Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 November Paper 2 Variant 2: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Find the set of values of x for which (3)

$$x^2 < 6 - 5x.$$

Solution

Well,

$$x^2 < 6 - 5x \Rightarrow x^2 + 5x - 6 < 0$$

$$\left.\begin{array}{ll} \text{add to:} & +5 \\ \text{multiply to:} & -6 \end{array}\right\} -1,\ +6$$

$$\Rightarrow (x-1)(x+6) < 0.$$

We need a 'table of signs':

	$x < -6$	$x = -6$	$-6 < x < 1$	$x = 1$	$x > 1$
$x + 6$	$-$	0	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	0	$+$
$(x+6)(x-1)$	$+$	0	$-$	0	$+$

Hence,

$$x^2 < 6 - 5x \Rightarrow \underline{\underline{-6 < x < 1}}.$$

2. **Do not use a calculator in this question.** (4)

Express

$$\frac{(4\sqrt{5}-2)^2}{\sqrt{5}-1}$$

in the form

$$p\sqrt{5}+q,$$

where p and q are integers.

Solution

Well,

$\times$	$4\sqrt{5}$	-2
$4\sqrt{5}$	80	$-8\sqrt{5}$
-2	$-8\sqrt{5}$	$+4$

and

$$\begin{aligned}\frac{(4\sqrt{5}-2)^2}{\sqrt{5}-1} &= \frac{84-16\sqrt{5}}{\sqrt{5}-1}\\ &= \frac{84-16\sqrt{5}}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}+1}\end{aligned}$$

$\times$	84	$-16\sqrt{5}$
$\sqrt{5}$	$84\sqrt{5}$	-80
$+1$	$+84$	$-16\sqrt{5}$

$\times$	$\sqrt{5}$	$+1$
$\sqrt{5}$	5	$+\sqrt{5}$
-1	$-\sqrt{5}$	-1

$$\begin{aligned}&= \frac{68\sqrt{5}+4}{4}\\ &= \underline{\underline{17\sqrt{5}+1}};\end{aligned}$$

hence, $\underline{\underline{p=17}}$ and $\underline{\underline{q=1}}$.

3. (a) Given that (2)

$$y = (\tfrac{1}{4}x - 5)^8,$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

Well,

$$y = (\tfrac{1}{4}x - 5)^8 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 8(\tfrac{1}{4}x - 5)^7 \times \tfrac{1}{4}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2(\tfrac{1}{4}x - 5)^7}}.$$

(b) Hence find the approximate change in y as x increases from 12 to $12 + p$ where p is small. (2)

Solution

Now,

$$x = 12 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -256$$

and

$$\delta \approx \frac{\mathrm{d}y}{\mathrm{d}x} \times \mathrm{d}x$$
$$= \underline{\underline{-256p}}.$$

4. Given that

$$\log_p X = 5 \text{ and that } \log_p Y = 2,$$

find

(a) $\log_p X^2$, (1)

Solution

$$\log_p X^2 = 2\log_p X$$
$$= 2 \times 5$$
$$= \underline{\underline{10}}.$$

(b) $\log_p \dfrac{1}{X}$, (1)

Solution

$$\begin{aligned}\log_p \frac{1}{X} &= \log_p X^{-1} \\ &= -\log_p X \\ &= \underline{\underline{-5}}.\end{aligned}$$

(c) $\log_{XY} p$. (2)

Solution

$$\begin{aligned}\log_{XY} p &= \frac{1}{\log_p XY} \\ &= \frac{1}{\log_p X + \log_p Y} \\ &= \frac{1}{5+2} \\ &= \underline{\underline{\tfrac{1}{7}}}.\end{aligned}$$

5. Solve the simultaneous equations (5)

$$\begin{aligned}\frac{4^x}{256^y} &= 1\,024, \\ 3^{2x} \times 9^y &= 243.\end{aligned}$$

