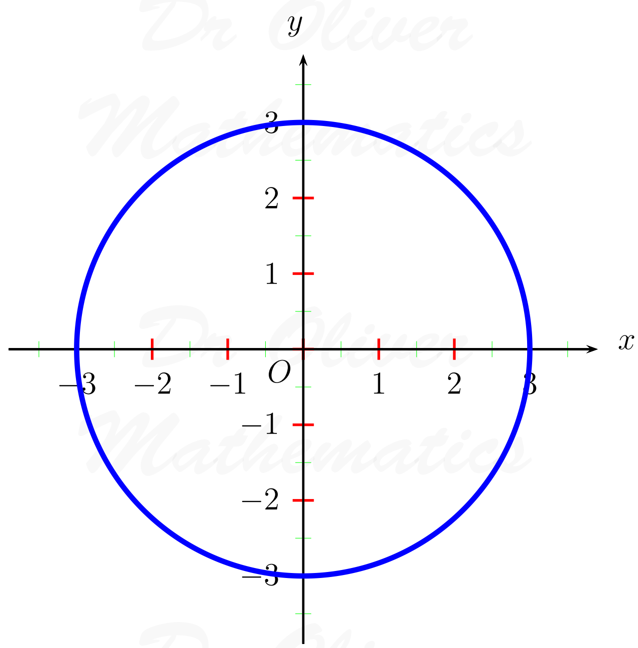
Dr Oliver Mathematics
Polar Curves

There are seven standard cases that you will be expected to recognise in the Further Pure Mathematics 2 examination. In these notes we have adopted the examination board's convention of only plotting points where $r \geqslant 0$; the reader should note, however, that this convention is not universally recognised and many graphical packages and websites will give polar graphs where points are plotted for negative values of r. In addition, all of the curves have been plotted using degrees: you should practise drawing these types of curves in both degrees and radians.

1 Circles

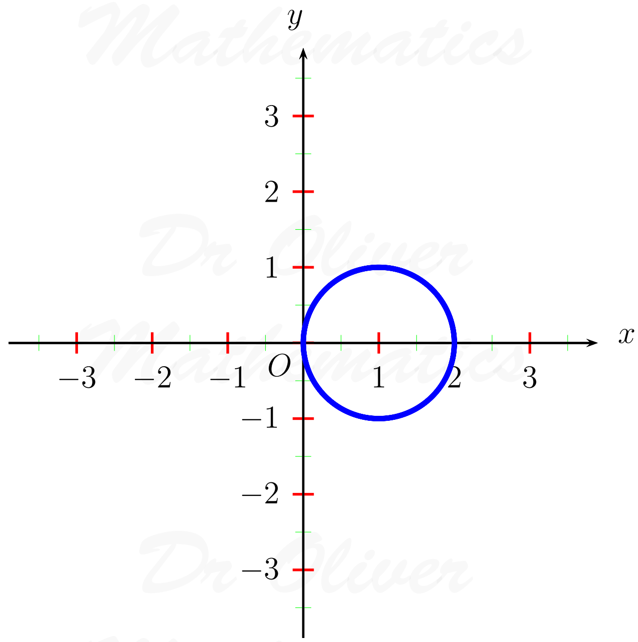
1.1 $r = 3$



Polar curves of the form $r = a$, for some constant a, are simply circles, centred at $(0, 0)$, with radius a:

$$r = a \Rightarrow r^2 = a^2 \Rightarrow x^2 + y^2 = a^2.$$

1.2 $r = 2\cos\theta$

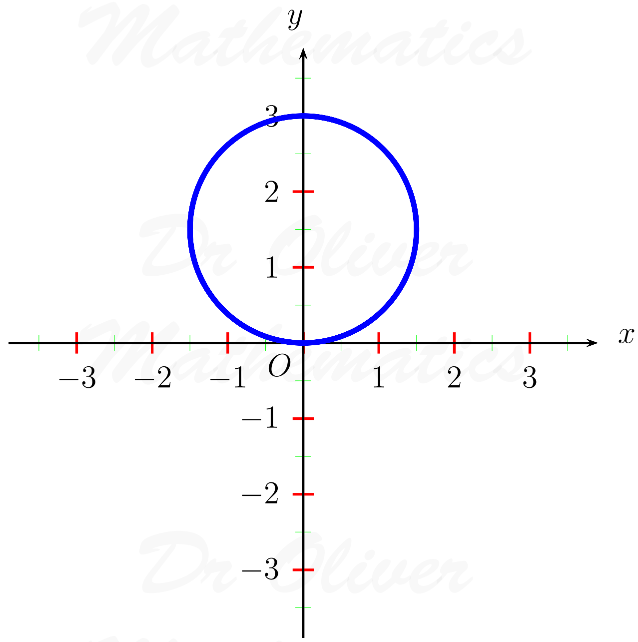


Polar curves of the form $r = a\cos\theta$, for some constant $a > 0$, are circles, centre $(\frac{1}{2}a, 0)$, radius $\frac{1}{2}a$.

$$\begin{aligned} r = a\cos\theta &\Rightarrow r = a\left(\frac{x}{r}\right) \\ &\Rightarrow r^2 = ax \\ &\Rightarrow x^2 + y^2 = ax \\ &\Rightarrow x^2 - ax + y^2 = 0 \\ &\Rightarrow x^2 - ax + \tfrac{1}{4}a^2 + y^2 = \tfrac{1}{4}a^2 \\ &\Rightarrow \left(x - \tfrac{1}{2}a\right)^2 + y^2 = \left(\tfrac{1}{2}a\right)^2, \end{aligned}$$

as required.

