Dr Oliver Mathematics
Mathematics: Advanced Higher
2007 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Express the binomial expansion of (4)

$$\left(x - \frac{2}{x}\right)^4$$

in the form

$$ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$$

for integers a, b, c, d, and e.

Solution

$\times$	x	$-\frac{2}{x}$
x	x^2	-2
$-\frac{2}{x}$	-2	$+\frac{4}{x^2}$

$$\left(x - \frac{2}{x}\right)^2 = x^2 - 4 + \frac{4}{x^2}$$

$\times$	x^2	-4	$+\frac{4}{x^2}$
x^2	x^4	$-4x^2$	$+4$
-4	$-4x^2$	$+16$	$-\frac{16}{x^2}$
$+\frac{4}{x^2}$	$+4$	$-\frac{16}{x^2}$	$+\frac{16}{x^4}$

$$\begin{aligned}\left(x - \frac{2}{x}\right)^4 &= \left(x - \frac{2}{x}\right)^2 \left(x - \frac{2}{x}\right)^2 \\ &= \underline{\underline{x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}}};\end{aligned}$$

hence, $\underline{\underline{a = 1}}$, $\underline{\underline{b = -8}}$, $\underline{\underline{c = 24}}$, $\underline{\underline{d = -32}}$, and $\underline{\underline{e = 16}}$.

2. Obtain the derivative of each of the following functions:

(a) $f(x) = \exp(\sin 2x)$, (3)

Solution

$$\begin{aligned} f(x) = \exp(\sin 2x) &\Rightarrow f'(x) = \exp(\sin 2x) \cdot \cos 2x \cdot 2 \\ &\Rightarrow \underline{\underline{f'(x) = 2\cos 2x \exp(\sin 2x)}}. \end{aligned}$$

(b) $y = 4^{(x^2+1)}$. (3)

Solution

$$\begin{aligned} y = 4^{(x^2+1)} &\Rightarrow \ln y = \ln 4^{(x^2+1)} \\ &\Rightarrow \ln y = (x^2+1)\ln 4 \\ &\Rightarrow \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 2x \ln 4 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy \ln 4 \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 2x(\ln 4)4^{(x^2+1)}}}. \end{aligned}$$

3. Show that (4)

$$z = 3 + 3\mathrm{i}$$

is a root of the equation

$$z^3 - 18z + 108 = 0$$

and obtain the remaining roots of the equation.

Solution

Another root is $\underline{\underline{z = 3 - 3\mathrm{i}}}$.

$\times$	z	-3	$-3\mathrm{i}$
z	z^2	$-3z$	$-3z\mathrm{i}$
-3	$-3z$	$+9$	$+9\mathrm{i}$
$+3\mathrm{i}$	$+3z\mathrm{i}$	$-9\mathrm{i}$	$+9$

Now,

$$(z-3-3\mathrm{i})(z-3+3\mathrm{i}) = z^2 - 6z + 18.$$

Next,

$$\frac{+108}{+18} = +6$$

and so we have

$$z^3 - 18z + 108 = (z^2 - 6z + 18)(z + 6)$$

and, finally, $\underline{\underline{z = -6}}$ is our third root.

4. (a) Express (3)

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} &\equiv \frac{2x^2 - 9x - 6}{x(x-3)(x+2)} \\ &\equiv \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2} \\ &\equiv \frac{A(x-3)(x+2) + Bx(x+2) + Cx(x-3)}{x(x-3)(x+2)}\end{aligned}$$

and so

$$2x^2 - 9x - 6 \equiv A(x-3)(x+2) + Bx(x+2) + Cx(x-3).$$

$\underline{x = 0}$: $-6 = -6A \Rightarrow A = 1$.
$\underline{x = -2}$: $20 = 10C \Rightarrow C = 2$.
$\underline{x = 3}$: $-15 = 15B \Rightarrow B = -1$.
Hence,

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \underline{\underline{\frac{1}{x} - \frac{1}{x-3} + \frac{2}{x+2}}}.$$

(b) Given that (3)

$$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \ln \frac{m}{n},$$

determine values for the integers m and n.

Solution

$$\begin{aligned}\int_4^6 \frac{2x^2-9x-6}{x(x^2-x-6)}\,\mathrm{d}x &= \int_4^6 \left(\frac{1}{x} - \frac{1}{x-3} + \frac{2}{x+2}\right)\,\mathrm{d}x \\ &= \left[\ln|x| - \ln|x-3| + 2\ln|x+2|\right]_{x=4}^{6} \\ &= (\ln 6 - \ln 3 + 2\ln 8) - (\ln 4 - \ln 1 + 2\ln 6) \\ &= \left(\ln 6 - \ln 3 + \ln 8^2\right) - \left(\ln 4 - \ln 1 + \ln 6^2\right) \\ &= \ln 128 - \ln 144 \\ &= \ln \tfrac{128}{144} \\ &= \underline{\underline{\ln \tfrac{8}{9}}};\end{aligned}$$

hence, $\underline{\underline{m = 8}}$ and $\underline{\underline{n = 9}}$.

