

Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2012 November Paper 2 Variant 2: Calculator
2 hours

The total number of marks available is 80.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You must write down all the stages in your working.

1. Solve the equation

$$|5x + 7| = 13.$$

(3)

Solution

$5x + 7 = 13$:

$$\begin{aligned} 5x + 7 = 13 &\Rightarrow 5x = 6 \\ &\Rightarrow \underline{\underline{x = 1\frac{1}{5}}}. \end{aligned}$$

$-(5x + 7) = 13$:

$$\begin{aligned} -(5x + 7) = 13 &\Rightarrow -5x - 7 = 13 \\ &\Rightarrow -5x = 20 \\ &\Rightarrow \underline{\underline{x = -4}}. \end{aligned}$$

2. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 7 & 8 \\ 4 & 6 \end{pmatrix},$$

(2)

find the inverse matrix, $\mathbf{A}^{-1}$.

Solution

Well,

$$\det \mathbf{A} = 42 - 32 = 10$$

and

$$\underline{\underline{\mathbf{A}^{-1} = \frac{1}{10} \begin{pmatrix} 6 & -8 \\ -4 & 7 \end{pmatrix}}}.$$

- (b) Use your answer to part (a) to solve the simultaneous equations (2)

$$7x + 8y = 39,$$

$$4x + 6y = 23.$$

Solution

Convert it into matrix form:

$$\begin{aligned} \begin{pmatrix} 7 & 8 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 39 \\ 23 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & -8 \\ -4 & 7 \end{pmatrix} \begin{pmatrix} 39 \\ 23 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 50 \\ 5 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0.5 \end{pmatrix}; \end{aligned}$$

hence,

$$\underline{\underline{x = 5 \text{ and } y = 0.5.}}$$

3. Without using a calculator, simplify (4)

$$\frac{(3\sqrt{3} - 1)^2}{2\sqrt{3} - 3},$$

giving your answer in the form

$$\frac{a\sqrt{3} + b}{3},$$

where a and b are integers.

Solution

Well,

$\times$	$3\sqrt{3}$	-1
$3\sqrt{3}$	27	$-3\sqrt{3}$
-1	$-3\sqrt{3}$	$+1$

and

$$\begin{aligned}\frac{(3\sqrt{3}-1)^2}{2\sqrt{3}-3} &= \frac{28-6\sqrt{3}}{2\sqrt{3}-3} \\ &= \frac{28-6\sqrt{3}}{2\sqrt{3}-3} \times \frac{2\sqrt{3}+3}{2\sqrt{3}+3}\end{aligned}$$

$\times$	$2\sqrt{3}$	-3
$2\sqrt{3}$	12	$-6\sqrt{3}$
$+3$	$+6\sqrt{3}$	-9

$\times$	$2\sqrt{3}$	$+3$
28	$56\sqrt{3}$	$+84$
$-6\sqrt{3}$	-36	$-18\sqrt{3}$

$$= \underline{\underline{\frac{38\sqrt{3}+48}{3}}};$$

hence, $a = 38$ and $b = 48$.

4. The points X , Y , and Z are such that $\overrightarrow{XY} = 3\overrightarrow{YZ}$. (5)

The position vectors of X and Z , relative to an origin O , are

$$\begin{pmatrix} 4 \\ -27 \end{pmatrix} \text{ and } \begin{pmatrix} 20 \\ -7 \end{pmatrix}$$

respectively.

Find the unit vector in the direction $\overrightarrow{OY}$.

Solution

Well,

$$\overrightarrow{XY} = \frac{3}{4}\overrightarrow{XZ}$$

and

$$\begin{aligned}\overrightarrow{OY} &= \overrightarrow{OX} + \overrightarrow{XY} \\ &= \overrightarrow{OX} + \frac{3}{4}\overrightarrow{XZ} \\ &= \begin{pmatrix} 4 \\ -27 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 20 - 4 \\ -7 - (-27) \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -27 \end{pmatrix} + \frac{3}{4} \begin{pmatrix} 16 \\ 20 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -27 \end{pmatrix} + \begin{pmatrix} 12 \\ 15\frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} 16 \\ -12 \end{pmatrix}.\end{aligned}$$

