Dr Oliver Mathematics
Probability and Quadratic Equations

Give your answers as exact fractions. You may use a calculator.

1. A bag contains n red discs and 7 blue discs.
   Two discs are chosen at random without replacement.
   The probability of obtaining two red discs is $\frac{4}{15}$.

   (a) How many red discs are there in the bag?

   Solution
   In total, there are $(n + 7)$ discs in the bag and hence, on the second selection, there will be $(n + 6)$ discs to choose from.

$$\begin{aligned}
\text{P(RR)} = \tfrac{4}{15} &\Rightarrow \frac{n}{n+7} \times \frac{n-1}{n+6} = \frac{4}{15} \\
&\Rightarrow 15n(n-1) = 4(n+7)(n+6) \\
&\Rightarrow 15n^2 - 15n = 4n^2 + 52n + 168 \\
&\Rightarrow 11n^2 - 67n - 168 = 0 \\
&\Rightarrow (11n + 21)(n - 8) = 0 \\
&\Rightarrow n = -\tfrac{21}{11} \text{ or } 8,
\end{aligned}$$

   and since n must be a positive integer there are $\underline{\underline{\text{8 red discs}}}$ in the bag.

   The two red discs are now replaced and the experiment is repeated.

   (b) What is the probability of getting one disc of each colour?

   Solution
   There are two possible combinations (RB and BR) and, since there is the same probability of each happening,

$$\text{P(one of each colour)} = 2 \times \frac{8}{15} \times \frac{7}{14} = \underline{\underline{\frac{8}{15}}}.$$

2. A bag contains n blue discs and 16 green discs.
   Two discs are chosen at random without replacement.
   The probability of getting one disc of each colour is $\frac{8}{21}$.

   (a) Show that $n^2 - 53n + 240 = 0$.

Solution

In total, there are $(n+16)$ discs in the bag and hence, on the second selection, there will be $(n+15)$ discs to choose from. There are two possible combinations (BG and GB) and, since there is the same probability of each happening,

$$\begin{aligned}\text{P(one of each colour)} = \tfrac{8}{21} &\Rightarrow 2 \times \frac{n}{n+16} \times \frac{16}{n+15} = \frac{8}{21} \\ &\Rightarrow 672n = 8(n+15)(n+16) \\ &\Rightarrow 84n = (n+15)(n+16) \\ &\Rightarrow 84n = n^2 + 31n + 240 \\ &\Rightarrow \underline{\underline{n^2 - 53n + 240 = 0}},\end{aligned}$$

as required.

(b) Hence, or otherwise, calculate how many blue discs the bag might contain.

Solution

$$n^2 - 53n + 240 = 0 \Rightarrow (n-48)(n-5) = 0 \Rightarrow n = 5 \text{ or } 48.$$

Hence there are either $\underline{\underline{\text{5 or 48 blue discs}}}$ in the bag.

3. A bag contains n white and $(2n+1)$ black discs.
Two discs are chosen at random without replacement.
The probability of obtaining two black discs is $\frac{11}{24}$.

(a) How many discs of each colour does the bag contain?

Solution

There are $(3n+1)$ discs in total so there will be $3n$ discs for the second selection.

$$\begin{aligned}\text{P(BB)} = \tfrac{11}{24} &\Rightarrow \frac{2n+1}{3n+1} \times \frac{2n}{3n} = \frac{11}{24} \\ &\Rightarrow \frac{2n+1}{3n+1} = \frac{11}{16} \\ &\Rightarrow 16(2n+1) = 11(3n+1) \\ &\Rightarrow 32n + 16 = 33n + 11 \\ &\Rightarrow n = 5\end{aligned}$$

and so there are $\underline{\underline{\text{5 white discs}}}$ and $\underline{\underline{\text{11 black discs}}}$ in the bag.

The discs are now replaced and the experiment is repeated.

(b) What is the probability of getting two white discs?

Solution

$$\mathrm{P(WW)} = \frac{5}{16} \times \frac{4}{15} = \underline{\underline{\frac{1}{12}}}.$$

4. A bag contains three more green discs than red discs.
Two discs are chosen at random without replacement.
The probability of obtaining one disc of each colour is $\frac{28}{55}$.

(a) If n represents the number of red discs, show that $n^2 - 25n + 84 = 0$.

Solution

There are $(n + 3)$ green discs and so $(2n + 3)$ discs in total. Hence

$$\begin{aligned}
\mathrm{P(one\ of\ each\ colour)} = \tfrac{28}{55} &\Rightarrow 2 \times \frac{n}{2n+3} \times \frac{n+3}{2(n+1)} = \frac{28}{55} \\
&\Rightarrow 55n(n+3) = 28(2n+3)(n+1) \\
&\Rightarrow 55n^2 + 165n = 56n^2 + 140n + 84 \\
&\Rightarrow \underline{\underline{n^2 - 25n + 84 = 0}},
\end{aligned}$$

as required.

(b) Hence, or otherwise, calculate all possible combination of green and red discs that the bag could have contained.

Solution

$$n^2 - 25n + 84 = 0 \Rightarrow (n-4)(n-21) = 0 \Rightarrow n = 4 \text{ or } n = 21$$

and hence there are either $\underline{\underline{\text{4 red and 7 green discs}}}$ or $\underline{\underline{\text{21 red and 24 green discs}}}$.

Assume that the bag contained the greater of the two possible totals of discs.
The two discs are replaced and the experiment is repeated.

(c) What is the probability of choosing two red discs?

