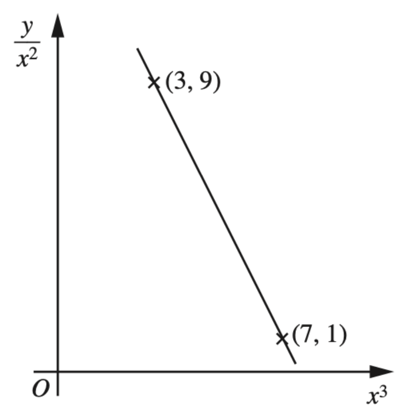
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 June Paper 2 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The variables x and y are related so that, when $\dfrac{y}{x^2}$ is plotted against x, a straight line graph passing through (3, 9) and (7, 1) is obtained. (4)



Express y in terms of x.

Solution

Well,

$$\begin{aligned} m &= \frac{9-1}{3-7} \\ &= \frac{8}{-4} \\ &= -2 \end{aligned}$$

and the equation is

$$\frac{y}{x^2} - 1 = -2(x-7) \Rightarrow \frac{y}{x^2} - 1 = -2x + 14$$
$$\Rightarrow \frac{y}{x^2} = 15 - 2x$$
$$\Rightarrow \underline{\underline{y = x^2(15-2x)}}.$$

2. In a singing competition there are 8 contestants.
Each contestant sings in the first round of this competition.

(a) In how many different orders could the contestants sing? (1)

Solution

$$8! = \underline{\underline{40\,320}}.$$

After the first round 5 contestants are chosen.

(b) In how many different ways can these 5 contestants be chosen? (2)

Solution

$$\binom{8}{5} = \underline{\underline{56}}.$$

These 5 contestants sing again and then First, Second, and Third prizes are awarded to three of them.

(c) In how many different ways can the prizes be awarded? (2)

Solution

$$5 \times 4 \times 3 = \underline{\underline{60}}.$$

3. It is given that $(x-1)$ is a factor of $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = x^3 - 6x^2 + ax + b.$$

(a) Express b in terms of a. (2)

Solution

$$\begin{aligned} f(1) = 0 &\Rightarrow 1 - 6 + a + b = 0 \\ &\Rightarrow \underline{\underline{b = 5 - a}}. \end{aligned}$$

(b) Show that the remainder when $f(x)$ is divided by $(x-3)$ is twice the remainder when $f(x)$ is divided $(x-2)$. (4)

Solution

Well,

$$\begin{aligned} f(3) &= 3^3 - 6(3^2) + 3a + b \\ &= 27 - 54 + 3a + b \\ &= 3a + b - 27 \end{aligned}$$

and

$$\begin{aligned} f(2) &= 2^3 - 6(2^2) + 2a + b \\ &= 8 - 24 + 2a + b \\ &= 2a + b - 16. \end{aligned}$$

Now,

$$\begin{aligned} 3a + b - 27 &= 3a + (5 - a) - 27 \\ &= 2a - 22 \\ &= 2(a - 11) \end{aligned}$$

and

$$\begin{aligned} 2a + b - 16 &= 2a + (5 - a) - 16 \\ &= a - 11; \end{aligned}$$

hence, the remainder when $f(x)$ is divided by $(x-3)$ is <u>twice</u> the remainder when $f(x)$ is divided $(x-2)$.

4. (a) Given that (3)

$$\sin x = p \text{ and } \cos x = 2p,$$

where x is acute, find the exact value of p and the exact value of $\operatorname{cosec} x$.

Solution
Well,

$$\begin{aligned}\sin^2 x + \cos^2 x = 1 &\Rightarrow p^2 + (2p)^2 = 1\\ &\Rightarrow p^2 + 4p^2 = 1\\ &\Rightarrow 5p^2 = 1\\ &\Rightarrow p^2 = \tfrac{1}{5}\\ &\Rightarrow \underline{\underline{p = \tfrac{\sqrt{5}}{5}}}\\ &\Rightarrow \underline{\underline{\operatorname{cosec} x = \tfrac{5}{\sqrt{5}}}}.\end{aligned}$$

(b) Prove that (3)

$$(\cot x + \tan x)(\cot x - \tan x) = \frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}.$$

Solution
Well,

$\times$	$\cot x$	$+\tan x$
$\cot x$	$\cot^2 x$	$+1$
$-\tan x$	-1	$-\tan^2 x$

so

$$\begin{aligned}(\cot x + \tan x)(\cot x - \tan x) &\equiv \cot^2 x - \tan^2 x\\ &\equiv (\operatorname{cosec}^2 x - 1) - (\sec^2 x - 1)\\ &\equiv \operatorname{cosec}^2 x - \sec^2 x\\ &\equiv \underline{\underline{\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}}},\end{aligned}$$

as required.

