Dr Oliver Mathematics
AQA Further Maths Level 2
June 2018 Paper 2
2 hours

The total number of marks available is 105.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. The nth term of a sequence is
$$\frac{1\,420 - 5n}{1\,420 + 5n}.$$

   (a) Work out the **position** of the term that has the value zero. (2)

   Solution
$$\begin{aligned}\frac{1\,420 - 5n}{1\,420 + 5n} = 0 &\Rightarrow 1\,420 - 5n = 0\\ &\Rightarrow 5n = 1\,420\\ &\Rightarrow \underline{\underline{n = 284}}.\end{aligned}$$

   (b) Write down the limiting value of the sequence as $n \to \infty$. (1)

   Solution
$$\begin{aligned}\frac{1\,420 - 5n}{1\,420 + 5n} &= \frac{\frac{1\,420}{n} - 5}{\frac{1\,420}{n} + 5}\\ &\to \frac{0 - 5}{0 + 5} \text{ (as } n \to \infty)\\ &= \underline{\underline{-1}}.\end{aligned}$$

2. $P(-3, -10)$ and $Q(a, b)$ are points on a straight line with gradient 12. (2)

   Work out one possible pair of integer values for a and b.

Solution

$$\begin{aligned} y + 10 = 12(x+3) &\Rightarrow y + 10 = 12x + 36 \\ &\Rightarrow y = 12x + 26. \end{aligned}$$

So, for example, $\underline{\underline{Q(0, 26)}}$; in this case, $\underline{\underline{a = 0}}$ and $\underline{\underline{b = 26}}$.

3.
$$p = \frac{m+2}{m^2+1}.$$

(a) Work out the value of p when $m = -5.5$. (1)

Solution

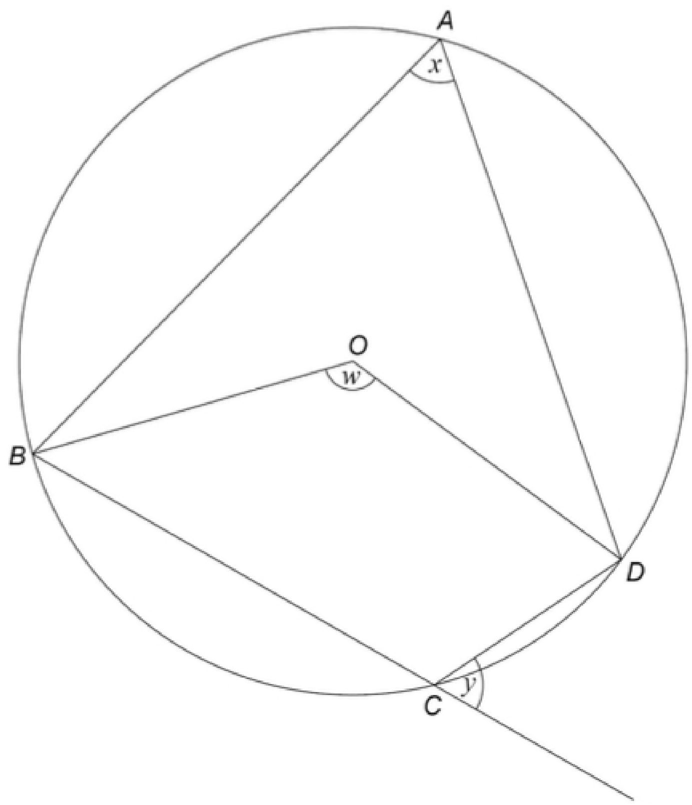
$$\begin{aligned} p &= \frac{-5.5+2}{(-5.5)^2+1} \\ &= \frac{-3.5}{31.25} \\ &= \underline{\underline{-\tfrac{14}{125}}}. \end{aligned}$$

(b) Work out the values of m when $p = 2$. (3)

Solution

$$\begin{aligned} \frac{m+2}{m^2+1} = 2 &\Rightarrow m + 2 = 2(m^2+1) \\ &\Rightarrow m + 2 = 2m^2 + 2 \\ &\Rightarrow 2m^2 - m = 0 \\ &\Rightarrow m(2m-1) = 0 \\ &\Rightarrow \underline{\underline{m = 0 \text{ or } m = \tfrac{1}{2}}} \end{aligned}$$

4. A, B, C, and D are points on a circle, centre O. (1)



Which statement is correct?
Tick **one** box.