Now,

$$\begin{aligned}OY &= \sqrt{16^2 + (-12)^2} \\ &= \sqrt{400} \\ &= 20\end{aligned}$$

so a unit vector is

$$\frac{1}{20} \begin{pmatrix} 16 \\ -12 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}}}.$$

5. Find the set of values of m for which the line

(6)

$$y = mx + 2$$

does not meet the curve

$$y = mx^2 + 7x + 11.$$

Solution

Well,

$$mx^2 + 7x + 11 = mx + 2 \Rightarrow mx^2 + (7 - m)x + 9 = 0.$$

Now, $a = m$, $b = 7 - m$, and $c = 9$ and

$$b^2 - 4ac < 0 \Rightarrow (7 - m)^2 - 4(m)(9) < 0$$

$\times$	7	$-m$
7	49	$-7m$
$-m$	$-7m$	$+m^2$

$$\Rightarrow (49 - 14m + m^2) - 36m < 0$$

$$\Rightarrow m^2 - 50m + 49 < 0$$

$$\left. \begin{array}{l} \text{add to:} \\ \text{multiply to:} \end{array} \right\} \begin{array}{l} -50 \\ +49 \end{array} \Rightarrow -49, -1$$

$$\Rightarrow (m - 1)(m - 49) < 0.$$

We need a ‘table of signs’:

	$m < 1$	$m = 1$	$-1 < m < 49$	$m = 49$	$m > 49$
$(m - 1)$	−	0	+	+	+
$(m - 49)$	−	−	−	0	+
$(m - 1)(m - 49)$	+	0	−	0	+

Hence,

$$\underline{\underline{1 < m < 49.}}$$

6. (a) Given that

$$\cos x = p,$$

(3)

find an expression, in terms of p , for $\tan^2 x$.

Solution

Well,

$$\begin{aligned}\cos x = p &\Rightarrow \cos^2 x = p^2 \\ &\Rightarrow 1 - \cos^2 x = 1 - p^2 \\ &\Rightarrow \sin^2 x = 1 - p^2\end{aligned}$$

and so

$$\begin{aligned}\tan^2 x &= \frac{\sin^2 x}{\cos^2 x} \\ &= \frac{1 - p^2}{p^2}.\end{aligned}$$

(b) Prove that

$$(\cot \theta + \tan \theta)^2 \equiv \sec^2 \theta + \operatorname{cosec}^2 \theta.$$

(3)

Solution

Well,

$\times$	$\cot \theta$	$+ \tan \theta$
$\cot \theta$	$\cot^2 \theta$	$+1$
$+ \tan \theta$	$+1$	$+ \tan^2 \theta$

and so

$$\begin{aligned}(\cot \theta + \tan \theta)^2 &\equiv \cot^2 \theta + 2 + \tan^2 \theta \\ &\equiv (\cot^2 \theta + 1) + (1 + \tan^2 \theta) \\ &\equiv (1 + \tan^2 \theta) + (\cot^2 \theta + 1) \\ &= \underline{\underline{\sec^2 \theta + \operatorname{cosec}^2 \theta}},\end{aligned}$$

as required.

7. (a) Find

$$\int (x + 3)\sqrt{x} \, dx.$$

(3)

Solution

Now,

$$\begin{aligned}\int (x+3)\sqrt{x} \, dx &= \int (x+3)x^{\frac{1}{2}} \, dx \\ &= \int \left(x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right) \, dx \\ &= \underline{\underline{\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + c}}.\end{aligned}$$

(b) Find

$$\int \frac{20}{(2x+5)^2} \, dx$$

(4)

and hence evaluate

$$\int_0^{10} \frac{20}{(2x+5)^2} \, dx.$$

Solution

$$\begin{aligned}\int \frac{20}{(2x+5)^2} \, dx &= \int 20(2x+5)^{-2} \, dx \\ &= \underline{\underline{-10(2x+5)^{-1} + c}}\end{aligned}$$

and

$$\begin{aligned}\int_0^{10} \frac{20}{(2x+5)^2} \, dx &= \left[-10(2x+5)^{-1}\right]_{x=0}^{10} \\ &= -\frac{2}{5} - (-2) \\ &= \underline{\underline{1\frac{3}{5}}}.\end{aligned}$$

8. **Solutions to this question by accurate drawing will not be accepted.**

(8)

The points $A(4, 5)$, $B(-2, 3)$, $C(1, 9)$, and D are the vertices of a trapezium in which BC is parallel to AD and angle BCD is 90° .