Solution

Well,

$$\begin{aligned}\frac{4^x}{256^y} = 1\,024 &\Rightarrow \frac{(2^2)^x}{(2^8)^y} = 2^{10} \\ &\Rightarrow \frac{2^{2x}}{2^{8y}} = 2^{10} \\ &\Rightarrow 2^{2x-8y} = 2^{10}\end{aligned}$$

take $\log_2$:

$$\Rightarrow 2x - 8y = 10 \quad (1)$$

and

$$\begin{aligned} 3^{2x} \times 9^y = 243 &\Rightarrow 3^{2x} \times (3^2)^y = 3^5 \\ &\Rightarrow 3^{2x} \times 3^{2y} = 3^5 \\ &\Rightarrow 3^{2x+2y} = 3^5 \end{aligned}$$

take $\log_3$:

$$\Rightarrow 2x + 2y = 5 \quad (2).$$

Do (2) − (1):

$$10y = -5 \Rightarrow \underline{\underline{y = -\tfrac{1}{2}}}$$

$$\Rightarrow \underline{\underline{x = 3}}.$$

6. (a) (i) Find the coefficient of x^3 in the expansion of (2)

$$(1 - 2x)^6.$$

Solution
Reading ahead, we need the coefficient of x^2 and x^3:

$$\begin{aligned} (1-2x)^6 &= \ldots + \binom{6}{2}(1)^4(-2x)^2 + \binom{6}{3}(1)^3(-2x)^3 + \ldots \\ &= \ldots + 60x^2 - 160x^3 + \ldots \end{aligned}$$

and so the coefficient of x^3 is $\underline{\underline{-160}}$.

(ii) find the coefficient of x^3 in the expansion of (3)

$$(1 + \tfrac{1}{2}x)(1 - 2x)^6.$$

Solution
Well,

$$(1+\tfrac{1}{2}x)(1-2x)^6 = (1+\tfrac{1}{2}x)(\ldots + 60x^2 - 160x^3 + \ldots)$$

$\times$	$60x^2$	$-160x^3$
1	$\ldots$	$-160x^3$
$+\frac{1}{2}x$	$+30x^3$	$\ldots$

$$= \ldots - 130x^3 + \ldots;$$

hence, the coefficient of x^3 is $\underline{\underline{-130}}$.

(b) Expand (3)

$$\left(2\sqrt{x} + \frac{1}{\sqrt{x}}\right)^4$$

in a series of powers of x with integer coefficients.

Solution

$$\left(2\sqrt{x} + \frac{1}{\sqrt{x}}\right)^4 = (2\sqrt{x})^4 + \binom{4}{1}(2\sqrt{x})^3\left(\frac{1}{\sqrt{x}}\right) + \binom{4}{2}(2\sqrt{x})^2\left(\frac{1}{\sqrt{x}}\right)^2 + \binom{4}{3}(2\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)^3 + \left(\frac{1}{\sqrt{x}}\right)^4$$

$$= \underline{\underline{16x^2 + 32x + 24 + \frac{8}{x} + \frac{1}{x^2}}}.$$

7. The diagram shows a box in the shape of a cuboid with a square cross-section of side x cm.

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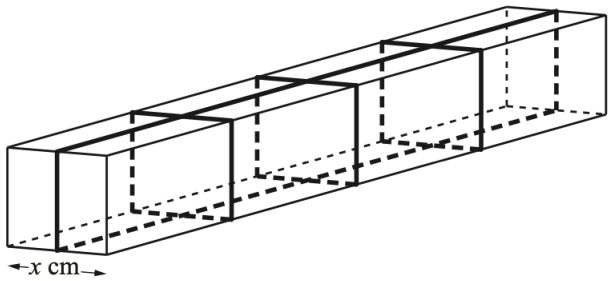


- The volume of the box is 3 500 cm^3.
- Four pieces of tape are fastened round the box as shown.
- The pieces of tape are parallel to the edges of the box.

(a) Given that the total length of the four pieces of tape is L cm, show that (3)

$$L = 14x + \frac{7\,000}{x^2}.$$

Solution

Let the length of each cuboid be l cm. Then

$$x^2 l = 3\,500 \Rightarrow l = \frac{3\,500}{x^2}$$

and

$$\begin{aligned} L &= \frac{3\,500}{x^2} + (3 \times 4x) + \frac{3\,500}{x^2} + x + x \\ &= \underline{\underline{14x + \frac{7\,000}{x^2}}}. \end{aligned}$$

(b) Given that x can vary, find the stationary value of L and determine the nature of this stationary value. (5)

Solution

Now,

$$\begin{aligned} L = 14x + \frac{7\,000}{x^2} &\Rightarrow L = 14x + 7\,000x^{-2} \\ &\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}x} = 14 - 14\,000x^{-3} \end{aligned}$$

and

$$\frac{\mathrm{d}L}{\mathrm{d}x} = 0 \Rightarrow 14 - 14\,000x^{-3} = 0$$
$$\Rightarrow 14 = 14\,000x^{-3}$$
$$\Rightarrow x^3 = 1\,000$$
$$\Rightarrow x = 10$$
$$\Rightarrow \underline{\underline{L = 210 \text{ cm}}}.$$

Next,

$$\frac{\mathrm{d}L}{\mathrm{d}x} = 14 - 14\,000x^{-3} \Rightarrow \frac{\mathrm{d}^2L}{\mathrm{d}x^2} = 42\,000x^{-4}$$

and

$$x = 10 \Rightarrow \frac{\mathrm{d}^2L}{\mathrm{d}x^2} = 4.2 > 0$$

so it is a $\underline{\underline{\text{minimum}}}$.