$x + y = 180°$ and $w = 2x$ ☐

$x + y = 180°$ and $x = 2w$ ☐

$x = y$ and $w = 2x$ ☐

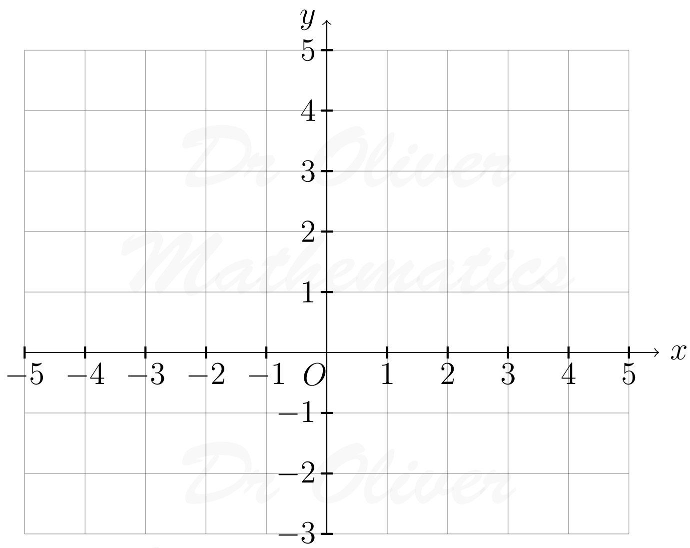
$x = y$ and $x = 2w$ ☐

Solution

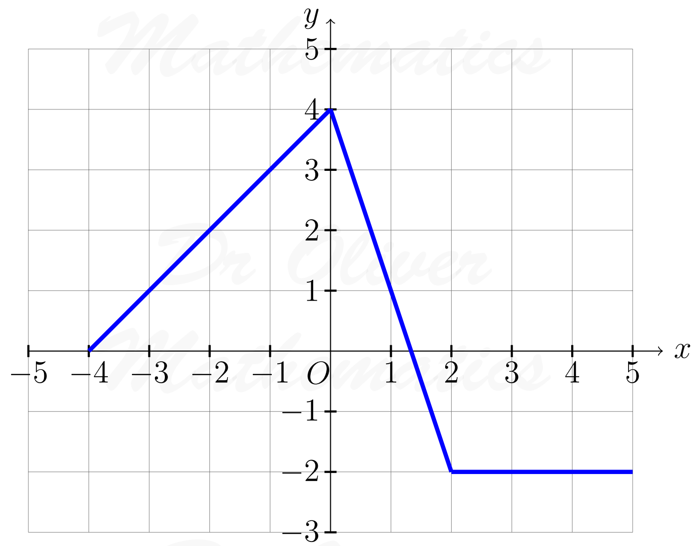
The <u>third</u> one: $x = y$ and $w = 2x$.

5. On the grid, draw the graph of $y = \mathrm{f}(x)$: (4)

$$\mathrm{f}(x) = \begin{cases} x + 4, & -4 \leqslant x < 0 \\ 4 - 3x, & 0 \leqslant x < 2 \\ -2, & 2 \leqslant x \leqslant 5. \end{cases}$$



Solution



6.

$$f(x) = x^2 - 7 \text{ for all values of } x$$
$$g(x) = 1 - 3x \text{ for } -4 \leqslant x \leqslant 4.$$

(a) Work out the range of $f(x)$. (1)
Give your answer as an inequality.

Solution
$f(x) \geqslant -7$.

(b) Work out the range of $g(x)$. (2)
Give your answer as an inequality.