5. Matrices **A** and **B** are defined by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

(a) Find the product **AB**. (2)

Solution

$$\mathbf{AB} = \underline{\underline{\begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}}}.$$

(b) Obtain the determinants of **A** and of **AB**. (2)

Solution

$$\begin{aligned}|\mathbf{A}| &= (1 \times 3) - (0 \times 0) + [(-1) \times 0] \\ &= \underline{\underline{3}}\end{aligned}$$

and

$$\begin{aligned}|\mathbf{AB}| &= (x \times 48) - [x \times (-48)] + (x \times 0) \\ &= \underline{\underline{96x}}.\end{aligned}$$

(c) Hence, or otherwise, obtain an expression for $\det \mathbf{B}$. (1)

Solution

$$\det \mathbf{B} = \frac{96x}{3} = \underline{\underline{32x}}.$$

6. (a) Find the Maclaurin series for $\cos x$ as far as the term in x^4. (2)

Solution

$$\cos x = \underline{\underline{1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \ldots}}.$$

(b) Deduce the Maclaurin series for $\mathrm{f}(x) = \frac{1}{2}\cos 2x$ as far as the term in x^4. (2)

Solution

$$\begin{aligned}\mathrm{f}(x) &= \tfrac{1}{2}\left[1 - \tfrac{1}{2}(2x)^2 + \tfrac{1}{24}(2x)^4 + \ldots\right] \\ &= \tfrac{1}{2}\left[1 - 2x^2 + \tfrac{2}{3}x^4 + \ldots\right] \\ &= \underline{\underline{\tfrac{1}{2} - x^2 + \tfrac{1}{3}x^4 + \ldots}}.\end{aligned}$$

(c) Hence write down the first three non-zero terms of the series for $\mathrm{f}(3x)$. (1)

Solution

$$\begin{aligned}\mathrm{f}(3x) &= \tfrac{1}{2} - (3x)^2 + \tfrac{1}{3}(3x)^4 + \ldots \\ &= \underline{\underline{\tfrac{1}{2} - 9x^2 + 27x^4 + \ldots}}.\end{aligned}$$

7. Use the Euclidean algorithm to find integers p and q such that (4)

$$599p + 53q = 1.$$

Solution

$$599 = 53 \times 11 + 16$$
$$55 = 16 \times 3 + 5$$
$$16 = 5 \times 3 + 1$$

and so we have

$$\begin{aligned} 1 &= 16 - 5 \times 3 \\ &= 16 - 3(55 - 16 \times 3) \\ &= 16 \times 10 - 55 \times 3 \\ &= 10(599 - 53 \times 11) - 55 \times 3 \\ &= \underline{\underline{599 \times 10 - 53 \times 113}}; \end{aligned}$$

hence, $\underline{\underline{p = 10}}$ and $\underline{\underline{q = -113}}$.

8. Obtain the general solution of the equation (6)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \mathrm{e}^{2x}.$$

Solution

Complementary function:

$$m^2 + 6m + 9 = 0 \Rightarrow (m+3)^2 = 0 \Rightarrow m = -3 \text{ (repeated root)}$$

and hence the complementary function is

$$y = (A + Bx)\mathrm{e}^{-3x}.$$

Particular integral: try

$$y = C\mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2C\mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4C\mathrm{e}^{2x}.$$

Substitute into the differential equation:

$$4C\mathrm{e}^{2x} + 12C\mathrm{e}^{2x} + 9C\mathrm{e}^{2x} \equiv \mathrm{e}^{2x} \Rightarrow C = \tfrac{1}{25}.$$

Hence the particular integral is $y = \frac{1}{25}\mathrm{e}^{2x}$.

General solution: hence the general solution is

$$\underline{\underline{y = (A + Bx)\mathrm{e}^{-3x} + \tfrac{1}{25}\mathrm{e}^{2x}}}.$$

9. (a) Show that (2)

$$\sum_{r=1}^{n}(4-6r) = n - 3n^2.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(4-6r) &= 4\sum_{r=1}^{n}1 - 6\sum_{r=1}^{n}r \\ &= 4n - 6 \times \tfrac{1}{2}n(n+1) \\ &= 4n - 3n(n+1) \\ &= 4n - 3n^2n - 3n \\ &= \underline{\underline{n - 3n^2}},\end{aligned}$$

as required.