1.3 $r = 3\sin\theta$

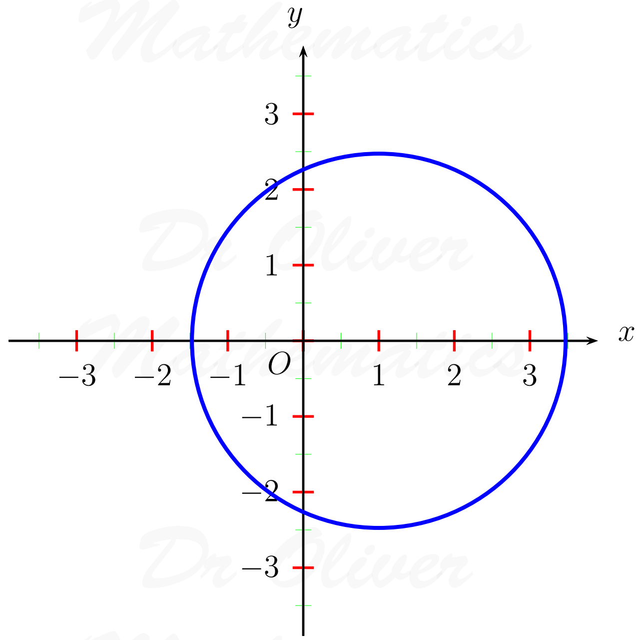


Polar curves of the form $r = a\sin\theta$, for some constant $a > 0$, are circles, centre $(0, \frac{1}{2}a)$, radius $\frac{1}{2}a$.

$$\begin{aligned} r = a\sin\theta &\Rightarrow r = a\left(\frac{y}{r}\right) \\ &\Rightarrow r^2 = ay \\ &\Rightarrow x^2 + y^2 = ay \\ &\Rightarrow x^2 + y^2 - ay = 0 \\ &\Rightarrow x^2 + y^2 - ay + \tfrac{1}{4}a^2 = \tfrac{1}{4}a^2 \\ &\Rightarrow x^2 + \left(y - \tfrac{1}{2}a\right)^2 = \left(\tfrac{1}{2}a\right)^2, \end{aligned}$$

as required.

1.4 $a^2 = r^2 + b^2 - 2br\cos\theta$



Polar curves of the form $a^2 = r^2 + b^2 - 2br\cos\theta$ are circles, centre $(b, 0)$, radius a.

$$\begin{aligned} a^2 = r^2 + b^2 - 2br\cos\theta &\Rightarrow a^2 = x^2 + y^2 + b^2 - 2bx \\ &\Rightarrow a^2 = (x-b)^2 + y^2, \end{aligned}$$

as required. In the above figure, $a = 2.5$ and $b = 1$.

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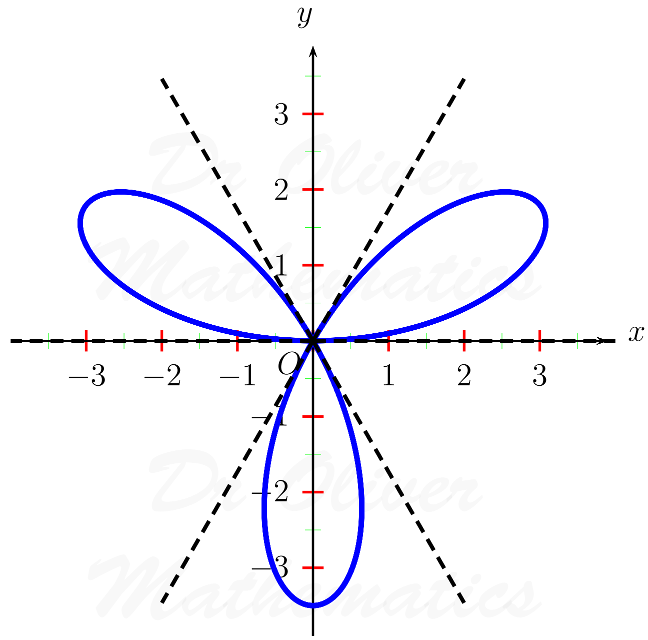
2 Rose or Petal Curves

2.1 $r = 3.5 \sin 3\theta^\circ$



Since the sine function is sometimes negative there are values of θ for which no points are plotted. Now,

$$0 \leqslant \theta \leqslant 360 \Rightarrow 0 \leqslant 3\theta \leqslant 1080.$$

Then

$$\sin 3\theta^\circ < 0 \Rightarrow 180 < 3\theta < 360,\ 540 < 3\theta < 720 \text{ or } 900 < 3\theta < 1080$$

and so

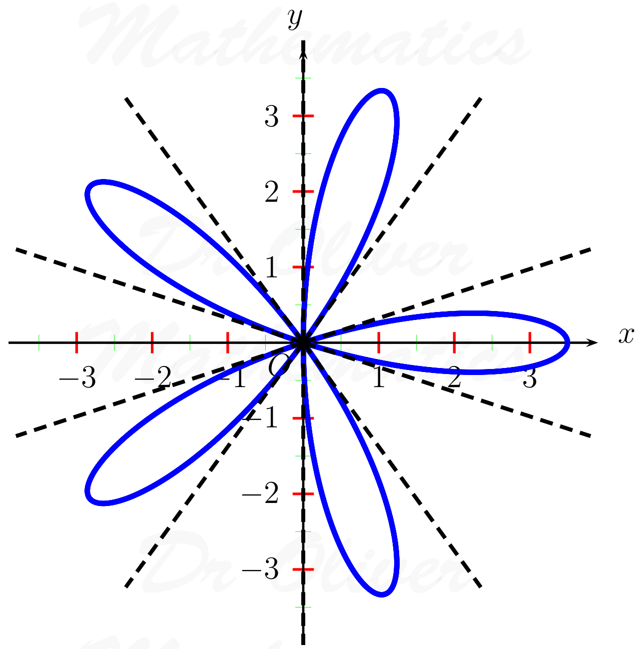
$$\sin 3\theta^\circ < 0 \Rightarrow 60 < \theta < 120,\ 180 < \theta < 240 \text{ or } 300 < \theta < 360.$$

This explains why there are three loops or 'petals' (because of the $3\theta^\circ$ in the formula) and that each loop or 'petal' has a maximum distance of 3.5 from the origin.