Solution
$g(-4) = 13$ and $g(4) = -11$ so $-11 \leqslant g(x) \leqslant 13$.

(c) Solve (4)

$$2\,f(x) = g(x).$$

You **must** show your working.
Give your answers to 3 decimal places.

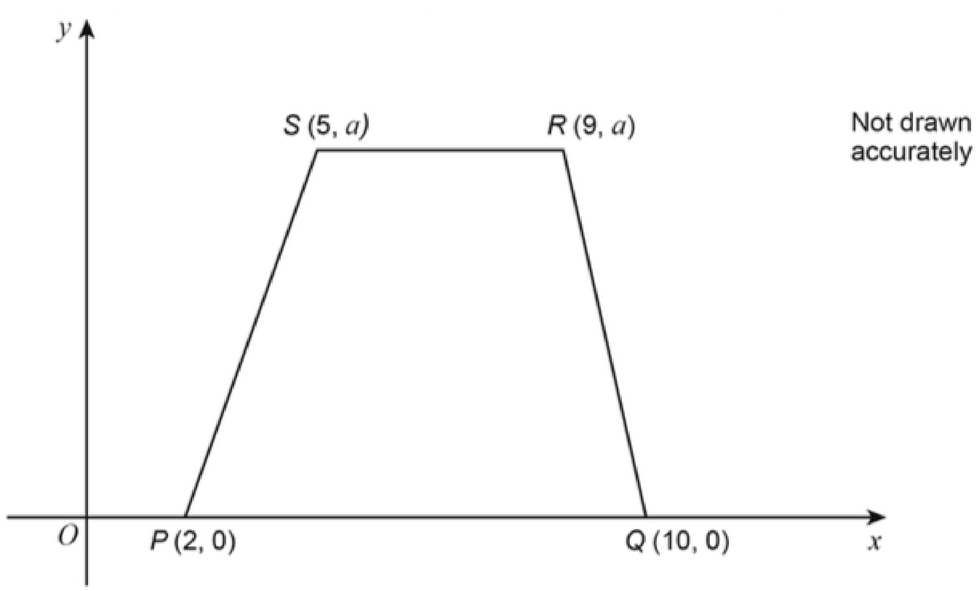
Solution

$$
\begin{aligned}
2\,\mathrm{f}(x) = \mathrm{g}(x) &\Rightarrow 2(x^2 - 7) = 1 - 3x \\
&\Rightarrow 2x^2 - 14 = 1 - 3x \\
&\Rightarrow 2x^2 + 3x - 15 = 0
\end{aligned}
$$

$a = 2$, $b = 3$, and $c = -15$:

$$
\begin{aligned}
&\Rightarrow x = \frac{-3 \pm \sqrt{(-3)^2 - 4 \times 2 \times (-15)}}{2 \times 2} \\
&\Rightarrow x = \frac{-3 \pm \sqrt{129}}{4} \\
&\Rightarrow x = -3.589\,454\,173,\ 2.089\,454\,173 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{x = -3.589,\ 2.089 \text{ (3 dp)}}}.
\end{aligned}
$$

