mathbf{j})$ cm s^{-1}, where k is a constant.

Given that the spider catches the fly, calculate the value of k.

Solution

Well, at time t,

$$\begin{aligned}\overrightarrow{OS} &= (85\mathbf{i} + 5\mathbf{j}) + t(-5\mathbf{i} + k\mathbf{j}) \\ &= (85 - 5t)\mathbf{i} + (5 + kt)\mathbf{j})\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{OF} &= (\mathbf{i} + 12\mathbf{j}) + t(3\mathbf{i} + 2\mathbf{j}) \\ &= (1 + 3t)\mathbf{i} + (12 + 2t)\mathbf{j}).\end{aligned}$$

Equate the $\mathbf{i}$:

$$\begin{aligned}85 - 5t = 1 + 3s &\Rightarrow 84 = 8t \\ &\Rightarrow t = 10.5\end{aligned}$$

and equate the $\mathbf{j}$:

$$\begin{aligned}5 + k(10.5) = 12 + 2(10.5) &\Rightarrow 5 + 10.5k = 12 + 21 \\ &\Rightarrow 10.5k = 28 \\ &\Rightarrow \underline{\underline{k = 2\tfrac{2}{3}}}.\end{aligned}$$

6. A particle starts from rest at a fixed point O and moves in a straight line towards a point A. The velocity, v ms^{-1}, of the particle, t seconds after leaving O, is given by

$$v = 6 - 6\mathrm{e}^{-3t}.$$

Given that the particle reaches A when $t = \ln 2$, find

(a) the acceleration of the particle at A, (3)

Solution

$$v = 6 - 6\mathrm{e}^{-3t} \Rightarrow a = 18\mathrm{e}^{-3t}$$

and

$$\begin{aligned} t = \ln 2 \Rightarrow a &= 18\mathrm{e}^{-3\ln 2} \\ \Rightarrow a &= 18\mathrm{e}^{\ln 2^{-3}} \\ \Rightarrow a &= 18 \times 2^{-3} \\ \Rightarrow \underline{\underline{a = 2\tfrac{1}{4} \text{ ms}^{-2}}}. \end{aligned}$$

(b) the distance OA. (4)

Solution

Now,

$$\begin{aligned} OA &= \int_0^{\ln 2} (6 - 6\mathrm{e}^{-3t})\,\mathrm{d}x \\ &= \left[6t + 2\mathrm{e}^{-3t}\right]_{t=0}^{\ln 2} \\ &= \left(6\ln 2 + 2\mathrm{e}^{-3\ln 2}\right) - (0 + 2) \\ &= 6\ln 2 + 2 \times \tfrac{1}{8} - 2 \\ &= \underline{\underline{\left(6\ln 2 - \tfrac{7}{4}\right) \text{ m}}}. \end{aligned}$$

7. (a) Solve (4)

$$\log_7(17y + 15) = 2 + \log_7(2y - 3).$$

Solution

$$
\begin{aligned}
\log_7(17y+15) = 2 + \log_7(2y-3) &\Rightarrow \log_7(17y+15) - \log_7(2y-3) = 2 \\
&\Rightarrow \log_7\left(\frac{17y+15}{2y-3}\right) = 2 \\
&\Rightarrow \frac{17y+15}{2y-3} = 7^2 \\
&\Rightarrow 17y+15 = 49(2y-3) \\
&\Rightarrow 17y+15 = 98y-147 \\
&\Rightarrow 162 = 81y \\
&\Rightarrow \underline{\underline{y = 2}}.
\end{aligned}
$$

(b) Evaluate (3)

$$\log_p 8 \times \log_{16} p.$$

Solution

$$
\begin{aligned}
\log_p 8 \times \log_{16} p &= \log_p 8 \div \log_p 16 \\
&= \frac{\log_p 8}{\log_p 16} \\
&= \frac{\log_p 2^3}{\log_p 2^4} \\
&= \frac{3\log_p 2}{4\log_p 2} \\
&= \underline{\underline{\tfrac{3}{4}}}.
\end{aligned}
$$

8. A curve has the equation

$$y = (x+2)\sqrt{x-1}.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx}{\sqrt{x-1}},$$

where k is a constant, and state the value of k.