5. Given that a curve has equation (7)

$$y = x^2 + 64\sqrt{x},$$

find the coordinates of the point on the curve where

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

Solution

$$\begin{aligned} y = x^2 + 64\sqrt{x} &\Rightarrow y = x^2 + 64x^{\frac{1}{2}} \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 32x^{-\frac{1}{2}} \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 - 16x^{-\frac{3}{2}} \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 &\Rightarrow 2 - 16x^{-\frac{3}{2}} = 0 \\ &\Rightarrow 16x^{-\frac{3}{2}} = 2 \\ &\Rightarrow x^{-\frac{3}{2}} = \tfrac{1}{8} \\ &\Rightarrow x^{\frac{3}{2}} = 8 \\ &\Rightarrow x = 8^{\frac{2}{3}} \\ &\Rightarrow x = 4 \\ &\Rightarrow y = 144; \end{aligned}$$

hence, the coordinates are $\underline{\underline{(4, 144)}}$.

6. The line (7)

$$y = x + 4$$

intersects the curve

$$2x^2 + 3xy - y^2 + 1 = 0$$

at the points A and B.

Find the length of the line AB.

Solution

Well,

$\times$	x	$+4$
x	x^2	$+4x$
$+4$	$+4x$	$+16$

and

$$\begin{aligned} & 2x^2 + 3xy - y^2 + 1 = 0 \\ \Rightarrow\ & 2x^2 + 3x(x+4) - (x+4)^2 + 1 = 0 \\ \Rightarrow\ & 2x^2 + 3x^2 + 12x - x^2 - 8x - 16 + 1 = 0 \\ \Rightarrow\ & 4x^2 + 4x - 15 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +4 \\ \text{multiply to:} & (+4)\times(-15) = -60 \end{array}\right\} +10,\ -6$$

e.g.,

$$\begin{aligned} \Rightarrow\ & 4x^2 + 10x - 6x - 15 = 0 \\ \Rightarrow\ & 2x(2x+5) - 3(2x+5) = 0 \\ \Rightarrow\ & (2x-3)(2x+5) = 0 \\ \Rightarrow\ & x = 1\tfrac{1}{2} \text{ or } x = -2\tfrac{1}{2} \\ \Rightarrow\ & y = 5\tfrac{1}{2} \text{ or } y = 1\tfrac{1}{2}; \end{aligned}$$

so, the two points are $(1\frac{1}{2}, 5\frac{1}{2})$ and $(-2\frac{1}{2}, 1\frac{1}{2})$.

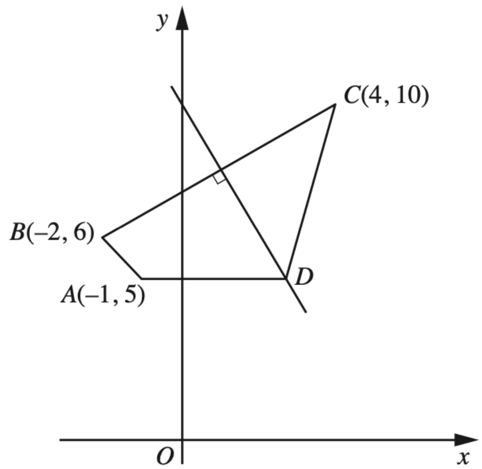
Finally,

$$\begin{aligned} AB &= \sqrt{[1\tfrac{1}{2} - (-2\tfrac{1}{2})]^2 + (5\tfrac{1}{2} - 1\tfrac{1}{2})^2} \\ &= \sqrt{4^2 + 4^2} \\ &= \underline{\underline{4\sqrt{2}}}. \end{aligned}$$

7. **Solutions to this question by accurate drawing will not be accepted.** (8)

In the diagram the points $A(-1, 5)$, $B(-2, 6)$, $C(4, 10)$, and D are the vertices of a quadrilateral in which AD is parallel to the x-axis.