The first petal is plotted between 0° and 60° inclusive and reaches its maximum distance from the origin midway between these values, i.e., at 30°. The other two petals are then symmetrically placed in the polar plane, with the three maxima begin spaced out by 120°. So once you know how to correctly draw the first petal, you can then place the others by symmetry.

Sometimes an examination question will expect you (although this may not be explicit in the question) to mark in the radial lines where the curve comes into the origin, i.e., at $\theta = 0, 60, 120, 180, 240,$ and 360 in this case; these six lines have been marked in on the diagram and you are advised to do the same.

2.2 $r = 3.5\cos 5\theta^\circ$



Note that, in this case, the first petal is symmetrically centred on the initial line: this makes it easier to place the first petal and the other four can then be placed symmetrically.

Just as with the previous polar curve, there are angles for which r would be negative and so no point is plotted. Now,

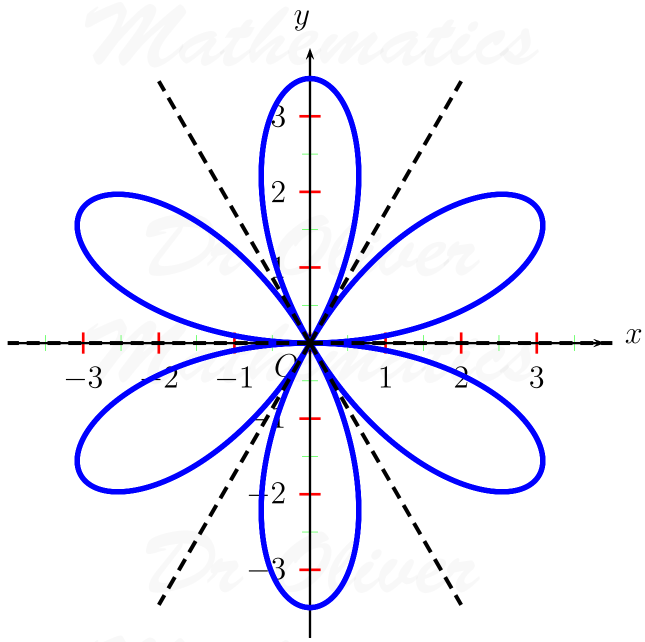
$$0 \leqslant \theta \leqslant 360 \Rightarrow 0 \leqslant 5\theta \leqslant 1800$$

and hence

$$\begin{aligned} \cos 5\theta < 0 &\Rightarrow 90 < 5\theta < 270,\ 450 < 5\theta < 630,\ 810 < 5\theta < 990, \\ &\quad 1170 < 5\theta < 1350,\ \text{ or } 1530 < 5\theta < 1710 \\ &\Rightarrow 18 < \theta < 54,\ 90 < \theta < 126,\ 162 < \theta < 198, \\ &\quad 234 < \theta < 270,\ \text{ or } 306 < \theta < 342. \end{aligned}$$

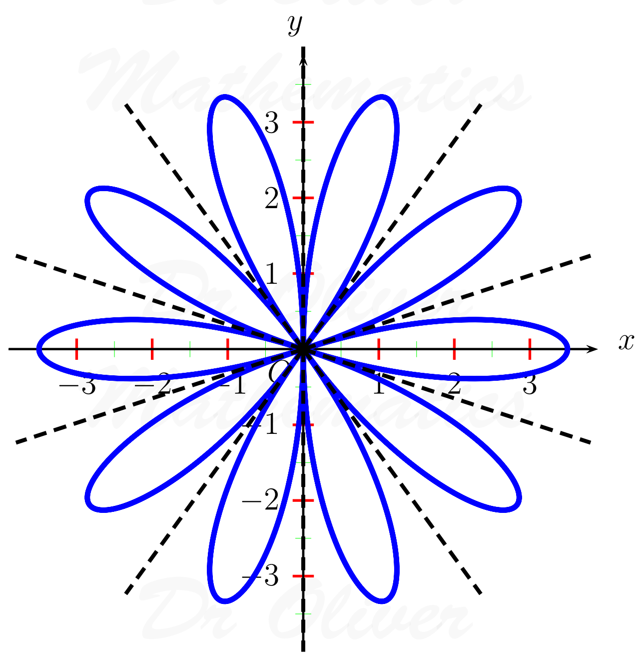
Again, we have marked in the radial lines (there are ten of them in this case) and, again, you are advised to do the same in each of the curves that you draw.

2.3 $r^2 = 12.25 \sin^2 3\theta^\circ$



Curves of the form $r^2 = a^2 \sin^2 b\theta^\circ$ are similar to the petal curves introduced in section 2.1 but, because of the squaring, there are $2b$ loops; the extra b loops are drawn where no values were previously plotted.