(b) Hence write down a formula for (1)

$$\sum_{r=1}^{2q}(4-6r).$$

Solution

$$\begin{aligned}\sum_{r=1}^{2q}(4-6r) &= (2q) - 3(2q)^2 \\ &= \underline{\underline{2q - 12q^2}}.\end{aligned}$$

(c) Show that (2)

$$\sum_{r=q+1}^{2n}(4-6r) = q - 9q^2.$$

Solution

$$
\begin{aligned}
\sum_{r=q+1}^{2q}(4-6r) &= \sum_{r=1}^{2q}(4-6r) - \sum_{r=1}^{q}(4-6r) \\
&= (2q-12q^2) - (q-3q^2) \\
&= 2q - 12q^2 - q + 3q^2 \\
&= \underline{\underline{q - 9q^2}},
\end{aligned}
$$

as required.

10. (a) Use the substitution $u = 1 + x^2$ to obtain (5)

$$\int_0^1 \frac{x^3}{(1+x^2)^4}\,\mathrm{d}x.$$

Solution

$$
\begin{aligned}
u = 1 + x^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \\
&\Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x
\end{aligned}
$$

and

$$x = 0 \Rightarrow u = 1 \text{ and } x = 1 \Rightarrow u = 2.$$

Now,

$$
\begin{aligned}
\int_0^1 \frac{x^3}{(1+x^2)^4}\,\mathrm{d}x &= \tfrac{1}{2}\int_0^1 \frac{x^2}{(1+x^2)^4}\,2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_0^1 \frac{(1+x^2)-1}{(1+x^2)^4}\,2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_1^2 \frac{u-1}{u^4}\,\mathrm{d}u \\
&= \tfrac{1}{2}\int_1^2 (u^{-3} - u^{-4})\,\mathrm{d}u \\
&= \tfrac{1}{2}\left[-\tfrac{1}{2}u^{-2} + \tfrac{1}{3}u^{-3}\right]_{u=1}^{2} \\
&= \tfrac{1}{2}\left\{\left(-\tfrac{1}{8} + \tfrac{1}{24}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right\} \\
&= \underline{\underline{\tfrac{1}{24}}}.
\end{aligned}
$$

A solid is formed by rotating the curve

$$y = \frac{x^{\frac{3}{2}}}{(1+x^2)^2}$$

between $x = 0$ and $x = 1$ through 360° about the x-axis.

(b) Write down the volume of this solid. (1)

Solution

$$\begin{aligned} \text{Volume} &= \int_0^1 \pi \left(\frac{x^{\frac{3}{2}}}{(1+x^2)^2} \right)^2 \, \mathrm{d}x \\ &= \pi \int_0^1 \frac{x^3}{(1+x^2)^4} \, \mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{24}\pi}}. \end{aligned}$$

11. (a) Given that (3)

$$|z-2| = |z+\mathrm{i}|,$$

where $z = x + \mathrm{i}y$, show that

$$ax + by + c = 0$$

for suitable values of a, b, and c.

Solution

$$\begin{aligned} |z-2| = |z+\mathrm{i}| &\Rightarrow |x+\mathrm{i}y-2|^2 = |x+\mathrm{i}y+\mathrm{i}|^2 \\ &\Rightarrow (x-2)^2 + y^2 = x^2 + (y+1)^2 \\ &\Rightarrow (x^2 - 4x + 4) + y^2 = x^2 + (y^2 + 2y + 1) \\ &\Rightarrow -4x + 4 = 2y + 1 \\ &\Rightarrow \underline{\underline{4x + 2y - 3 = 0}}; \end{aligned}$$

hence, $\underline{\underline{a = 4}}$, $\underline{\underline{b = 2}}$, and $\underline{\underline{c = -3}}$.

(b) Indicate on an Argand diagram the locus of complex numbers z which satisfy (1)

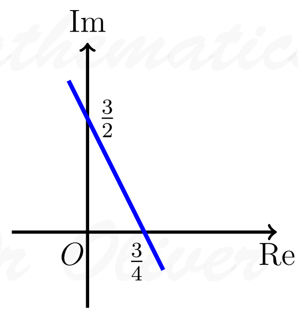
$$|z-2| = |z+\mathrm{i}|.$$

Solution

$$\begin{aligned} 4x + 2y - 3 = 0 &\Rightarrow 2y = -4x + 3 \\ &\Rightarrow y = -2x + \tfrac{3}{2} \end{aligned}$$

and

$$0 = -2x + \tfrac{3}{2} \Rightarrow x = \tfrac{3}{4}.$$



12. Prove by induction that, for $a > 0$, (5)

$$(1 + a)^n \geqslant 1 + na$$

for all positive integers n.

Solution

$\underline{n = 1}$: LHS $= (1 + a)^1 = 1 + a$ and RHS $= 1 + (1 \times a) = 1 + a$ and so the solution is true for $n = 1$.