The perpendicular bisector of BC passes through D.



Find the area of the quadrilateral $ABCD$.

Solution

Well,

$$\begin{aligned} m_{BC} &= \frac{10-6}{4-(-2)} \\ &= \tfrac{2}{3} \end{aligned}$$

and

$$m_{\text{normal}} = -\tfrac{3}{2}.$$

Now, the midpoint of BC — let us call it E — is

$$\left(\frac{-2+4}{2}, \frac{6+10}{2}\right) = E(1, 8)$$

so the equation of the line perpendicular to BC is

$$y - 8 = -\tfrac{3}{2}(x-1).$$

Next, AD is $y = 5$ so they intersect at

$$\begin{aligned} 5 - 8 = -\tfrac{3}{2}(x-1) &\Rightarrow -3 = -\tfrac{3}{2}(x-1) \\ &\Rightarrow 2 = x - 1 \\ &\Rightarrow x = 3; \end{aligned}$$

so, $D(3, 5)$.

Let, $S(-2, 10)$, $T(4, 5)$, and $W(-2, 5)$ so that $SCTW$ encloses the area of the quadrilateral $ABCD$.

Finally,

$$\begin{aligned}\text{area of } ABCD &= \text{area of } SCTW - \text{area of } BSC - \text{area of } CDT - \text{area of } ABW \\ &= (6 \times 5) - (\tfrac{1}{2} \times 4 \times 6) - (\tfrac{1}{2} \times 1 \times 5) - (\tfrac{1}{2} \times 1 \times 1) \\ &= 30 - 12 - 2\tfrac{1}{2} - \tfrac{1}{2} \\ &= \underline{\underline{15}}.\end{aligned}$$

8. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 7 \\ 1 & -5 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 8 & 6 \end{pmatrix},$$

calculate

(b) (i) $2\mathbf{A}$, (1)

Solution

$$2\mathbf{A} = \underline{\underline{\begin{pmatrix} 4 & 6 & 14 \\ 2 & -10 & 8 \end{pmatrix}}}.$$

(ii) $\mathbf{B}^2$, (2)

Solution

$$\begin{aligned}\mathbf{B}^2 &= \begin{pmatrix} 2 & 1 \\ 8 & 6 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 8 & 6 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 12 & 8 \\ 64 & 44 \end{pmatrix}}}.\end{aligned}$$

(iii) $\mathbf{BA}$. (2)

Solution

$$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} 2 & 3 & 7 \\ 1 & -5 & 4 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} 5 & 1 & 18 \\ 21 & -6 & 80 \end{pmatrix}}}.$$

(c) (i) Given that (2)

$$\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 7 & 6 \end{pmatrix},$$

find $\mathbf{C}^{-1}$.

Solution
Now,

$$\det \mathbf{C} = 12 - 7 = 5$$

and

$$\mathbf{C}^{-1} = \underline{\underline{\tfrac{1}{5} \begin{pmatrix} 6 & -1 \\ -7 & 2 \end{pmatrix}}}.$$

(ii) Given also that (2)

$$\mathbf{D} = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix},$$

find the matrix $\mathbf{X}$ such that

$$\mathbf{XC} = \mathbf{D}.$$

Solution

$$\mathbf{XC} = \mathbf{D} \Rightarrow \mathbf{X} = \mathbf{DC}^{-1}$$
$$= \tfrac{1}{5} \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 6 & -1 \\ -7 & 2 \end{pmatrix}$$
$$= \underline{\underline{\tfrac{1}{5} \begin{pmatrix} 3 & 2 \\ -5 & 0 \end{pmatrix}}}.$$

9. A particle starts from rest and moves in a straight line so that, t seconds after leaving a fixed point O, its velocity, v ms^{-1}, is given by

$$v = 4 \sin 2t.$$

(a) Find the distance travelled by the particle before it first comes to instantaneous rest. (5)

Solution

Well,

$$\begin{aligned} v = 0 &\Rightarrow 4\sin 2t = 0 \\ &\Rightarrow 2t = \pi \\ &\Rightarrow t = \tfrac{1}{2}\pi. \end{aligned}$$