2.4 $r^2 = 12.25 \cos^2 5\theta^\circ$



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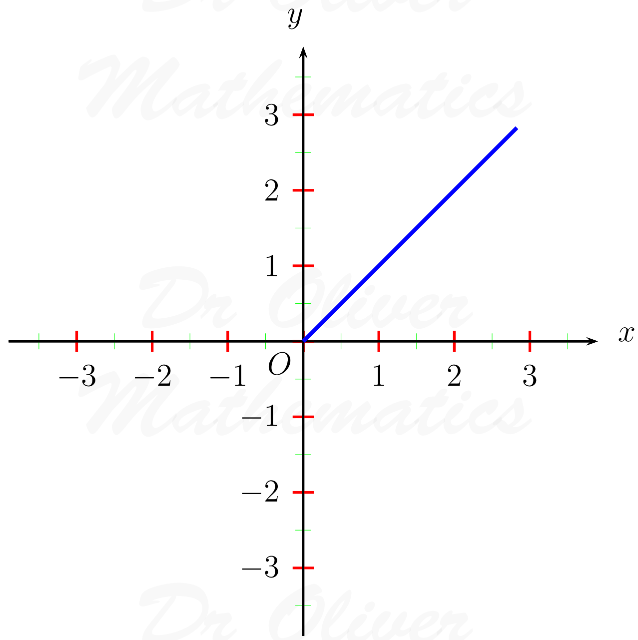
Curves of the form $r^2 = a^2 \cos^2 b\theta^\circ$ are similar to the petal curves introduced in section 2.2; just as in the previous section we have an additional b loops drawn where no loops were drawn previously.

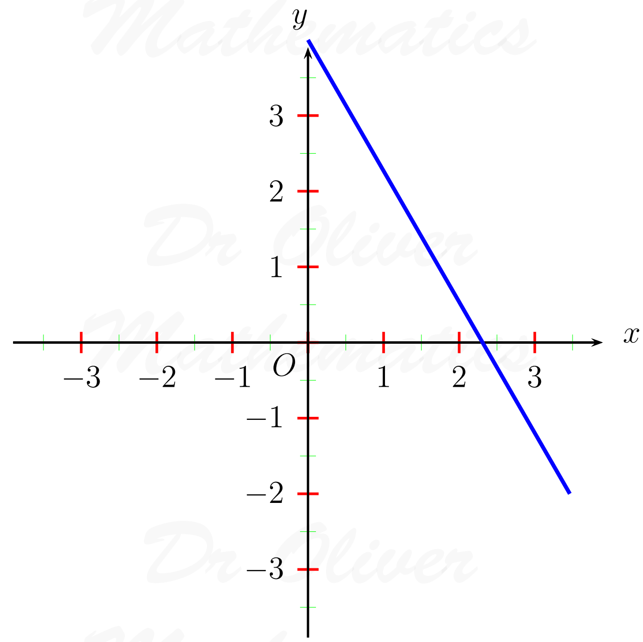
3 Lines and Half-Lines

3.1 $\theta = 45^\circ$



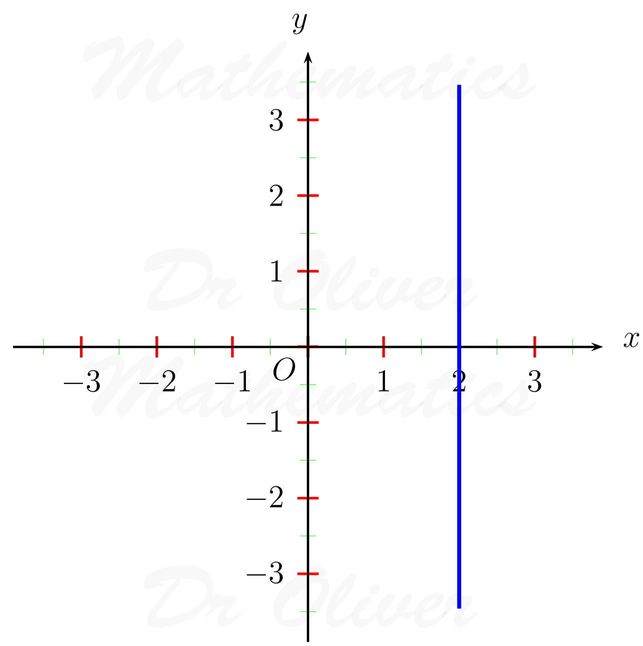
A line of the form $\theta = \alpha$ is a half-line though the origin. It is not, in this case, the whole line $y = x$: the point $(-1, -1)$ has a polar angle of -135°.

3.2 $2 = r\cos(30 - \theta)^\circ$ **or** $r = 2\sec(30 - \theta)^\circ$



A line of the form $p = r\cos(\alpha - \theta)$ or $r = p\sec(\alpha - \theta)$ is the straight line which is perpendicular to a radial line of length p through the origin at an angle α to the initial line. The line does, of course, extend infinitely far in each direction.

3.3 $r\cos\theta^\circ = 2$



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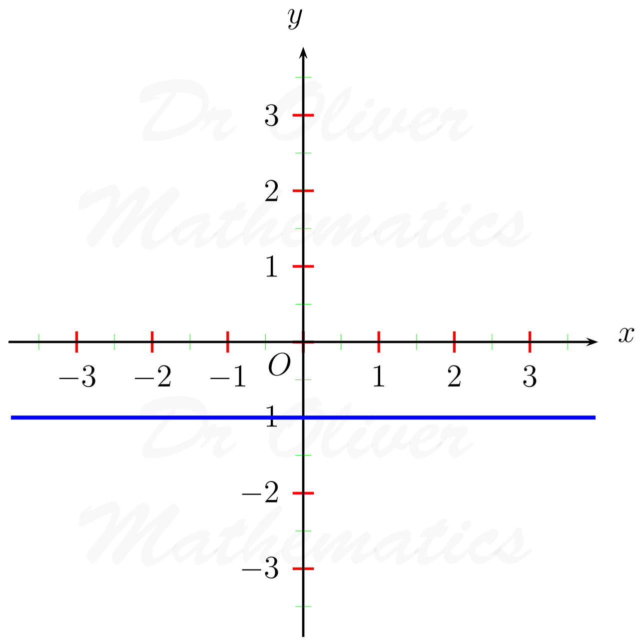
Since $x = r\cos\theta^\circ$ vertical lines have the form $r\cos\theta^\circ = a$ for some constant a; in the above figure $a = 2$.