Solution

$$\mathrm{P(RR)} = \frac{21}{45} \times \frac{20}{44} = \underline{\underline{\frac{7}{33}}}.$$

5. There are 10 boys in a mixed gender class.
Two pupils are chosen at random to come out and work at the board.
The probability that one boy and one girl are chosen is $\frac{40}{77}$.

(a) If x represents the total number of students in the group, show that

$$2x^2 - 79x + 770 = 0.$$

Solution
There are $(x - 10)$ girls in the class. So

$$\begin{aligned} \text{P(one of each gender)} = \tfrac{40}{77} &\Rightarrow 2 \times \frac{10}{x} \times \frac{x-10}{x-1} = \frac{40}{77} \\ &\Rightarrow 1540(x-10) = 40x(x-1) \\ &\Rightarrow 77(x-10) = 2x(x-1) \\ &\Rightarrow 77x - 770 = 2x^2 - 2x \\ &\Rightarrow \underline{\underline{2x^2 - 79x + 770 = 0}}, \end{aligned}$$

as required.

(b) Hence, or otherwise, find the total number of students in the group.

Solution

$$2x^2 - 79x + 770 = 0 \Rightarrow (2x-35)(x-22) = 0 \Rightarrow x = 22 \text{ or } 17\tfrac{1}{2}$$

and hence there are $\underline{\underline{\text{22 students}}}$ in the group.

The two students return to their desks and another two students are chosen from the class at random.

(c) Calculate the probability that the two chosen are both girls.

Solution

$$\text{P(GG)} = \frac{12}{22} \times \frac{11}{21} = \underline{\underline{\frac{2}{7}}}.$$

6. A word contains five vowels and c consonants.
Two letters are chosen at random.
The probability of choosing two consonants is $\frac{1}{6}$.

(a) How many letters did the word contain?

Solution

$$\begin{aligned}
\mathrm{P(CC)} = \tfrac{1}{6} &\Rightarrow \frac{c}{c+5} \times \frac{c-1}{c+4} = \frac{1}{6} \\
&\Rightarrow 6c(c-1) = (c+4)(c+5) \\
&\Rightarrow 6c^2 - 6c = c^2 + 9c + 20 \\
&\Rightarrow 5c^2 - 15c - 20 = 0 \\
&\Rightarrow c^2 - 3c - 4 = 0 \\
&\Rightarrow (c-4)(c+1) = 0 \\
&\Rightarrow c = 4 \text{ (since it must be a positive integer)}
\end{aligned}$$

Hence there are 9 letters in the word.

Another selection of two letters is made at random.

(b) What is the probability that one vowel and one consonant are selected?

Solution

There are two ways (CV and VC) that this can be done and each has the same probability so

$$\mathrm{P(one\ of\ each)} = 2 \times \frac{4}{9} \times \frac{5}{8} = \frac{5}{9}.$$

7. A bag contains twice as many blue discs as red discs.
Two discs are chosen at random without replacement.
The probability of obtaining one disc of each colour is $\frac{16}{33}$.

(a) How many of the discs are blue?

Solution

Suppose that there are $2x$ blue discs. Then there are x red discs and hence $3x$

discs in total.

$$\begin{aligned}
\text{P(one of each colour)} = \tfrac{16}{33} &\Rightarrow 2 \times \frac{2x}{3x} \times \frac{x}{3x-1} = \frac{16}{33} \\
&\Rightarrow \frac{4}{3} \times \frac{x}{3x-1} = \frac{16}{33} \\
&\Rightarrow 132x = 48(3x-1) \\
&\Rightarrow 132x = 144x - 48 \\
&\Rightarrow 12x = 48 \\
&\Rightarrow \underline{\underline{x = 4}},
\end{aligned}$$

and hence there are $\underline{\underline{\text{8 blue discs}}}$ in the bag.

The two discs are now replaced and the experiment is repeated.

(b) Calculate the probability of getting two red discs.

Solution

$$\text{P(RR)} = \frac{4}{12} \times \frac{3}{11} = \underline{\underline{\frac{1}{11}}}.$$

8. A bag contains four more black discs than green discs.
Two discs are chosen at random without replacement.
The probability of obtaining two green discs is $\frac{12}{77}$. Given that g is the number of green discs in the bag,

(a) show that $29g^2 - 245g - 144 = 0$.

Solution

There are g green discs, $(g+4)$ black discs, and hence $(2g+4)$ discs in total.

$$\begin{aligned}
\text{P(GG)} = \tfrac{12}{77} &\Rightarrow \frac{g}{2g+4} \times \frac{g-1}{2g+3} = \frac{12}{77} \\
&\Rightarrow 77g(g-1) = 12(2g+3)(2g+4) \\
&\Rightarrow 77g^2 - 77g = 48g^2 + 168g + 144 \\
&\Rightarrow \underline{\underline{29g^2 - 245g - 144 = 0}},
\end{aligned}$$

as required.

(b) Hence, or otherwise, calculate the number of black discs.

Solution

$$29g^2 - 245g - 144 = 0 \Rightarrow (29g - 16)(g - 9) = 0 \Rightarrow g = \tfrac{16}{29} \text{ or } 9.$$

Since g is a positive integer, $g = 9$ and hence there are $\underline{\underline{\text{13 black discs}}}$ in the bag

The two discs are now replaced and the experiment is repeated.

(c) Calculate the probability of getting one disc of each colour.

Solution

There are two ways (BG and GB) that this can be done and each has the same probability so

$$\text{P(one of each)} = 2 \times \frac{9}{22} \times \frac{13}{21} = \underline{\underline{\frac{39}{77}}}.$$