8. The table shows experimental values of two variables x and y.

x	2	4	6	8
y	9.6	38.4	105	232

It is known that x and y are related by the equation

$$y = ax^3 + bx,$$

where a and b are constants.

(a) A straight line graph is to be drawn for this information with $\frac{y}{x}$ on the vertical axis. (1)

State the variable which must be plotted on the horizontal axis.

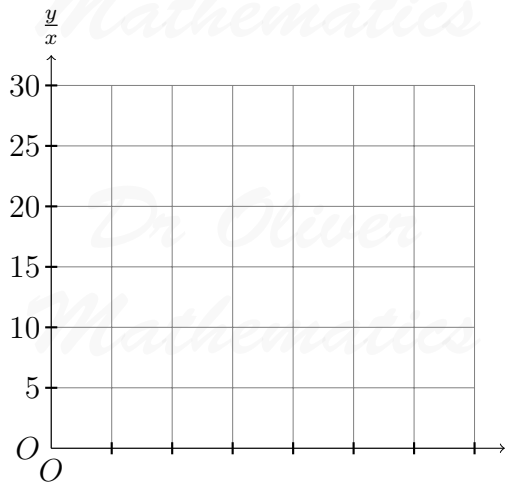
Solution

Well,

$$y = ax^3 + bx \Rightarrow \frac{y}{x} = ax^2 + b$$
$$\Rightarrow \frac{y}{x} = a(x^2) + b$$

and so we have $\underline{\underline{x^2}}$ on the horizontal axis.

(b) Draw this straight line graph on the grid below. (2)

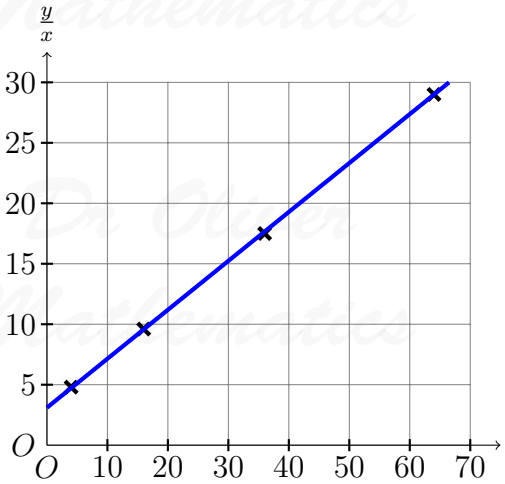


Solution

We complete the table

x^2	4	16	36	64
$\frac{y}{x}$	4.8	9.6	17.5	29

and draw the graph:



(c) Use your graph to estimate the value of a and of b. (3)

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Solution

Well,

$$\begin{aligned} a &= \frac{30 - 3.1}{66.5 - 0} \\ &= 0.404\,511\,278 \text{ (FCD)} \\ &= \underline{\underline{0.405 \text{ (3 sf)}}} \end{aligned}$$

and

$$\underline{\underline{b = 3.1}}.$$

(d) Estimate the value of x for which (2)

$$2y = 25x.$$

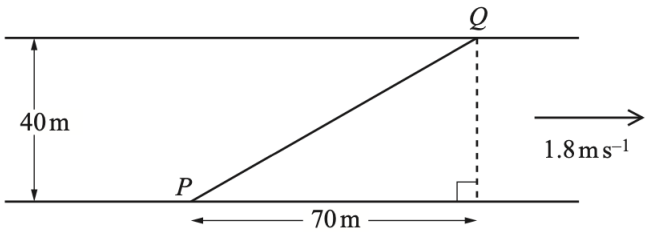
Solution

Now,

$$\begin{aligned} 2y = 25x &\Rightarrow \frac{y}{x} = 12.5 \\ &\Rightarrow 0.405(x^2) + 3.1 = 12.5 \\ &\Rightarrow 0.405(x^2) = 9.4 \\ &\Rightarrow x^2 = 23\tfrac{17}{81} \\ &\Rightarrow x = 4.817\,662\,975 \text{ (FCD)}; \end{aligned}$$

hence, the estimate the value is $\underline{\underline{x = 4.8}}$.