Solution

Well,

$$u = x + 2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$v = (x-1)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \tfrac{1}{2}(x-1)^{-\frac{1}{2}}$$

and

$$\begin{aligned} y = (x+2)(x-1)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= (x+2)(\tfrac{1}{2}(x-1)^{-\frac{1}{2}}) + (1)((x-1)^{\frac{1}{2}}) \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+2}{2\sqrt{x-1}} + \sqrt{x-1} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+2}{2\sqrt{x-1}} + \frac{2(x-1)}{2\sqrt{x-1}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+2+2(x-1)}{2\sqrt{x-1}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+2+2x-2}{2\sqrt{x-1}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x}{2\sqrt{x-1}}; \end{aligned}$$

so, $k = 3$.

(b) Hence evaluate (4)

$$\int_2^5 \frac{x}{\sqrt{x-1}}\,\mathrm{d}x.$$

Solution

Now,

$$\begin{aligned} \int_2^5 \frac{x}{\sqrt{x-1}}\,\mathrm{d}x &= \tfrac{2}{3}\int_2^5 \frac{3x}{2\sqrt{x-1}}\,\mathrm{d}x \\ &= \tfrac{2}{3}\left[(x+2)\sqrt{x-1}\right]_{x=2}^5 \\ &= \tfrac{2}{3}\,(14-4) \\ &= 6\tfrac{2}{3}. \end{aligned}$$

9. (a) Find all the angles between 0° and 360° which satisfy the equation (5)

$$3\cos x = 8\tan x.$$

Solution

$$\begin{aligned} 3\cos x = 8\tan x &\Rightarrow 3\cos x = \frac{8\sin x}{\cos x} \\ &\Rightarrow 3\cos^2 x = 8\sin x \\ &\Rightarrow 3(1-\sin^2 x) = 8\sin x \\ &\Rightarrow 3 - 3\sin^2 x = 8\sin x \\ &\Rightarrow 3\sin^2 x + 8\sin x - 3 = 0 \end{aligned}$$

e.g.,

$$\left.\begin{array}{ll}\text{add to:} & +8 \\ \text{multiply to:} & (+3)\times(-3) = -9\end{array}\right\} +9,\ -1$$

$$\begin{aligned} &\Rightarrow 3\sin^2 x + 9\sin x - \sin x - 3 = 0 \\ &\Rightarrow 3\sin x(\sin x + 3) - 1(\sin x + 3) = 0 \\ &\Rightarrow (3\sin x - 1)(\sin x + 3) = 0 \\ &\Rightarrow \sin x = \tfrac{1}{3} \end{aligned}$$

(as $\sin x = -3$ has no real solutions)

$$\begin{aligned} &\Rightarrow x = 19.471\,220\,63 \text{ or } 160.528\,779\,4 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 19.5 \text{ or } 161 \text{ (3 sf)}}}. \end{aligned}$$

(b) Given that $4 \leqslant y \leqslant 6$, find the value of y for which (3)

$$2\cos(\tfrac{2}{3}y) + \sqrt{3} = 0.$$

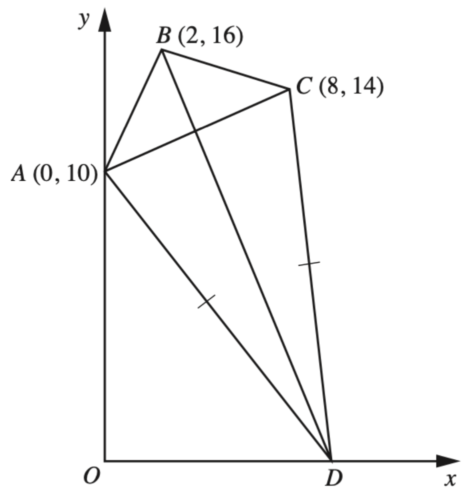
Solution

$$\begin{aligned} 2\cos(\tfrac{2}{3}y) + \sqrt{3} = 0 &\Rightarrow 2\cos(\tfrac{2}{3}y) = -\sqrt{3} \\ &\Rightarrow \cos(\tfrac{2}{3}y) = -\tfrac{\sqrt{3}}{2} \\ &\Rightarrow \tfrac{2}{3}y = \tfrac{5}{6}\pi,\ \tfrac{7}{6}\pi,\ \tfrac{17}{6}\pi,\ \tfrac{20}{6}\pi \\ &\Rightarrow y = \tfrac{15}{12}\pi,\ \tfrac{21}{12}\pi,\ \tfrac{51}{12}\pi,\ \tfrac{60}{12}\pi. \end{aligned}$$