7. PQRS is a trapezium. (2)

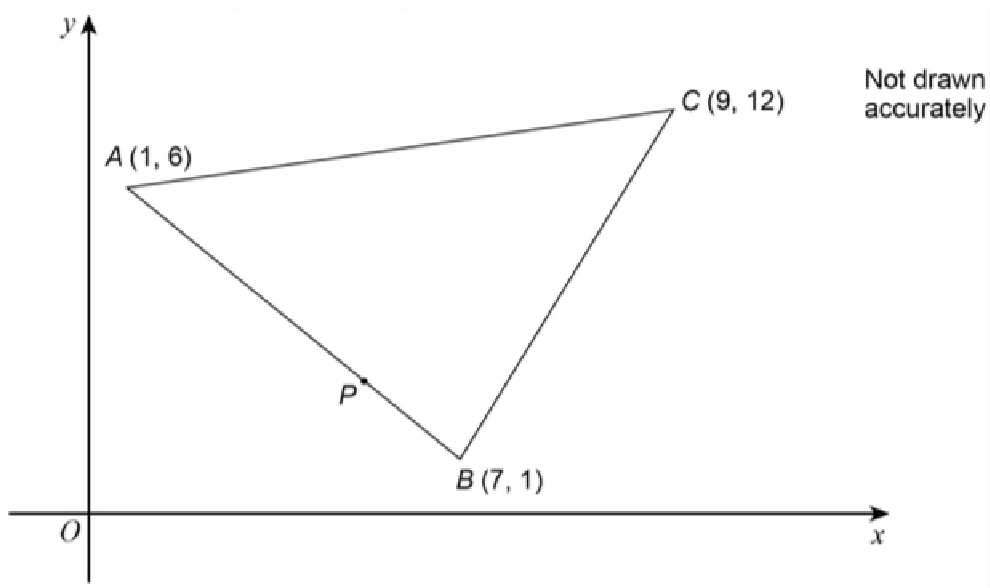


The area of the trapezium is 63 square units.
Work out the value of a.

Solution

$$\begin{aligned}\text{Area} = 63 &\Rightarrow \tfrac{1}{2}[(10-2)+(9-5)]a = 63\\ &\Rightarrow \tfrac{1}{2}(8+4)a = 63\\ &\Rightarrow 6a = 63\\ &\Rightarrow \underline{\underline{a = 10\tfrac{1}{2}}}.\end{aligned}$$

8. Here is a sketch of triangle ABC. (4)
   P is a point on AB.



$AP : PB$ is $3 : 1$.

Work out the length PC.
Give your answer to 4 significant figures.

Solution

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \overrightarrow{OA} + \tfrac{3}{4}\overrightarrow{AB} \\ &= \overrightarrow{OA} + \tfrac{3}{4}\begin{pmatrix} 7-1 \\ 1-6 \end{pmatrix} \\ &= \overrightarrow{OA} + \tfrac{3}{4}\begin{pmatrix} 6 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 6 \end{pmatrix} + \begin{pmatrix} 4.5 \\ -3.75 \end{pmatrix} \\ &= \begin{pmatrix} 5.5 \\ 2.25 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}PC &= \sqrt{(9-5.5)^2 + (12-2.25)^2} \\ &= 10.359\,174\,68 \text{ (FCD)} \\ &= \underline{\underline{10.36 \text{ (4 sf)}}}.\end{aligned}$$

9. (4)

$$y = \frac{2x^7 + 15x^2}{3x}.$$

Work out the value of x when

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 133.$$

Solution

$$y = \frac{2x^7 + 15x^2}{3x} = \tfrac{2}{3}x^6 + 5x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 4x^5 + 5.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 133 &\Rightarrow 4x^5 + 5 = 133 \\ &\Rightarrow x^5 = 32 \\ &\Rightarrow \underline{\underline{x = 2}}.\end{aligned}$$

10. The transformation matrix (5)

$$\begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$$

maps the point $(1, -3)$ onto the point $(1, 4)$.

Work out the values of a and b.
You must show your working.

Solution

$$\begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} a - 3b \\ 2a - 9b \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

which means

$$\begin{aligned} a - 3b &= 1 \quad (1) \\ 2a - 9b &= 4 \quad (2). \end{aligned}$$

Now, do $3 \times (1)$:

$$3a - 9b = 3 \quad (3)$$

and add $(3) - (2)$:

$$\underline{\underline{a = -1}} \Rightarrow -1 - 3b = 1$$

$$\begin{aligned} &\Rightarrow 3b = -2 \\ &\Rightarrow \underline{\underline{b = -\tfrac{2}{3}}}. \end{aligned}$$

11. Expand and simplify fully (3)

$$(x + 2)(x + 3)(x + 4).$$

Solution

$\times$	x	$+2$
x	x^2	$+2x$
$+3$	$+3x$	$+6$

×	x^2	$+5x$	$+6$
x	x^3	$+5x^2$	$+6x$
$+4$	$+4x^2$	$+20x$	$+24$

Hence,

$$(x+2)(x+3)(x+4) = \underline{\underline{x^3 + 9x^2 + 26x + 24}}.$$

12. (a) Write (3)

$$\frac{7}{9x} + \frac{2}{3x^2}$$

as a single fraction in its simplest form.