9. The diagram shows a river with parallel banks. (8)



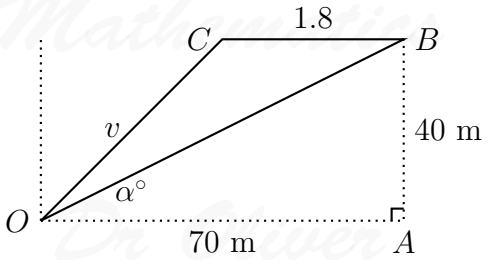
- The river is 40 m wide and is flowing with a speed of 1.8 ms^{-1}.

- A canoe travels in a straight line from a point P on one bank to a point Q on the opposite bank 70 m downstream from P.

Given that the canoe takes 12 s to travel from P to Q, calculate the speed of the canoe in still water and the angle to the bank that the canoe was steered

Solution

Let v ms^{-1} be the speed of the canoe in still water and let the angle be α°.



Well,

$$\begin{aligned}(\rightarrow): &\quad 12(v\cos\alpha + 1.8) = 70\\ (\uparrow): &\quad 12v\sin\alpha = 40 \quad (1).\end{aligned}$$

Now,

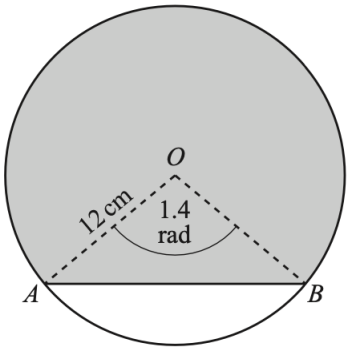
$$\begin{aligned}12(v\cos\alpha + 1.8) = 70 &\Rightarrow 12v\cos\alpha + 21.6 = 70\\ &\Rightarrow 12v\cos\alpha = 48.4 \quad (2).\end{aligned}$$

Divide (1) $\div$ (2):

$$\begin{aligned}\frac{12v\sin\alpha}{12v\cos\alpha} = \frac{40}{48.4} &\Rightarrow \tan\alpha = \tfrac{100}{121}\\ &\Rightarrow \angle\alpha = 39.571\,902\,81 \text{ (FCD)}\\ &\Rightarrow v = 5.232\,484\,008 \text{ (FCD)};\end{aligned}$$

so, the speed of the canoe in still water is $\underline{\underline{5.23 \text{ ms}^{-1}}}$ and the angle is $\underline{\underline{39.6^\circ}}$ (to 3 sf).

10. The diagram shows a circle with centre O and a chord AB.



The radius of the circle is 12 cm and angle AOB is 1.4 radians.

(a) Find the perimeter of the shaded region. (5)

Solution

Well,

$$\begin{aligned} & AB^2 = OA^2 + OB^2 - 2 \times OA \times OB \times \cos AOB \\ \Rightarrow \quad & AB^2 = 12^2 + 12^2 - 2 \times 12 \times 12 \times \cos 1.4 \\ \Rightarrow \quad & AB^2 = 239.049\,462\,8 \text{ (FCD)} \\ \Rightarrow \quad & AB = 15.461\,224\,49 \text{ (FCD)} \end{aligned}$$

and

$$\begin{aligned} \text{arc } AB &= (2\pi - 1.4) \times 12 \\ &= 58.598\,223\,69 \text{ (FCD)}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{perimeter} &= AB + \text{arc } AB \\ &= 15.461\ldots + 58.598\ldots \\ &= 74.059\,448\,18 \text{ (FCD)} \\ &= \underline{\underline{74.1\text{cm (3 sf)}}}. \end{aligned}$$

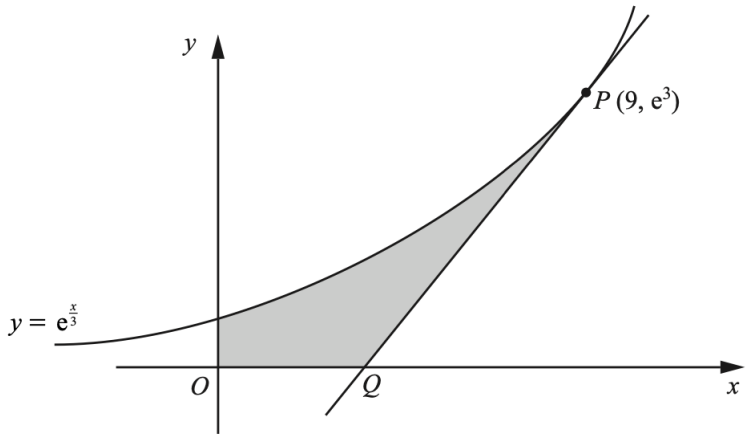
(b) Find the area of the shaded region. (4)