Now,

$$v = 4\sin 2t \Rightarrow s = -2\cos 2t + c,$$

for some constant c. Next,

$$\begin{aligned} t = 0, s = 0 &\Rightarrow -2 + c = 0 \\ &\Rightarrow c = 2, \end{aligned}$$

and

$$s = -2\cos 2t + 2.$$

Finally,

$$\begin{aligned} \text{distance travelled} &= s(\tfrac{1}{2}\pi) - s(0) \\ &= (-2\cos\pi + 2) + 0 \\ &= \underline{\underline{4\text{ m}}}. \end{aligned}$$

(b) Find the acceleration of the particle when $t = 3$. (3)

Solution

Well,

$$v = 4\sin 2t \Rightarrow a = 8\cos 2t$$

and

$$\begin{aligned} t = 3 &\Rightarrow a = 7.681\,362\,293 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{a = 7.68\text{ ms}^{-2}\text{ (3 sf)}}}. \end{aligned}$$

10. In this question,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a unit vector due east and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is a unit vector due north.

A lighthouse has position vector

$$\begin{pmatrix} 27 \\ 48 \end{pmatrix} \text{ km}$$

relative to an origin O.

A boat moves in such a way that its position vector is given by

$$\begin{pmatrix} 4 + 8t \\ 12 + 6t \end{pmatrix} \text{ km},$$

where t is the time, in hours, after 1200.

(a) Show that at 1400 the boat is 25 km from the lighthouse. (4)

Solution

Well,

$$\begin{aligned} t = 2 \Rightarrow & \begin{pmatrix} 20 - 27 \\ 24 - 48 \end{pmatrix} \\ \Rightarrow & \begin{pmatrix} -7 \\ -24 \end{pmatrix} \text{ km} \end{aligned}$$

and

$$\begin{aligned} \text{distance} &= \sqrt{(-7)^2 + (-24)^2} \\ &= \sqrt{49 + 576} \\ &= \sqrt{625} \\ &= \underline{\underline{25 \text{ km}}}, \end{aligned}$$

as required.

(b) Find the length of time for which the boat is less than 25 km from the lighthouse. (4)

Solution

Now,

$$\begin{aligned} & [(4 + 8t) - 27]^2 + [(12 + 6t) - 48]^2 < 25^2 \\ \Rightarrow \quad & (-23 + 8t)^2 + (-36 + 6t)^2 < 625 \end{aligned}$$

$\times$	-23	$+8t$
-23	529	$-184t$
$+8t$	$-184t$	$+64t^2$

$\times$	-36	$+6t$
-36	1 296	$-216t$
$+6t$	$-216t$	$+36t^2$

$$\Rightarrow \quad (529 - 368t + 64t^2) + (1\,296 - 432t + 36t^2) < 625$$
$$\Rightarrow \quad 100t^2 - 800t + 1\,200 < 0$$
$$\Rightarrow \quad 100(t^2 - 8t + 12) < 0$$

$$\left.\begin{array}{ll}\text{add to:} & -8 \\ \text{multiply to:} & +12\end{array}\right\} -6,\ -2$$

$$\Rightarrow \quad 100(t-6)(t-2) \leqslant 0.$$

We need a 'table of signs':

	$t < 2$	$t = 2$	$2 < t < 6$	$t = 6$	$t > 6$
$t-2$	$-$	0	$+$	$+$	$+$
$t-6$	$-$	$-$	$-$	0	$+$
$(t-2)(t-6)$	$+$	0	$-$	0	$+$

Hence,

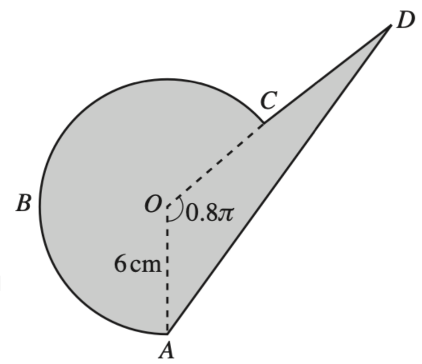
$$6 - 2 = \underline{\underline{4 \text{ hours}}}.$$

EITHER

11. The diagram represents a company logo $ABCDA$, consisting of

- a sector $OABCO$ of a circle, centre O and radius 6 cm and

- a triangle AOD.



Angle $AOC = 0.8\pi$ radians and C is the mid-point of OD.