Find the area of the trapezium.

Solution

BC :

$$\begin{aligned}m_{BC} &= \frac{9-3}{1-(-2)} \\&= \frac{6}{3} \\&= 2.\end{aligned}$$

Now,

$$\begin{aligned}BC &= \sqrt{[1-(-2)]^2 + (9-3)^2} \\&= \sqrt{3^2 + 6^2} \\&= \sqrt{45} \\&= 3\sqrt{5}.\end{aligned}$$

AD : Now,

$$m_{AD} = m_{BC} = 2$$

and the line has equation

$$\begin{aligned}y-5 &= 2(x-4) \Rightarrow y-5 = 2x-8 \\&\Rightarrow y = 2x-3 \quad (1).\end{aligned}$$

CD : Now,

$$m_{CD} = -\frac{1}{2}$$

and the line has equation

$$\begin{aligned}y-9 &= -\frac{1}{2}(x-1) \Rightarrow y-9 = -\frac{1}{2}x + \frac{1}{2} \\&\Rightarrow y = -\frac{1}{2}x + \frac{19}{2} \quad (2).\end{aligned}$$

D : So,

$$\begin{aligned}2x-3 &= -\frac{1}{2}x + \frac{19}{2} \Rightarrow \frac{5}{2}x = \frac{25}{2} \\&\Rightarrow x = 5 \\&\Rightarrow y = 7;\end{aligned}$$

hence, $D(5, 7)$.

CD :

$$\begin{aligned} CD &= \sqrt{(5-1)^2 + (7-9)^2} \\ &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5}. \end{aligned}$$

AD :

$$\begin{aligned} AD &= \sqrt{(5-4)^2 + (7-5)^2} \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5}. \end{aligned}$$

Area:

$$\begin{aligned} \text{area of the trapezium} &= \frac{1}{2}h(a+b) \\ &= \frac{1}{2} \times CD \times (BC + AD) \\ &= \frac{1}{2} \times 2\sqrt{5} \times (3\sqrt{5} + \sqrt{5}) \\ &= \underline{\underline{20}}. \end{aligned}$$

9. The table shows experimental values of two variables x and y .

x	1	2	3	4
y	9.41	1.29	-0.69	-1.77

It is known that x and y are related by the equation

$$y = \frac{a}{x^2} + bx,$$

where a and b are constants.

- (a) A straight line graph is to be drawn to represent this information. (1)

Given that x^2y is plotted on the vertical axis, state the variable to be plotted on the horizontal axis.

Solution

Well,

$$\begin{aligned}
 y &= \frac{a}{x^2} + bx \Rightarrow y = \frac{a}{x^2} + \frac{bx^3}{x^2} \\
 &\Rightarrow y = \frac{a + bx^3}{x^2} \\
 &\Rightarrow x^2y = a + bx^3;
 \end{aligned}$$

hence, the variable to be plotted on the horizontal axis is x^3 .

(b) Draw this straight line graph.

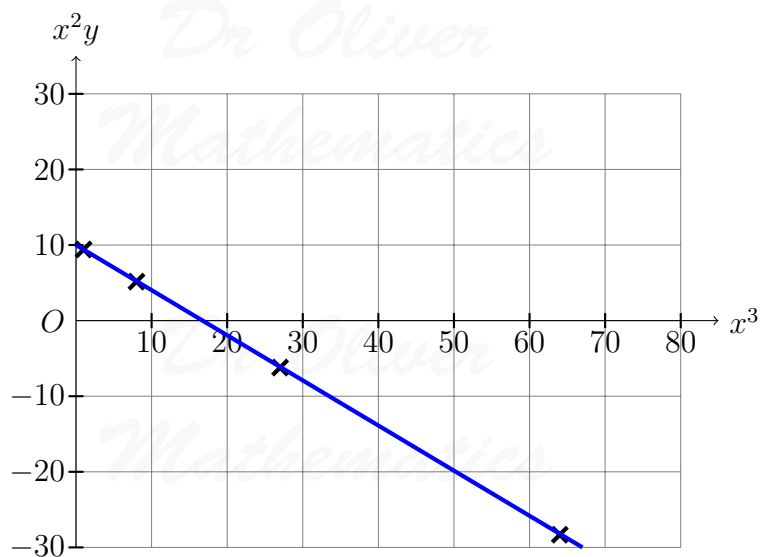
(3)

Solution

The table is

x^3	1	8	27	64
x^2y	9.41	5.16	-6.21	-28.32

and the graph is



(c) Use your graph to estimate the value of a and of b .