Of these,

$$4 \leqslant y \leqslant 6 \Rightarrow \underline{\underline{y = \tfrac{7}{4}\pi}}.$$

10. The diagram, which is not drawn to scale, shows a quadrilateral $ABCD$ in which A is $(0, 10)$, B is $(2, 16)$, and C is $(8, 14)$.



(a) Show that triangle ABC is isosceles. (2)

Solution

$$\begin{aligned} AB &= \sqrt{(2-0)^2 + (16-10)^2} \\ &= \sqrt{4+36} \\ &= 2\sqrt{10} \end{aligned}$$

and

$$\begin{aligned} BC &= \sqrt{(8-2)^2 + (16-14)^2} \\ &= \sqrt{36+4} \\ &= 2\sqrt{10}; \end{aligned}$$

hence, the triangle ABC is $\underline{\underline{\text{isosceles}}}$.

The point D lies on the x-axis and is such that $AD = CD$.

Find

(b) the coordinates of D, (4)

Solution

Now,

$$m_{AC} = \frac{14-10}{8-0} = \tfrac{1}{2}$$

and

$$m_{BD} = -\frac{1}{\frac{1}{2}} = -2.$$

Next, the equation of BD is

$$y - 16 = -2(x-2) \Rightarrow y - 16 = -2x + 4 \Rightarrow y = -2x + 20.$$

Finally,

$$y = 0 \Rightarrow 0 = -2x + 20 \Rightarrow 2x = 20 \Rightarrow x = 10;$$

hence, $\underline{\underline{D(10, 0)}}$.

(c) the ratio of the area of triangle ABC to the area of triangle ACD. (3)

Solution

Well, the midpoint of AC is

$$\left(\frac{0+8}{2}, \frac{10+14}{2}\right) = E(4, 12).$$

Now,

$$BE = \sqrt{(2-4)^2 + (16-12)^2} = \sqrt{4+16} = 2\sqrt{5}$$

and

$$\begin{aligned} DE &= \sqrt{(10-4)^2 + (0-12)^2} \\ &= \sqrt{36+144} \\ &= 6\sqrt{5}. \end{aligned}$$

Finally,

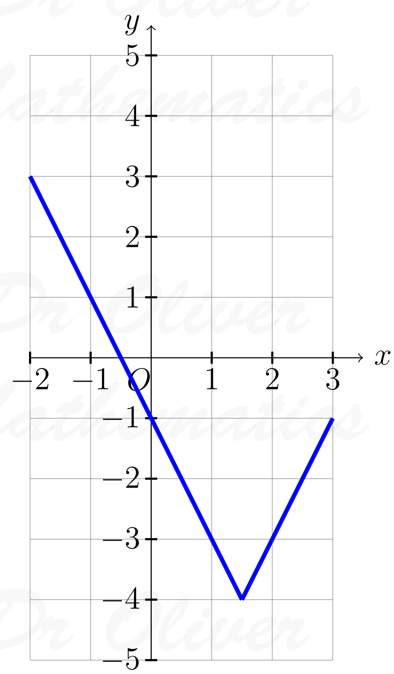
$$\begin{aligned} & \text{the area of triangle } ABC : \text{the area of triangle } ACD \\ = \ & \tfrac{1}{2} \times BE \times AC = \tfrac{1}{2} \times DE \times AC \\ = \ & BE : DE \\ = \ & 2\sqrt{5} : 6\sqrt{5} \\ = \ & \underline{\underline{1 : 3}}. \end{aligned}$$

11. A function f is defined by

$$\mathrm{f} : x \mapsto |2x-3| - 4, \text{ for } -2 \leqslant x \leqslant 3.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$. (2)

Solution



(b) State the range of f. (2)

Solution

$\underline{\underline{-4 \leqslant \mathrm{f}(x) \leqslant 3}}$.