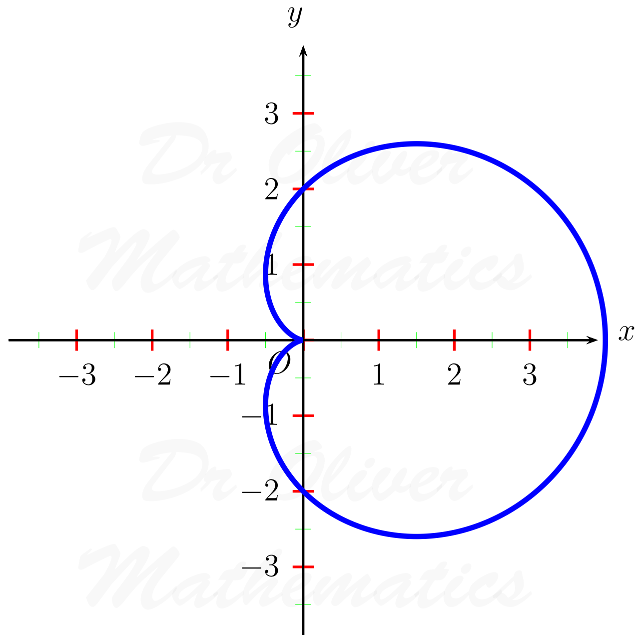
3.4 $r\sin\theta^\circ = -1$



Since $y = r\sin\theta^\circ$ horizontal lines have the form $r\sin\theta^\circ = a$ for some constant a; in the above diagram, $a = -1$.

4 Cardioids

4.1 $r = 2(1 + \cos\theta)$



Note that the initial line is a tangent to the curve as it approaches the pole. (The *Further Pure Mathematics 2* textbook, in common with many other books and internet resources, do not get this feature correct.) This is because $r \geqslant 0$ for all values of θ.

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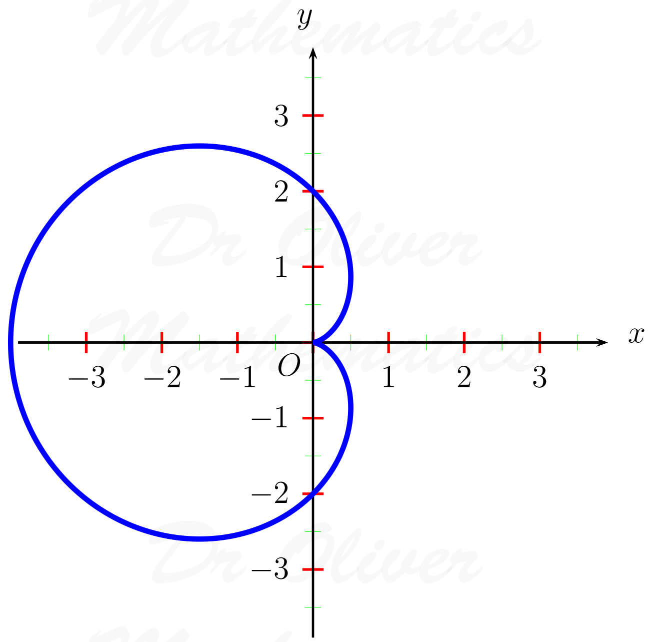
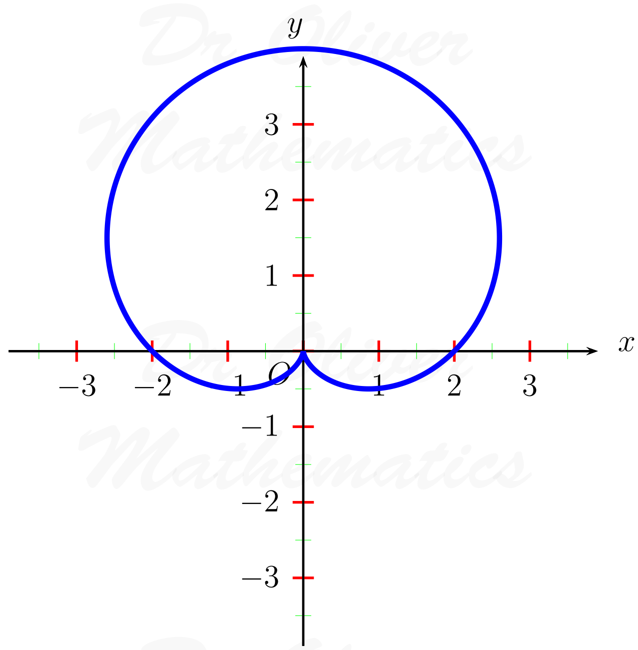
4.2 $r = 2(1 - \cos\theta)$



This is simply a reflection of the previous case in the y-axis.