Solution

$$\begin{aligned} \frac{7}{9x} + \frac{2}{3x^2} &= \frac{7x}{9x^2} + \frac{6}{9x^2} \\ &= \underline{\underline{\frac{7x+6}{9x^2}}}. \end{aligned}$$

(b) Show that (4)

$$\frac{x^4}{x+4} \times \frac{x+2}{x} \div \frac{x^2}{3x+12}$$

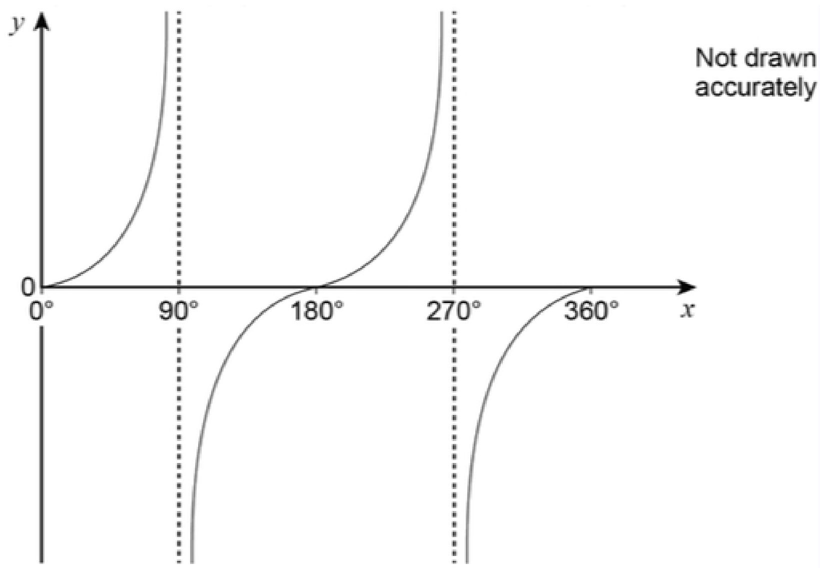
simplifies to the form $ax^2 + bx$ where a and b are integers.

Solution

$$\begin{aligned} \frac{x^4}{x+4} \times \frac{x+2}{x} \div \frac{x^2}{3x+12} &= \frac{x^4}{x+4} \times \frac{x+2}{x} \times \frac{3(x+4)}{x^2} \\ &= \frac{x}{1} \times \frac{x+2}{1} \times \frac{3}{1} \\ &= 3x(x+2) \\ &= \underline{\underline{3x^2 + 6x}}; \end{aligned}$$

hence, $\underline{\underline{a = 3}}$ and $\underline{\underline{b = 6}}$.

13. Here is a sketch of $y = \tan x$ for $0^\circ \leqslant x \leqslant 360^\circ$.



(a) **How many** solutions of $\tan x = k$ where $k > 0$ are between 90° and 360°? (1)

> **Solution**
> $\underline{\underline{1}}$.

$0 < p < 1$,

(b) **How many** solutions of $\sin x = p - 1$ are between 0° and 180°? (1)
You may use a sketch graph to help you.

> **Solution**
> $\underline{\underline{0}}$.

(c) State the coordinates of each point where the graph (2)

$$y = \cos x$$

for $0° \leqslant x \leqslant 360°$ meets or intersects an axis.

> **Solution**
> $\underline{\underline{(0, 1)}}$, $\underline{\underline{(90, 0)}}$, and $\underline{\underline{(270, 0)}}$.