(c) Solve the equation $\mathrm{f}(x) = -2$. (3)

Solution

$\underline{y = (2x - 3) - 4}$:

$$\begin{aligned}(2x-3)-4 = -2 &\Rightarrow 2x = 5\\ &\Rightarrow x = 2\tfrac{1}{2}.\end{aligned}$$

$\underline{y = -(2x - 3) - 4}$:

$$\begin{aligned}-(2x-3)-4 = -2 &\Rightarrow -2x = -1\\ &\Rightarrow x = \tfrac{1}{2}.\end{aligned}$$

Hence,

$$\mathrm{f}(x) = -2 \Rightarrow \underline{\underline{x = \tfrac{1}{2} \text{ or } x = 2\tfrac{1}{2}}}.$$

A function g is defined by

$$\mathrm{g} : x \mapsto |2x - 3| - 4, \text{ for } -2 \leqslant x \leqslant k.$$

(d) State the largest value of k for which g has an inverse. (1)

Solution

$\underline{\underline{k = 1\tfrac{1}{2}}}$.

(e) Given that g has an inverse, express g in the form (2)

$$\mathrm{g} : x \mapsto ax + b,$$

where a and b are constants.

Solution

$$\begin{aligned}y = -(2x-3)-4 &\Rightarrow y = -2x + 3 - 4\\ &\Rightarrow y = -2x - 1\end{aligned}$$

and

$$\underline{\underline{g(x) = -2x - 1}}.$$

EITHER

12. Variables x and y are related by the equation

$$yx^n = a,$$

where a and n are constants.

The table below shows measured values of x and y.

x	1.5	2	2.5	3	3.5
y	7.3	3.5	2.0	1.3	0.9

(a) On graph paper plot $\log y$ against $\log x$. (3)

Draw a straight line graph to represent the equation

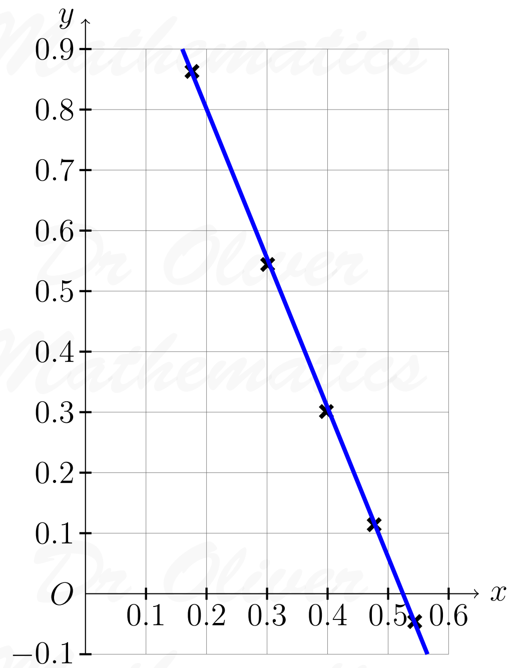
$$yx^n = a.$$

Solution

Well,

$\log x$	0.176	0.301	0.398	0.477	0.544
$\log y$	0.863	0.544	0.301	0.114	-0.046

so



(b) Use your graph to estimate the value of a and of n. (4)

Solution

$$\begin{aligned} yx^n = a &\Rightarrow \log(yx^n) = \log a \\ &\Rightarrow \log y + \log x^n = \log a \\ &\Rightarrow \log y + n \log x = \log a \\ &\Rightarrow \log y = \log a - n \log x \end{aligned}$$

Pick two points on the straight line: given my line, I will choose $(0.2, 0.8)$ and $(0.5, 0.05)$. Now,

$$\begin{aligned} m &= \frac{0.8 - 0.05}{0.2 - 0.5} \\ &= -2.5 \end{aligned}$$

and that implies

$$\log y = \log a - 2.5 \log x.$$

Now,

$$\begin{aligned} \log x = 0.2, \log y = 0.8 &\Rightarrow 0.8 = \log a - 2.5 \times 0.2 \\ &\Rightarrow 0.8 = \log a - 0.5 \\ &\Rightarrow \log a = 1.3 \\ &\Rightarrow a = 19.952\ldots; \end{aligned}$$

hence, $\underline{\underline{a = 20}}$ and $\underline{\underline{n = 2.5}}$.