Find

(a) the perimeter of the logo, (7)

Solution

Well,

$$\begin{aligned}\text{perimeter of } ABCOA &= (2\pi - 0.8\pi) \times 6 \\ &= 7.2\pi.\end{aligned}$$

Now,

$$\begin{aligned} && AD^2 &= OA^2 + OD^2 - 2 \times OA \times OD \times \cos AOD \\ &\Rightarrow & AD^2 &= 6^2 + 12^2 - 2 \times 6 \times 12 \times \cos 0.8\pi \\ &\Rightarrow & AD^2 &= 296.498\,447\,2 \text{ (FCD)} \\ &\Rightarrow & AD &= 17.219\,130\,27 \text{ (FCD)}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{perimeter} &= \text{perimeter of } ABCOA + AD + CD \\ &= 7.2\pi + 17.219\ldots + 6 \\ &= 45.838\,597\,37 \text{ (FCD)} \\ &= \underline{\underline{45.9 \text{ cm (3 sf)}}}.\end{aligned}$$

(b) the area of the logo. (5)

Solution

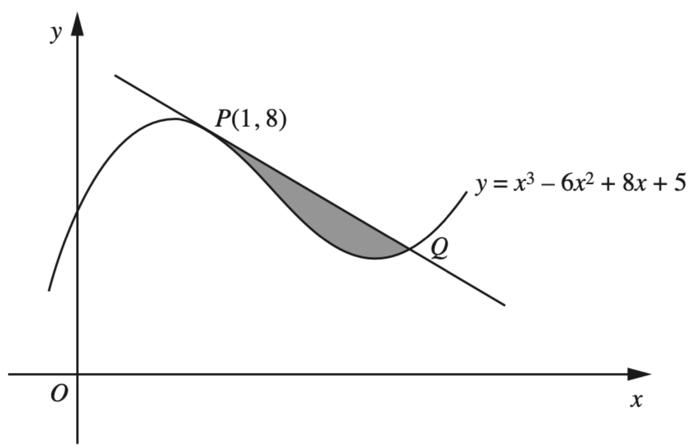
Now,

$$\begin{aligned}\text{area} &= \text{area of } ABCOA + \text{area of } OAD \\ &= (\tfrac{1}{2} \times 6^2 \times 1.2\pi) + (\tfrac{1}{2} \times 6 \times 12 \times \sin 0.8\pi) \\ &= 89.018\,670\,4 \text{ (FCD)} \\ &= \underline{\underline{89.0 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

OR

12. The diagram shows part of the curve

$$y = x^3 - 6x^2 + 8x + 5.$$



The tangent to the curve at the point $P(1, 8)$ cuts the curve at the point Q.

(a) Show that the x-coordinate of Q is 4. (6)

Solution

Well,

$$y = x^3 - 6x^2 + 8x + 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 12x + 8$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -1.$$

Now, the equation of the tangent through P is

$$y - 8 = -(x-1) \Rightarrow y - 8 = -x + 1$$
$$\Rightarrow y = -x + 9.$$

Next, we have some simultaneous equations:

$$x^3 - 6x^2 + 8x + 5 = -x + 9 \Rightarrow x^3 - 6x^2 + 9x - 4 = 0.$$

We use synthetic division (we know $(x-1)$ is a factor):

1	1	−6	9	−4
	↓	1	−5	4
	1	−5	4	0

So,

$$x^3 - 6x^2 + 8x + 5 = 0 = (x-1)(x^2 - 5x + 4) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -5 \\ \text{multiply to:} & +4\end{array}\right\} -1, -4$$

$$\Rightarrow (x-1)^2(x-4) = 0$$
$$\Rightarrow x = 1 \text{ (repeated twice) or } x = 4.$$

Hence, the x-coordinate of Q is $\underline{\underline{4}}$.

(b) Find the area of the shaded region. (6)

Solution

Now,

$$\begin{aligned}
\text{area of the shaded region} &= \int_1^4 \left[(-x+9) - (x^3 - 6x^2 + 8x + 5)\right] \mathrm{d}x \\
&= \int_1^4 (-x^3 + 6x^2 - 9x + 4)\,\mathrm{d}x \\
&= \left[-\tfrac{1}{4}x^4 + 2x^3 - \tfrac{9}{2}x^2 + 4x\right]_{x=1}^{4} \\
&= (-64 + 128 - 72 + 16) - \left(-\tfrac{1}{4} + 2 - \tfrac{9}{2} + 4\right) \\
&= \underline{\underline{6\tfrac{3}{4}}}.
\end{aligned}$$