(3)

Solution

The line of best fit goes through $(0, 10)$ and $(67, -30)$. Now, $c = 10$ and

$$\begin{aligned} m &= \frac{10 - (-30)}{0 - 67} \\ &= -\frac{40}{67}. \end{aligned}$$

So,

$$x^2 y = 10 - \frac{40}{67} x^3 \Rightarrow y = \frac{10}{x^2} - \frac{40}{67} x;$$

hence,

$$\underline{\underline{a = 10 \text{ and } b = -\frac{40}{67}}}.$$

(d) Estimate the value of y when x is 3.7.

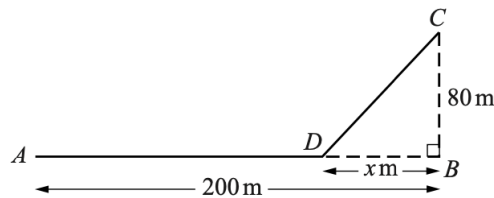
(2)

Solution

$$x = 3.7 \Rightarrow y = -1.478\,495\,034 \text{ (FCD)}$$

$$\Rightarrow \underline{\underline{y = -1.48 \text{ (3 sf)}}}.$$

10. A track runs due east from A to B , a distance of 200 m.



- The point C is 80 m due north of B .
 - A cyclist travels on the track from A to D , where D is x m due west of B .
 - The cyclist then travels in a straight line across rough ground from D to C .
 - The cyclist travels at 10 ms^{-1} on the track and at 6 ms^{-1} across rough ground.
- (a) Show that the time taken, T s, for the cyclist to travel from A to C is given by

(2)

$$T = \frac{200 - x}{10} + \frac{\sqrt{x^2 + 6\,400}}{6}.$$

Solution

Let t s be the time it goes at 6 ms^{-1} and then $(T - t)$ s be the time it goes at 10 ms^{-1} . Now,

$$\begin{aligned} CD &= \sqrt{x^2 + 80^2} \\ &= \sqrt{x^2 + 6400}. \end{aligned}$$

Next,

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

and

$$\begin{aligned} T &= (T - t) + t \\ &= \frac{200 - x}{10} + \frac{\sqrt{x^2 + 6400}}{6}, \end{aligned}$$

as required.

- (b) Given that x can vary, find the value of x for which T has a stationary value and the corresponding value of T . (6)

Solution

Well,

$$\begin{aligned} T &= \frac{200 - x}{10} + \frac{\sqrt{x^2 + 6400}}{6} \\ \Rightarrow T &= \frac{1}{10}(200 - x) + \frac{1}{6}(x^2 + 6400)^{\frac{1}{2}} \\ \Rightarrow \frac{dT}{dx} &= -\frac{1}{10} + \frac{1}{6}x(x^2 + 6400)^{-\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned}
 \frac{dT}{dx} = 0 &\Rightarrow -\frac{1}{10} + \frac{1}{6}x(x^2 + 6400)^{-\frac{1}{2}} = 0 \\
 &\Rightarrow \frac{1}{6}x(x^2 + 6400)^{-\frac{1}{2}} = \frac{1}{10} \\
 &\Rightarrow (x^2 + 6400)^{-\frac{1}{2}} = \frac{3}{5x} \\
 &\Rightarrow (x^2 + 6400)^{\frac{1}{2}} = \frac{5}{3}x \\
 &\Rightarrow x^2 + 6400 = \frac{25}{9}x^2 \\
 &\Rightarrow \frac{16}{9}x^2 = 6400 \\
 &\Rightarrow x^2 = 3600 \\
 &\Rightarrow \underline{x = 60} \\
 &\Rightarrow \underline{T = 30\frac{2}{3}}.
 \end{aligned}$$

11. (a) Solve

$$(2^{x-2})^{\frac{1}{2}} = 100,$$

(3)

giving your answer to 1 decimal place.