14. (a) Factorise fully (2)

$$12pq^3r - 18pq^2r^2 + 24pq^2r.$$

Solution

$$12pq^3r - 18pq^2r^2 + 24pq^2r = \underline{\underline{6pq^2r(2q - 3r + 4)}}.$$

(b) Factorise fully (3)

$$6(y+3)^5 + 4(y+3)^4.$$

Give your answer in its simplest form.
Do **not** attempt to expand $(y+3)^5$ or $(y+3)^4$.

Solution

$$\begin{aligned} 6(y+3)^5 + 4(y+3)^4 &= 2(y+3)^4[3(y+3)+2] \\ &= \underline{\underline{2(y+3)^4(3y+11)}}. \end{aligned}$$

(c) Factorise fully (2)

$$48 - 75x^2.$$

Solution

The difference of two squares:

$$\begin{aligned} 48 - 75x^2 &= 3(16 - 25x^2) \\ &= 3[4^2 - (5x)^2] \\ &= \underline{\underline{3(4-5x)(4+5x)}}. \end{aligned}$$

15. Work out the rate of change of y with respect to x at the point on the curve (4)

$$y = x^2(x^2 - 9)$$

where $x = -2$.
You $\underline{\underline{\text{must}}}$ show your working.

Solution

$$\begin{aligned} y = x^2(x^2 - 9) &\Rightarrow y = x^4 - 9x^2 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 18x. \end{aligned}$$

Finally,

$$\begin{aligned} x = -2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 4[(-2)^3] - 18(-2) \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -32 + 36 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 4. \end{aligned}$$

16. (4)

$$A = 2 - 5x \qquad B = 3x - 1 \qquad C = x^2.$$

Show that

$$(2A + 3B)^2 \equiv A + B + C.$$

Solution

$$\begin{aligned} (2A + 3B)^2 &\equiv [2(2 - 5x) + 3(3x - 1)]^2 \\ &\equiv [4 - 10x - 9x - 3]^2 \\ &\equiv (1 - x)^2 \end{aligned}$$

$\times$	1	$-x$
1	1	$-x$
$-x$	$-x$	$+x^2$

$$\begin{aligned} &= 1 - 2x + x^2 \\ &= (2 - 5x) + (3x - 1) + x^2 \\ &\equiv \underline{\underline{A + B + C}}, \end{aligned}$$

as required.

17. A circle has equation

$$x^2 + y^2 = 29.$$

P is the point $(-5, 2)$.

(a) Show that P is on the circle. (1)

Solution

$$\begin{aligned} x^2 + y^2 &= (-5)^2 + 2^2 \\ &= 25 + 4 \\ &= 29; \end{aligned}$$

hence, P is on the circle.

The tangent to the circle at P intersects the x-axis at point Q.

(b) Work out the x-coordinate of Q. You **must** show your working. (4)

Solution

$$\begin{aligned} m_{OP} &= \frac{-2-0}{5-0} \\ &= -\tfrac{2}{5} \end{aligned}$$

which implies

$$m_{\text{normal}} = \tfrac{5}{2}.$$

Now, the normal to P is

$$y - 2 = \tfrac{5}{2}(x + 5)$$

and

$$\begin{aligned} y = 0 &\Rightarrow -\tfrac{4}{5} = x + 5 \\ &\Rightarrow \underline{\underline{x = -5\tfrac{4}{5}}}. \end{aligned}$$

18. (a) Work out all the integer values of x for which (3)

$$-5 < 4x + 3 \leqslant 13.$$

Solution

$$\begin{aligned} -5 < 4x + 3 \leqslant 13 &\Rightarrow -8 < 4x \leqslant 10 \\ &\Rightarrow -2 < x \leqslant 2\tfrac{1}{2}; \end{aligned}$$

hence, the integer values are $\underline{\underline{-1,\ 0,\ 1,\ \text{or}\ 2}}$.

(b) Work out the range of values of x for which (3)

$$x^2 - 11x + 28 > 0.$$

You **must** show your working.