4.3 $r = 2(1 + \sin\theta)$



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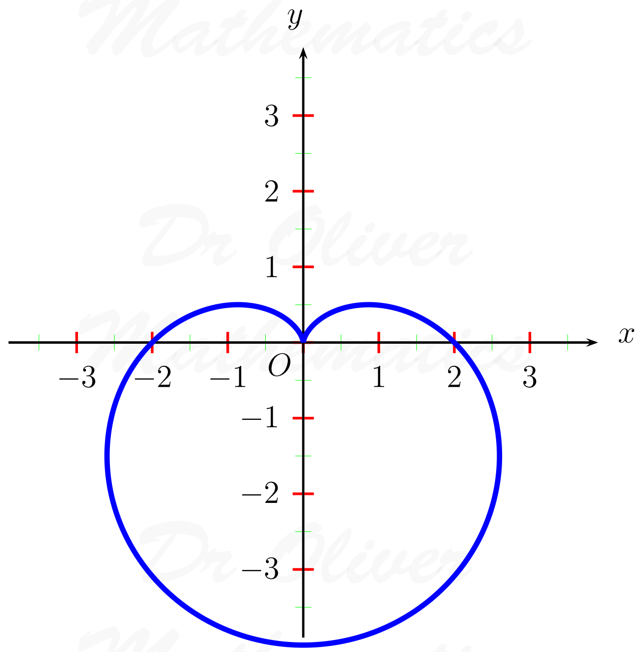
4.4 $r = 2(1 - \sin\theta)$

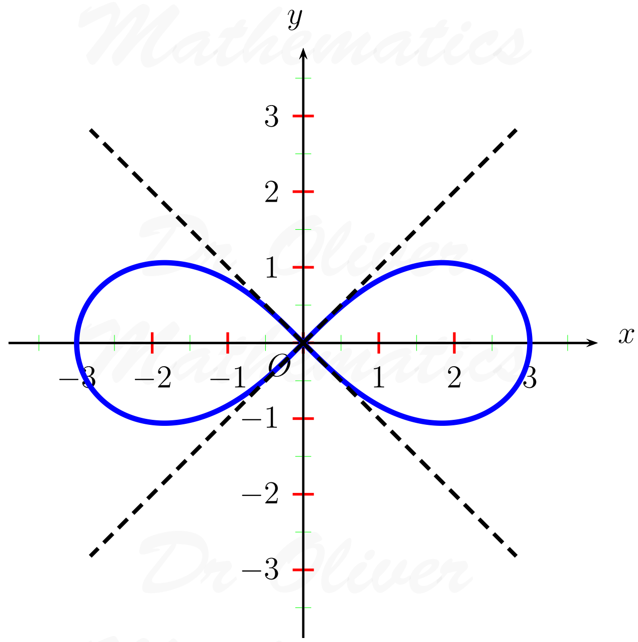


This is simply a reflection of the previous case in the x-axis.

5 Lemniscates

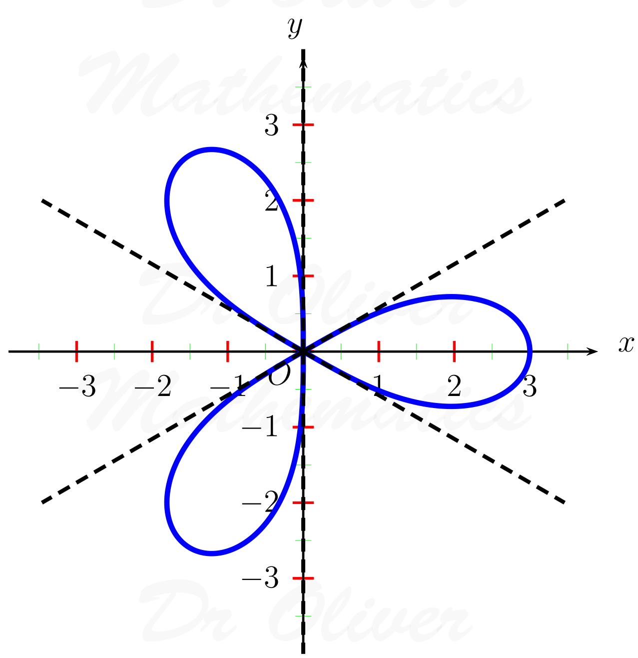
There are clear similarities between this family of curves and the rose or petal curves that were dealt with in section 2.

5.1 $r^2 = 9\cos 2\theta$



Curves of the form $r^2 = a^2 \cos 2\theta$ (sometimes referred to as the lemniscate of (Jakob) Bernoulli) can be thought of as the locus of all points, the product of whose distances from two fixed points a distance $2a$ apart, is a^2.