Solution

$$\begin{aligned}
 (2^{x-2})^{\frac{1}{2}} = 100 &\Rightarrow 2^{x-2} = 100^2 \\
 &\Rightarrow 2^{x-2} = 100^2 \\
 &\Rightarrow x - 2 = \log_2 100^2 \\
 &\Rightarrow x = 2 \log_2 100 + 2 \\
 &\Rightarrow x = 15.28771238 \text{ (FCD)} \\
 &\Rightarrow \underline{x = 15.3 \text{ (1 dp)}}.
 \end{aligned}$$

(b) Solve

$$\log_y 2 = 3 - \log_y 256.$$

(3)

Solution

$$\begin{aligned}
 \log_y 2 &= 3 - \log_y 256 \Rightarrow \log_y 2 + \log_y 256 = 3 \\
 &\Rightarrow \log_y 2 + \log_y 256 = 3 \\
 &\Rightarrow \log_y 512 = 3 \\
 &\Rightarrow 512 = y^3 \\
 &\Rightarrow y = \sqrt[3]{512} \\
 &\Rightarrow \underline{\underline{y = 8.}}
 \end{aligned}$$

(c) Solve

$$\frac{6^{5z-2}}{36^z} = \frac{216^{z-1}}{36^{3-z}}.$$

(4)

Solution

$$\begin{aligned}
 \frac{6^{5z-2}}{36^z} &= \frac{216^{z-1}}{36^{3-z}} \Rightarrow \frac{6^{5z-2}}{(6^2)^z} = \frac{(6^3)^{z-1}}{(6^2)^{3-z}} \\
 &\Rightarrow \frac{6^{5z-2}}{6^{2z}} = \frac{6^{3(z-1)}}{6^{2(3-z)}} \\
 &\Rightarrow 6^{(5z-2)-2z} = 6^{(3z-3)-(6-2z)} \\
 &\Rightarrow 6^{3z-2} = 6^{5z-9}
 \end{aligned}$$

take $\log_6$:

$$\begin{aligned}
 &\Rightarrow 3z - 2 = 5z - 9 \\
 &\Rightarrow 2z = 7 \\
 &\Rightarrow \underline{\underline{z = 3\frac{1}{2}}}.
 \end{aligned}$$

EITHER

12. (a) Express

$$4x^2 + 32x + 55$$

(3)

in the form

$$(ax + b)^2 + c,$$

where a , b , and c are constants and a is positive.

Solution

Well,

$$\begin{aligned}
 4x^2 + 32x + 55 &= 4(x^2 + 8x) + 55 \\
 &= 4[(x^2 + 8x + 16) - 16] + 55 \\
 &= 4(x^2 + 8x + 16) - 64 + 55 \\
 &= 4(x + 4)^2 - 9 \\
 &= \underline{\underline{(2x + 8)^2 - 9;}}
 \end{aligned}$$

hence, $a = 2$, $b = 8$, and $c = -9$.

The functions f and g are defined by

$$\begin{aligned}
 f : x &\mapsto 4x^2 + 32x + 55 \text{ for } x > -4, \\
 g : x &\mapsto \frac{1}{x} \text{ for } x > 0.
 \end{aligned}$$

(b) Find $f^{-1}(x)$.

(3)

Solution

Well,

$$\begin{aligned}
 y = 4x^2 + 32x + 55 &\Rightarrow y = 4(x + 4)^2 - 9 \\
 &\Rightarrow y + 9 = 4(x + 4)^2 \\
 &\Rightarrow \frac{y + 9}{4} = (x + 4)^2 \\
 &\Rightarrow \sqrt{\frac{y + 9}{4}} = x + 4 \\
 &\Rightarrow \sqrt{\frac{y + 9}{4}} - 4 = x,
 \end{aligned}$$

and so

$$\underline{\underline{y = \sqrt{\frac{x + 9}{4}} - 4.}}$$

(c) Solve the equation

$$f g(x) = 135.$$

(4)