Solution

$$\left.\begin{array}{ll}\text{add to:} & -11 \\ \text{multiply to:} & +28\end{array}\right\} -7, -4$$

$$\begin{aligned} x^2 - 11x + 28 > 0 &\Rightarrow (x-7)(x-4) > 0 \\ &\Rightarrow \underline{\underline{x < 4 \text{ or } x > 7}}. \end{aligned}$$

19. Use matrix multiplication to show that, in the $x - y$ plane, (5)

- a reflection in the line $y = -x$, followed by
- a rotation, 90° anticlockwise about the origin, followed by
- a reflection in the x-axis

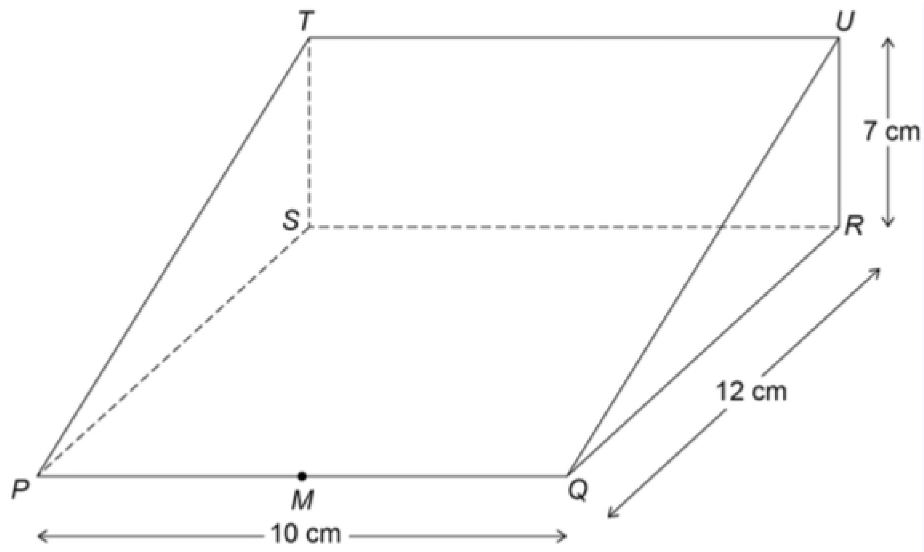
is equivalent to a transformation by the identity matrix.

Solution

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \end{aligned}$$

hence, the three transformations are equivalent to a transformation by the identity matrix.

20. $PQRSTU$ is a triangular prism.
$PQRS$ is a rectangle and angle $QRU = 90°$.
$PQ = 10$ cm.
$QR = 12$ cm.
$UR = 7$ cm.
M is the midpoint of PQ.



(a) Calculate the size of the angle between the line UM and the plane $PQRS$. (4)

Solution

Three-dimensional Pythagaros:

$$\begin{aligned} UM &= \sqrt{UR^2 + QR^2 + MQ^2} \\ &= \sqrt{7^2 + 12^2 + 5^2} \\ &= \sqrt{49 + 144 + 25} \\ &= \sqrt{218} \end{aligned}$$

and

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin\text{angle} = \frac{7}{\sqrt{218}} \\ &\Rightarrow \text{angle} = 28.300\,755\,77 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\text{angle} = 28.3^\circ \text{ (3 sf)}}}. \end{aligned}$$

(b) Calculate the size of the angle between the planes UMR and UQR. (2)

Solution

$$\begin{aligned} MR &= \sqrt{5^2 + 12^2} \\ &= 13 \end{aligned}$$

and

$$\sin = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin \text{angle} = \tfrac{5}{13}$$
$$\Rightarrow \text{angle} = 22.619\,864\,95 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{\text{angle} = 22.6^\circ \text{ (3 sf)}}}.$$

21. The continuous curve $y = \mathrm{f}(x)$ has exactly two stationary points. (3)
Here is some information about the curve.

$x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$x > 2$
$\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive	$\frac{\mathrm{d}y}{\mathrm{d}x}$ is zero	$\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative	$\frac{\mathrm{d}y}{\mathrm{d}x}$ is zero	$\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative

$\mathrm{f}(-1) = 3$ and $\mathrm{f}(2) = 1$.
State the coordinates and the nature of each of the stationary points.