5.2 $r^2 = 9\cos 3\theta$



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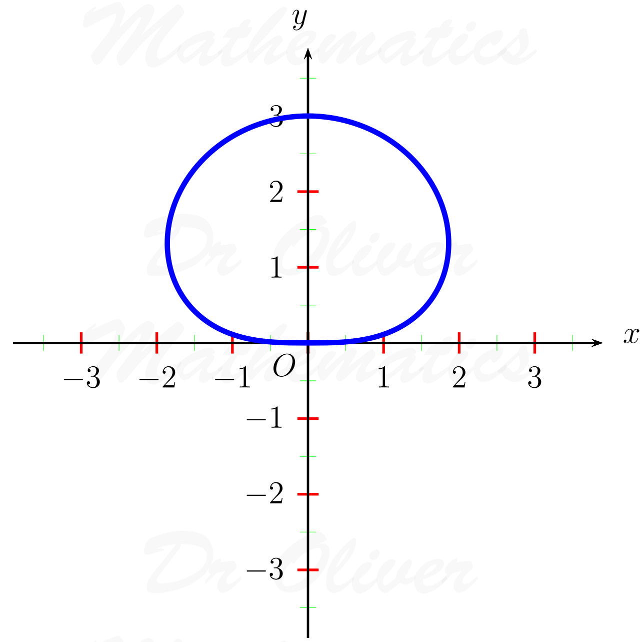
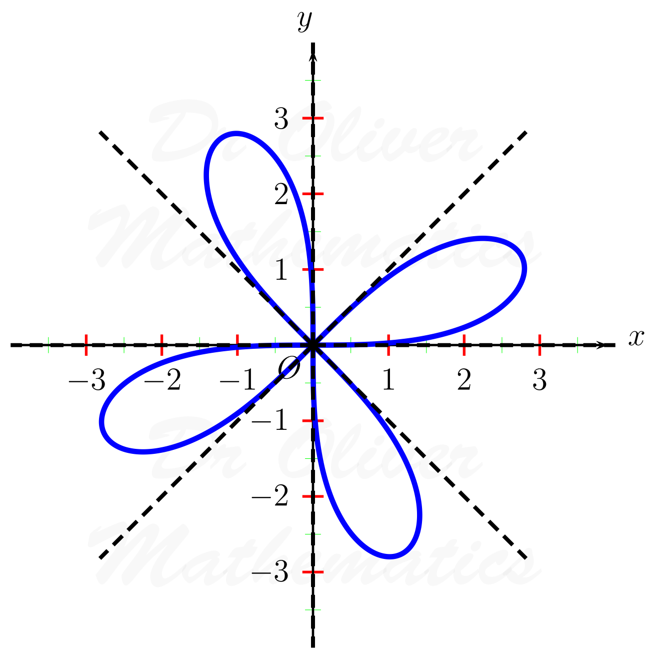
5.3 $r^2 = 9\sin\theta$



5.4 $r^2 = 9\sin 4\theta$



6 Limaçons

There are essentially four cases to consider for curves of the form $r = a + b\cos\theta$.

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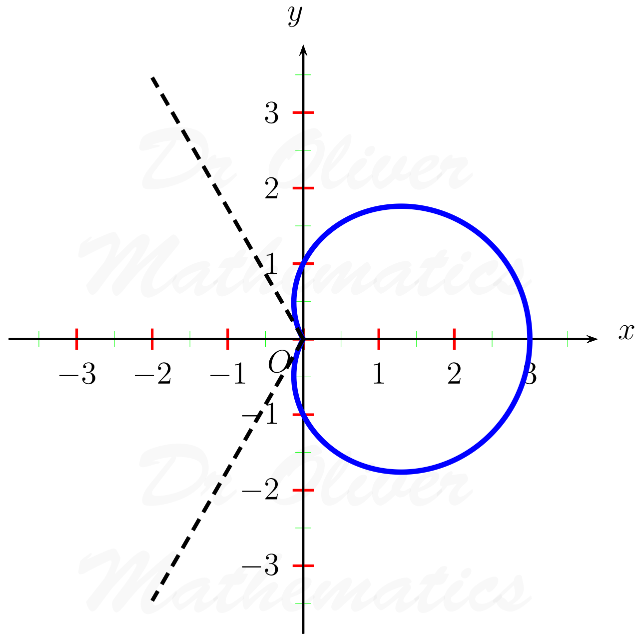
6.1 $a < b$

In this case, r will sometimes be negative and so no points are plotted when this happens. For example, the curve described by $r = 1 + 2\cos\theta$ is shown below.



Note that, in this case, the curve does come into the pole at an angle (unlike the cardioid) and we have drawn in the radial lines at both $120°$ and $240°$ to show this.

6.2 $a = b$

In this case, the curve has the form $r = a(1 + \cos\theta)$ and so we simply have a cardioid.

6.3 $b < a < 2b$

In this case, we have the 'dimple' that is correctly explained in the textbook; the example of $r = 2 + 1.5\cos\theta$ is given below. Note that, for this range of values, r is always positive.

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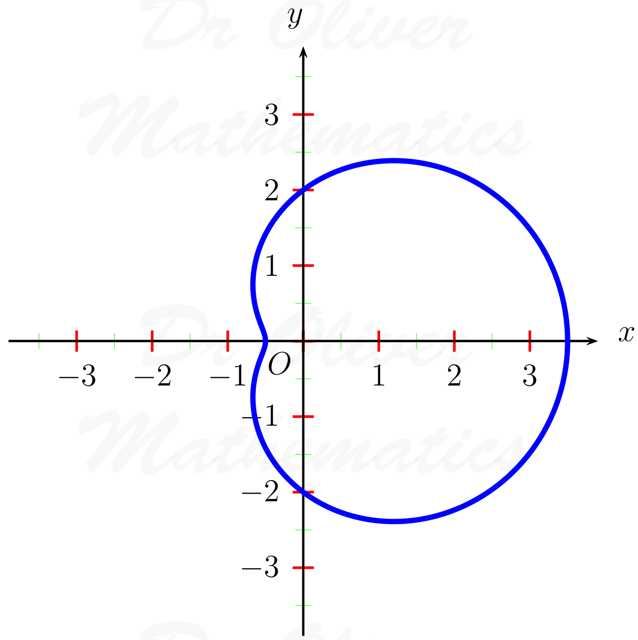
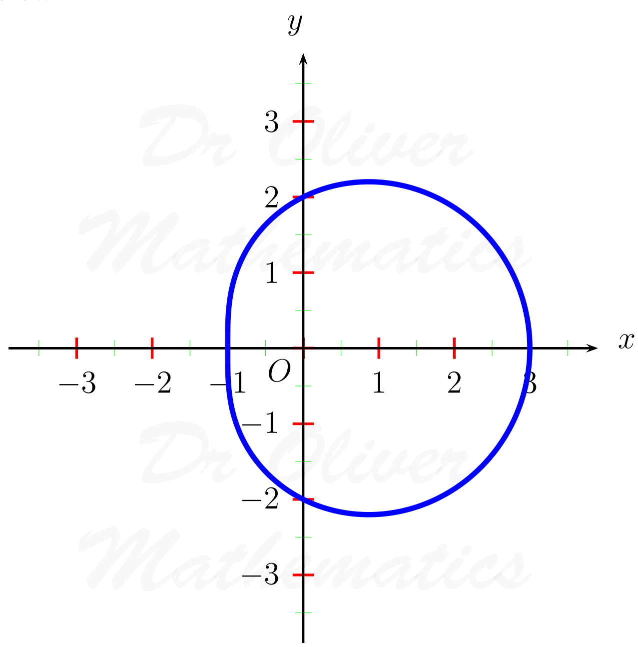


