Core Mathematics 4

Dr Oliver
Mathematics

Partial fractions

For the product of distinct linear brackets,

$$\frac{5x+2}{(x+4)(x-5)} \equiv \frac{A}{x+4} + \frac{B}{x-5} \equiv \frac{A(x-5)+B(x+4)}{(x+4)(x-5)},$$

and, since the denominators are the same, the numerators must be identically equal, i.e.,

$$5x+2 \equiv A(x-5)+B(x+4).$$

You can either compare coefficients ($A+B=5$, $-5A+4B=2$ and solve) or substitute appropriate values ($x=5 \Rightarrow 27=9B$ and so $B=3$). For repeated linear factors, repeat the denominators according to multiplicity, i.e.,

$$\frac{5x+2}{(x+4)(x-5)^2} \equiv \frac{A}{x+4} + \frac{B}{x-5} + \frac{C}{(x-5)^2}.$$

An algebraic fraction is *improper* is the highest power of the numerator is at least as great as the highest power of the denominator. In order to express an improper algebraic fraction in partial fractions it is necessary first to do polynomial division.

Parametric equations

Parametric equations are when the variables, such as x and y are each defined in terms of a third variable, e.g.,

$$x = 3\cos\theta,\ y = 3\sin\theta,\ 0 \le \theta \le \pi$$

are parametric equations that describe the top semi-circle of a circle, centred at the origin, with radius 3.

Parametric differentiation

When a curve is given in parametric form we can find the derivative in terms of the parameter using

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

rather than trying to eliminate the parameter t.

Binomial expansion

For any real number n,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

This expansion is finite and exact if n is a positive integer; if n is not, then the expansion is infinite and is convergent for $|x| < 1$.

For an expression of the form $(a+bx)^n$ you can either use

$$(a+bx)^n = \left[a\left(1+\frac{b}{a}x\right)\right]^n = a^n\left(1+\frac{b}{a}x\right)^n$$

and proceed as above or apply the following general formula

$$(a+bx)^n = a^n + na^{n-1}bx + \frac{n(n-1)}{2!}a^{n-2}b^2x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3x^3 + \ldots;$$

this method usually appears explicitly in the exam board's mark schemes but is *not* given in the formula book — so, if you are going to use it, you need to learn it!

You should check that you can use the binomial expansion in conjunction with partial fractions and that you have practised getting numerical approximations in a variety of contexts.

Implicit differentiation

We apply the chain rule, differentiating terms in x with respect to x and terms in y with respect to y and then multiplying by $\frac{dy}{dx}$:

$$\begin{aligned} 3x^2y^3 = 5 &\Rightarrow \frac{d}{dx}\left(3x^2y^3\right) = \frac{d}{dx}(5) \\ &\Rightarrow \frac{d}{dx}(3x^2) \times y^3 + 3x^2 \times \frac{d}{dx}(y^3) = 0 \\ &\Rightarrow 6x \times y^3 + 3x^2 \times \frac{d}{dy}(y^3) \times \frac{dy}{dx} = 0 \\ &\Rightarrow 6xy^3 + 9x^2y^2\frac{dy}{dx} = 0. \end{aligned}$$

Logarithmic differentiation

We use this technique when the exponent is a function of x:

$$\begin{aligned} y = 5^{\cos x} &\Rightarrow \ln y = \ln 5^{\cos x} \\ &\Rightarrow \ln y = (\ln 5)\cos x \\ &\Rightarrow \frac{1}{y}\frac{dy}{dx} = -(\ln 5)\sin x \\ &\Rightarrow \frac{dy}{dx} = -(\ln 5)y\sin x \\ &\Rightarrow \frac{dy}{dx} = -(\ln 5)\sin x(5^{\cos x}) \end{aligned}$$

Connected rates of change

A sphere's volume, V cm^3, is given by $V = \frac{4}{3}\pi r^3$. If the radius is increasing at 2 cm $\cdot$ s^{-1}, then we can find the rate at which the volume is increasing by

$$\frac{dV}{dt} = \frac{d}{dt}\left(\tfrac{4}{3}\pi r^3\right) = \frac{d}{dr}\left(\tfrac{4}{3}\pi r^3\right) \times \frac{dr}{dt} = 4\pi r^2\frac{dr}{dt} = 8\pi r^2.$$

Scalar product

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3,$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

Straight lines

The vector equation of a straight line is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position of any point on the line, $\mathbf{b}$ is any direction vector parallel to the line, and λ is a scalar.

In three dimensions, lines may be parallel, skew, or intersect, and you need to be able to determine which category two given lines fall into.

Given lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$ the angle between them is simply the angle between the direction vectors $\mathbf{b}$ and $\mathbf{d}$ and it is found using the scalar product.

Standard integrals

You should know the integrals of the six trigonometric functions, the exponential function, and the natural logarithm.

Integration by inspection

$$\int f'(x)\,[f(x)]^n\,dx = \tfrac{1}{n+1}[f(x)]^{n+1} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

Integration by substitution

In Core Mathematics 4 you will be given the substitution, usually in the form $u = \ldots$. If you are given the substitution in a different form, such as $u^2 = 3x-4$ then use the chain rule rather than rearranging to get $u = \ldots$:

$$u^2 = 3x - 4 \Rightarrow 2u\frac{du}{dx} = 3.$$

Integration using identities

$$\begin{aligned} \int \tan^3 x\,dx &= \int \tan x\tan^2 x\,dx \\ &= \int \tan x(\sec^2 x - 1)\,dx \\ &= \int \sec^2 x\tan x\,dx - \int \tan x\,dx \\ &= \tfrac{1}{2}\tan^2 x - \ln|\sec x| + c. \end{aligned}$$

Parametric integration

Area: $\int_a^b y\,dx = \int_{t_1}^{t_2} y\frac{dx}{dt}\,dt$

VoR: $\pi\int_a^b y^2\,dx = \pi\int_{t_1}^{t_2} y^2\frac{dx}{dt}\,dt,$

where t_1 and t_2 are the parametric values such that $x(t_1) = a$ and $x(t_2) = b$.