(c) On the same diagram, draw the line representing the equation (3)

$$y = x^2$$

and hence find the value of x for which

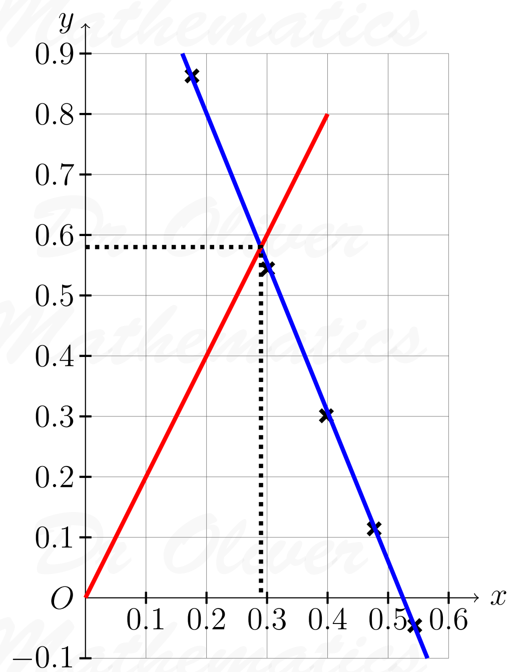
$$x^{n+2} = a.$$

Solution

Well,

$$y = x^2 \Rightarrow \log y = \log x^2$$
$$\Rightarrow \log y = 2 \log x$$

and so we have

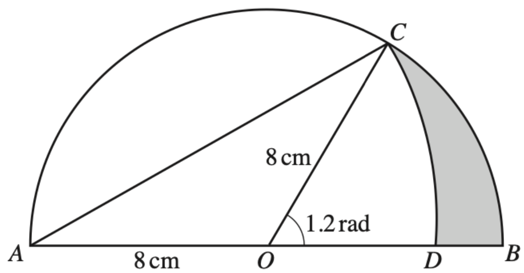


Now,

$$\log x = 0.29 \Rightarrow x = 1.949\,844\,6 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 1.95 \text{ (3 sf)}}}.$$

OR

13. The diagram shows a semicircle, centre O, of radius 8 cm.



The radius OC makes an angle of 1.2 radians with the radius OB.

The arc CD of a circle has centre A and the point D lies on OB.

Find the area of

(a) sector COB, (2)

Solution

$$\begin{aligned} \text{Area of sector } COB &= \tfrac{1}{2} \times 8^2 \times 1.2 \\ &= \underline{\underline{38.4 \text{ cm}^2}}. \end{aligned}$$

(b) sector CAD, (5)

Solution

Well,

$$\begin{aligned} AC^2 &= OA^2 + OC^2 - 2 \times OA \times OC \times \cos AOC \\ \Rightarrow \quad AC^2 &= 8^2 + 8^2 - 2 \times 8 \times 8 \times \cos(\pi - 1.2) \\ \Rightarrow \quad AC^2 &= 128 - 128\cos(\pi - 1.2) \\ \Rightarrow \quad AC &= 13.205\,369\,84 \text{ (FCD)}. \end{aligned}$$

Now,

$$\angle OAC = \tfrac{1}{2}(\pi - \cos(\pi - 1.2)) = 0.6$$

and

$$\begin{aligned} \text{area of sector } CAD &= \tfrac{1}{2} \times 13.205\ldots^2 \times 0.6 \\ &= 52.314\,537\,77 \text{ (FCD)} \\ &= \underline{\underline{52.3 \text{ cm}^2 \text{ (3 sf)}}}. \end{aligned}$$

(c) the shaded region. (3)

Solution

Now,

$$\begin{aligned} \text{area of } \triangle OAC &= \tfrac{1}{2} \times 8 \times 8 \sin(\pi - 1.2) \\ &= 32 \sin(\pi - 1.2) \end{aligned}$$

and, hence,

$$\begin{aligned} \text{shaded area} &= 32 \sin(\pi - 1.2) + 38.4 - 52.314\ldots \\ &= 15.910\,712\,98 \text{ (FCD)} \\ &= \underline{\underline{15.9 \text{ cm}^2 \text{ (3 sf)}}}. \end{aligned}$$