6.4 $a \geqslant 2b$

In this case the curve does not have the 'dimple' of the previous case; the example of $r = 2 + \cos\theta$ is given below.



As the value of a begins to significantly exceed that of $2b$ then the curve loses its apparent flatness and begins to look genuinely more rounded, as can be seen in the example of $r = 2 + 0.5\cos\theta$ below.

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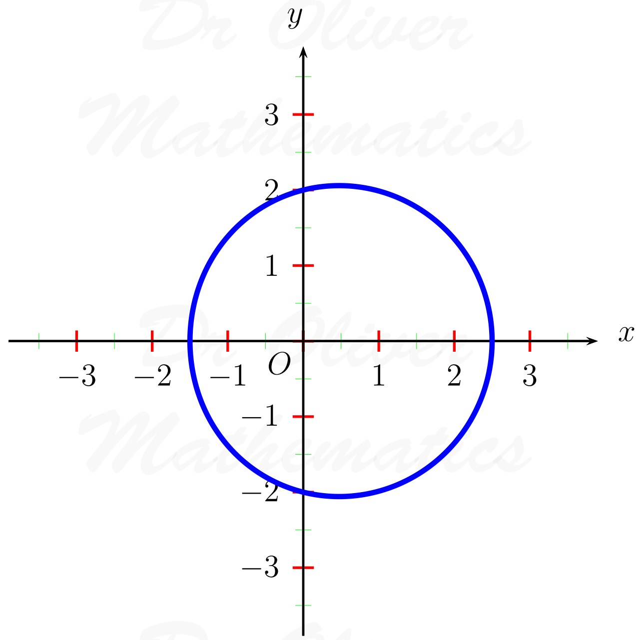
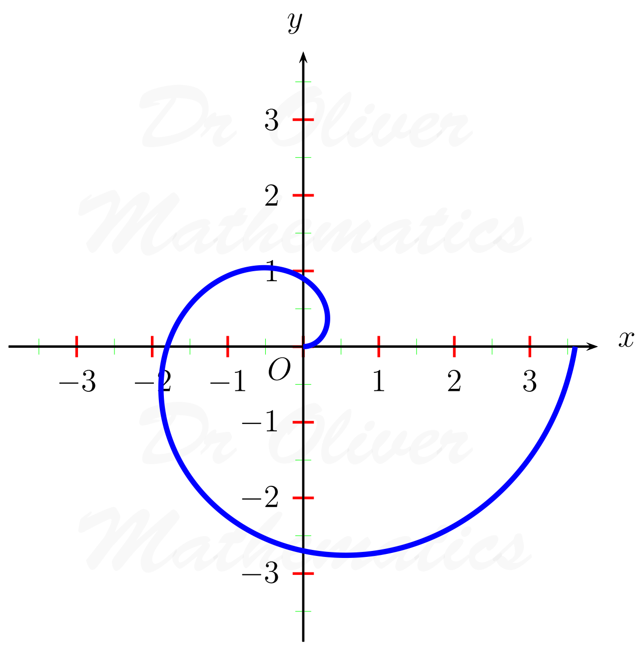


7 Spirals

7.1 The spiral of Archimedes: $r = 0.01\theta$



In general these spirals have the form $r = a + b\theta$. Any ray from the origin intersects successive turnings of the spiral with a constant separation distance (the distance is $2\pi b$ is the angle θ is measured in radians).

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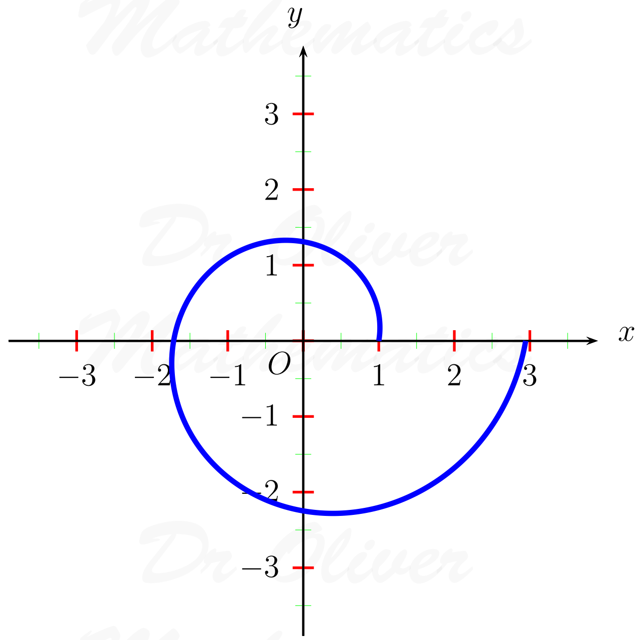
7.2 The equiangular sprial: $r = e^{0.003\theta}$



These spirals get their name because the angle between the tangent to the curve and the radial line at the point (r, θ) is is constant.