Solution

Stationary point $\underline{\underline{(-1, 3)}}$, nature a maximum turning point.

Stationary point $\underline{\underline{(2, 1)}}$, nature a point of inflexion.

22.

$$8 \cos x + 5 \sin x = 0$$

where $90^\circ < x < 180^\circ$.

(a) Work out the size of angle x. (3)

Solution

$$8 \cos x + 5 \sin x = 0 \Rightarrow 5 \sin x = -8 \cos x$$
$$\Rightarrow \tan x = -\tfrac{8}{5}$$
$$\Rightarrow x = 122.005\,383\,2 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 122^\circ \text{ (3 sf)}}}.$$

$$6\sin^2 x + 4\cos^2 x \equiv A + B\cos^2 x,$$

where A and B are integers.

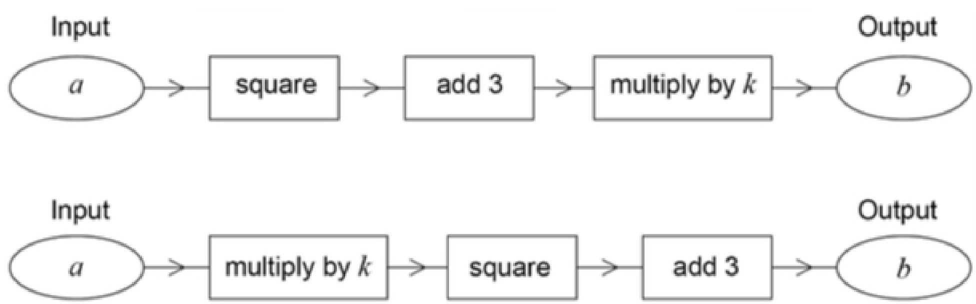
(b) Work out the values of A and B. (2)
You must show your working.

Solution

$$\begin{aligned} 6\sin^2 x + 4\cos^2 x &\equiv 6(1-\cos^2 x) + 4\cos^2 x \\ &\equiv 6 - 6\cos^2 x + 4\cos^2 x \\ &\equiv \underline{\underline{6 - 2\cos^2 x}}; \end{aligned}$$

hence, $\underline{\underline{A = 6}}$ and $\underline{\underline{B = -2}}$.

23. For each of these two function machines, when the input is a the output is b. (6)
$k > 0$ and $k \neq 1$ and $a > 0$.



Work out an expression for a in terms of k.
Give your answer in its simplest form.

Solution

$$\begin{aligned} b &= k(a^2+3) \quad (1) \\ b &= (ka)^2 + 3 \quad (2). \end{aligned}$$

Now, (1) = (2):

$$\begin{aligned} k(a^2+3) = (ka)^2 + 3 &\Rightarrow ka^2 + 3k = k^2a^2 + 3 \\ &\Rightarrow k(k-1)a^2 = 3(k-1) \\ &\Rightarrow ka^2 = 3 \text{ (as } k \neq 1) \\ &\Rightarrow a^2 = \frac{3}{k} \\ &\Rightarrow \underline{\underline{a = \sqrt{\frac{3}{k}}}}, \end{aligned}$$

as $a > 0$.

24. Work out the value of p when (4)

$$9^{0.5p} \times 81 = 27^{2p-1}.$$

Solution

$$\begin{aligned} 9^{0.5p} \times 81 = 27^{2p-1} &\Rightarrow (3^2)^{0.5p} \times 3^4 = (3^3)^{2p-1} \\ &\Rightarrow 3^p \times 3^4 = 3^{3(2p-1)} \\ &\Rightarrow 3^{p+4} = 3^{3(2p-1)} \\ &\Rightarrow p + 4 = 3(2p-1) \\ &\Rightarrow p + 4 = 6p - 3 \\ &\Rightarrow 5p = 7 \\ &\Rightarrow \underline{\underline{p = 1.4}}. \end{aligned}$$