Dr Oliver Mathematics
Mathematics: Advanced Higher
2022 Paper 2: Calculator
2 hours

The total number of marks available is 65.
You must write down all the stages in your working.

1. Express (3)

$$\frac{3x^2 - 3x + 5}{x(x^2 + 5)}$$

in partial fractions.

2. Find the exact value of (2)

$$\int_0^3 \frac{4}{2x + 1}\,\mathrm{d}x.$$

3. Use the Euclidean algorithm to find integers a and b such that (3)

$$634a + 87b = 1.$$

4. Use integration by parts to find (3)

$$\int (x + 2)(2x + 7)^{\frac{1}{2}}\,\mathrm{d}x.$$

5. Matrix $\mathbf{A}$ is given by (3)

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & k & 3 \\ k & 18 & -7 \end{pmatrix}.$$

Find the values of k so that the matrix $\mathbf{A}$ is singular.

6. The first three terms of a sequence are defined algebraically by

$$x + 5,\ 3x + 2,\ 5x - 1,$$

where $x \in \mathbb{N}$.

(a) Show that these three terms form the start of an arithmetic sequence. (2)

(b) Find a simplified expression for the 15th term of this sequence. (2)

(c) Given that the sum of the first 20 terms of this sequence is 1 130, find the value of x. (2)

7. The complex number

$$z = 3 + \mathrm{i}$$

is a root of

$$z^2 - 6z + a = 0,$$

where a is a real number.

(a) State the second root of (1)

$$z^2 - 6z + a = 0.$$

(b) Hence, or otherwise, find the value of a. (2)

The expression

$$z^2 - 6z + a$$

is a factor of

$$z^3 - z^2 - 20z + b,$$

where b is a real number.

(c) Find the value of b. (1)

8. (a) Differentiate (2)

$$x \ln x - x$$

with respect to x.

(b) Hence find the general solution of the differential equation (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \ln x = x^{-x}.$$

9. The matrix $\mathbf{A}$ is given by (5)

$$\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}.$$

Prove by induction that

$$\mathbf{A}^n = \begin{pmatrix} 3^n & 1 - 3^n \\ 0 & 1 \end{pmatrix}.$$

10. Solve the differential equation (9)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 9\sin x + 13\cos x,$$

given that $y = 5$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

11. A curve defined parametrically has the following properties: (4)

- $x = \tan^{-1} 2t$,
- $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6t(1 + 4t^2)$, and
- $y = 5$ when $t = 1$.

Find y in terms of t.

12. Let

$$z = \cos\theta + \mathrm{i}\sin\theta.$$

(a) Use de Moivre's theorem to state an expression for z^4. (1)

(b) State and simplify the binomial expansion of (3)

$$(\cos\theta + \mathrm{i}\sin\theta)^4.$$

(c) Hence show that:

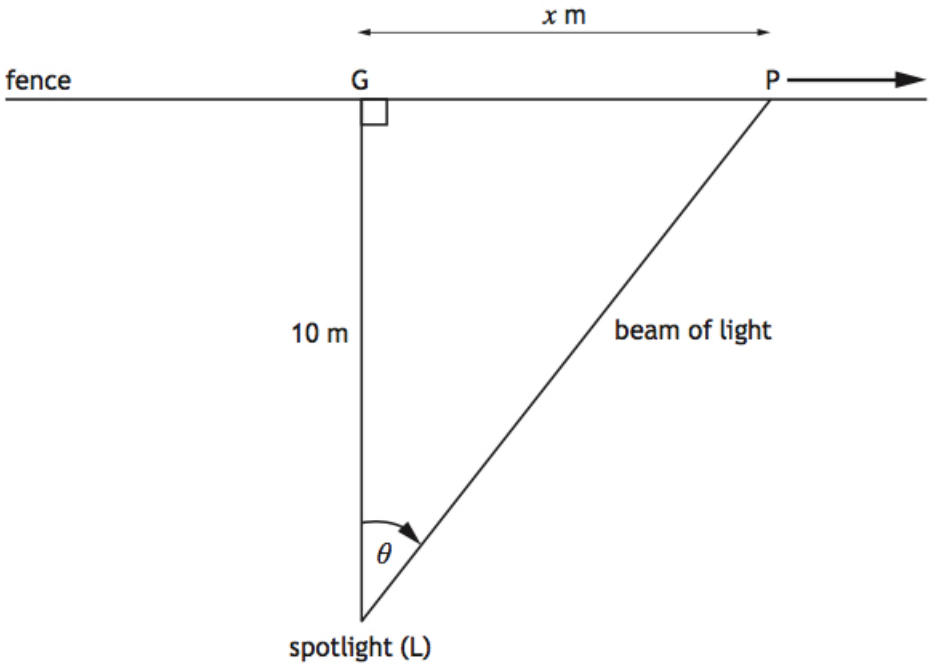
(i) $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. (2)

(ii) $\sin\theta\cot 4\theta$ can be written in terms of $\cos\theta$ only. (2)

13. A security spotlight is situated 10 metres from a straight fence. The spotlight rotates at a constant speed and makes one full revolution every 12 seconds.

The situation at time t seconds is modelled in the diagram below, where:

- L is the position of the spotlight,
- G is the point on the fence nearest to the spotlight,
- P is the position where the light hits the fence,
- θ is the angle between LG and LP, and
- x is the distance in metres from G to P.



(a) Show that:

(i) $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{6}\pi$ radians per second, (1)

(ii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \frac{5}{3}\pi \sec^2\theta$ metres per second. (4)

(b) Prove that (1)

$$1 + \tan^2\theta = \sec^2\theta.$$

(c) Hence, or otherwise, find the exact value of $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ when P is 5 metres from G. (3)