Solution

Well,

$$\begin{aligned} f g(x) &= f(g(x)) \\ &= f\left(\frac{1}{x}\right) \\ &= \frac{4}{x^2} + \frac{32}{x} + 55 \end{aligned}$$

and so

$$f g(x) = 135 \Rightarrow \frac{4}{x^2} + \frac{32}{x} + 55 = 135$$

multiply by x^2 :

$$\begin{aligned} \Rightarrow 4 + 32x + 55x^2 &= 135x^2 \\ \Rightarrow 80x^2 - 32x - 4 &= 0 \\ \Rightarrow 4(20x^2 - 8x - 1) &= 0 \end{aligned}$$

$$\left. \begin{array}{l} \text{add to:} \quad \quad \quad -8 \\ \text{multiply to: } (+20) \times (-1) = -20 \end{array} \right\} -10, +2$$

e.g.,

$$\begin{aligned} \Rightarrow 4[20x^2 - 10x + 2x - 1] &= 0 \\ \Rightarrow 4[10x(2x - 1) + 1(2x - 1)] &= 0 \\ \Rightarrow 4(10x + 1)(2x - 1) &= 0 \\ \Rightarrow 10x + 1 = 0 \text{ or } 2x - 1 &= 0 \\ \Rightarrow x = -\frac{1}{10} \text{ or } x = \frac{1}{2}. \end{aligned}$$

But $x \neq -\frac{1}{10}$ (why?) so $x = \frac{1}{2}$.

OR

13. The functions h and k are defined by

$$\begin{aligned} h : x &\mapsto \sqrt{2x - 7} \text{ for } x \geq c, \\ k : x &\mapsto \frac{3x - 4}{x - 2} \text{ for } x > 2. \end{aligned}$$

(a) State the least possible value of c .

(1)

Solution

Well,

$$\begin{aligned}\sqrt{2x-7} = 0 &\Rightarrow 2x-7 = 0 \\ &\Rightarrow 2x = 7 \\ &\Rightarrow \underline{\underline{x = 3\frac{1}{2}}}.\end{aligned}$$

(b) Find $h^{-1}(x)$.

(2)

Solution

Now,

$$\begin{aligned}y = \sqrt{2x-7} &\Rightarrow y^2 = 2x-7 \\ &\Rightarrow y^2 + 7 = 2x \\ &\Rightarrow \frac{y^2 + 7}{2} = x\end{aligned}$$

and so

$$\underline{\underline{h^{-1}(x) = \frac{x^2 + 7}{2}}}.$$

(c) Solve the equation

(3)

$$k(x) = x.$$

Solution

Well,

$$\begin{aligned}k(x) = x &\Rightarrow \frac{3x-4}{x-2} = x \\ &\Rightarrow 3x-4 = x(x-2) \\ &\Rightarrow 3x-4 = x^2-2x \\ &\Rightarrow x^2-5x+4 = 0\end{aligned}$$

$$\left. \begin{array}{l} \text{add to:} \\ \text{multiply to:} \end{array} \right\} \begin{array}{l} -5 \\ +4 \end{array} \quad -4, -1$$

$$\begin{aligned} \Rightarrow (x-4)(x-1) &= 0 \\ \Rightarrow x-4 &= 0 \text{ or } x-1 = 0 \\ \Rightarrow x &= 4 \text{ or } x = 1. \end{aligned}$$

But $x \neq 1$ (why?) so $x = 4$.

- (d) Find an expression for the function k^2 , in the form (4)

$$k^2 : x \mapsto a + \frac{b}{x},$$

where a and b are constants.

Solution

Now,

$$\begin{aligned} k^2(x) &= k(k(x)) \\ &= k\left(\frac{3x-4}{x-2}\right) \\ &= \frac{3\left(\frac{3x-4}{x-2}\right) - 4}{\left(\frac{3x-4}{x-2}\right) - 2} \end{aligned}$$

multiply by $(x-2)$ on the top and bottom:

$$\begin{aligned} &= \frac{3(3x-4) - 4(x-2)}{(3x-4) - 2(x-2)} \\ &= \frac{9x - 12 - 4x + 8}{3x - 4 - 2x + 4} \\ &= \frac{5x - 4}{x} \\ &= \underline{\underline{5 - \frac{4}{x}}}; \end{aligned}$$

hence, $a = 5$ and $b = -4$.