Solution

$$\begin{aligned}\text{Area} &= \text{whole circle} - \text{segment } OAB \\ &= (\pi \times 12^2) - [\tfrac{1}{2} \times 12^2 \times (1.4 - \sin 1.4)] \\ &= 422.541\,722\,7 \text{ (FCD)} \\ &= \underline{\underline{423 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

11. The diagram shows part of the curve

$$y = \mathrm{e}^{\frac{1}{3}x}.$$



The tangent to the curve at $P(9, \mathrm{e}^3)$ meets the x-axis at Q.

(a) Find the coordinates of Q. (4)

Solution

Well,

$$y = \mathrm{e}^{\frac{1}{3}x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{3}\mathrm{e}^{\frac{1}{3}x}$$

and

$$x = 9 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{3}\mathrm{e}^3.$$

Now, the tangent is

$$y - \mathrm{e}^3 = \tfrac{1}{3}\mathrm{e}^3(x - 9)$$

and

$$\begin{aligned} y = 0 &\Rightarrow -\mathrm{e}^3 = \tfrac{1}{3}\mathrm{e}^3(x-9) \\ &\Rightarrow -3 = x - 9 \\ &\Rightarrow x = 6; \end{aligned}$$

hence, $\underline{\underline{Q(6,0)}}$.

(b) Find the area of the shaded region bounded by the curve, the coordinate axes, and the tangent to the curve at P. (6)

Solution

Now,

$$\begin{aligned} \text{shaded region} &= \text{integral} - \text{triangle} \\ &= \int_0^9 \mathrm{e}^{\frac{1}{3}x}\,\mathrm{d}x - (\tfrac{1}{2} \times 3 \times \mathrm{e}^3) \\ &= \left[3\mathrm{e}^{\frac{1}{3}x}\right]_{x=0}^{9} - \tfrac{3}{2}\mathrm{e}^3 \\ &= (3\mathrm{e}^3 - 3) - \tfrac{3}{2}\mathrm{e}^3 \\ &= \underline{\underline{\tfrac{3}{2}\mathrm{e}^3 - 3}}. \end{aligned}$$

12. (a) Solve the equation (4)

$$2\operatorname{cosec} x + \frac{7}{\cos x} = 0 \text{ for } 0^\circ \leqslant x \leqslant 360^\circ.$$

Solution

$$\begin{aligned} 2\operatorname{cosec} x + \frac{7}{\cos x} = 0 &\Rightarrow \frac{2}{\sin x} = \frac{7}{\cos x} \\ &\Rightarrow \frac{\sin x}{2} = -\frac{\cos x}{7} \\ &\Rightarrow \tan x = -\tfrac{2}{7} \\ &\Rightarrow x = 164.054\,604\,1,\ 344.054\,604\,1 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{x = 164,\ 344 \text{ (3 sf)}}}. \end{aligned}$$

(b) Solve the equation (5)

$$7\sin(2y-1) = 5 \text{ for } 0 \leqslant y \leqslant 5 \text{ radians.}$$

Solution

Well,

$$0 \leqslant y \leqslant 5 \Rightarrow 0 \leqslant 2y \leqslant 10$$
$$\Rightarrow -1 \leqslant 2y - 1 \leqslant 9$$

and

$$\begin{aligned}
& 7\sin(2y-1) = 5 \\
\Rightarrow\quad & \sin(2y-1) = \tfrac{5}{7} \\
\Rightarrow\quad & 2y - 1 = 0.795\,602\,953\,5,\ 2.345\,989\,7,\ 7.078\,788\,261,\ 8.629\,175\,007 \text{ (FCD)} \\
\Rightarrow\quad & 2y = 1.795\,602\,953\,5,\ 3.345\,989\,7,\ 8.078\,788\,261,\ 9.629\,175\,007 \text{ (FCD)} \\
\Rightarrow\quad & y = 0.897\,801\,476\,7,\ 1.672\,994\,85,\ 4.039\,394\,13,\ 4.814\,587\,504 \text{ (FCD)} \\
\Rightarrow\quad & \underline{\underline{y = 0.898,\ 1.67,\ 4.04,\ 4.81 \text{ (3 sf)}}}.
\end{aligned}$$

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