Suppose the solution is true for $n = k$, i.e.,

$$(1 + a)^k \geqslant 1 + ka.$$

Then

$$\begin{aligned} (1 + a)^{k+1} &= (1 + a)(1 + a)^k \\ &\geqslant (1 + a)(1 + ka) \\ &= 1 + (k + 1)a + ka^2 \\ &> 1 + (k + 1)a \quad \text{(since } ka^2 > 0\text{)} \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, by mathematical induction, the expression is true for all $n \in \mathbb{N}$, as required.

13. A curve is defined by the parametric equations

$$x = \cos 2t,\ y = \sin 2t,\ 0 < t < \tfrac{\pi}{2}.$$

(a) Use parametric differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2\sin 2t \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 2\cos 2t.$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{2\cos 2t}{-2\sin 2t} \\ &= \underline{\underline{-\cot 2t}}.\end{aligned}$$

(b) Hence find the equation of the tangent when $t = \frac{\pi}{8}$. (2)

Solution

$$t = \tfrac{\pi}{8} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -1, x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$$

and the equation of the tangent is

$$\begin{aligned} y - \frac{\sqrt{2}}{2} = -(x - \frac{\sqrt{2}}{2}) &\Rightarrow y - \frac{\sqrt{2}}{2} = -x + \frac{\sqrt{2}}{2} \\ &\Rightarrow \underline{\underline{y = -x + \sqrt{2}}}.\end{aligned}$$

(c) Obtain an expression for $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and hence show that (5)

$$\sin 2t \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = k,$$

where k is an integer.
State the value of k.

Solution

$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}t}{\mathrm{d}x}\\ &= 2\operatorname{cosec}^2 2t \cdot (-\tfrac{1}{2}\operatorname{cosec} 2t)\\ &= -\operatorname{cosec}^3 2t.\end{aligned}$$

Finally,

$$\begin{aligned}\sin 2t\,\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= \sin 2t(-\operatorname{cosec}^3 2t) + (-\cot 2t)^2\\ &= -\operatorname{cosec}^2 2t + \cot^2 2t\\ &= -\frac{1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t}\\ &= \frac{\cos^2 2t - 1}{\sin^2 2t}\\ &= \frac{-\sin^2 2t}{\sin^2 2t}\\ &= \underline{\underline{-1}}.\end{aligned}$$

14. A garden centre advertises young plants to be used as hedging. After planting, the growth G metres (i.e., the increase in height) after t years is modelled by the differential equation

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{25k - G}{25},$$

where k is a constant and $G = 0$ when $t = 0$.

(a) Express G in terms of t and k. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{25k - G}{25} &\Rightarrow \frac{1}{25k - G}\,\mathrm{d}G = \frac{1}{25}\,\mathrm{d}t\\ &\Rightarrow \int \frac{1}{25k - G}\,\mathrm{d}G = \int \frac{1}{25}\,\mathrm{d}t\\ &\Rightarrow -\ln|25k - G| = \tfrac{1}{25}t + c.\end{aligned}$$

Now,

$$G = 0, t = 0 \Rightarrow -\ln 25k = c$$

and

$$
\begin{aligned}
-\ln|25k - G| = \tfrac{1}{25}t - \ln 25k &\Rightarrow \ln 25k - \ln|25k - G| = \tfrac{1}{25}t \\
&\Rightarrow \ln\left(\frac{25k}{|25k - G|}\right) = \tfrac{1}{25}t \\
&\Rightarrow \frac{25k}{|25k - G|} = \mathrm{e}^{\frac{1}{25}t} \\
&\Rightarrow \underline{\underline{25k\mathrm{e}^{-\frac{1}{25}t} = |25k - G|}}.
\end{aligned}
$$

Unless we ditch the absolute value signs:

$$
\begin{aligned}
25k\mathrm{e}^{-\frac{1}{25}t} = 25k - G &\Rightarrow G = 25k - 25k\mathrm{e}^{-\frac{1}{25}t} \\
&\Rightarrow \underline{\underline{G = 25k(1 - \mathrm{e}^{-\frac{1}{25}t})}}.
\end{aligned}
$$

(b) Given that a plant grows 0.6 metres by the end of 5 years, find the value of k correct to 3 decimal places. (2)

Solution

$$
\begin{aligned}
0.6 = 25k(1 - \mathrm{e}^{-\frac{1}{5}}) &\Rightarrow k = \frac{0.6}{25(1 - \mathrm{e}^{-\frac{1}{5}})} \\
&\Rightarrow k = 0.132\,399\,733\,6 \text{ (FCD)} \\
&\Rightarrow \underline{\underline{k = 0.132 \text{ (3 sf)}}}.
\end{aligned}
$$

(c) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? (2)

Solution

$$
\begin{aligned}
G &= 25k(1 - \mathrm{e}^{-\frac{2}{5}}) \\
&= 1.091\,238\,452 \text{ (FCD)}
\end{aligned}
$$

so, yes, this is claim justified

(d) Given that the initial height of the plants was 0.3 m, what is the likely long-term height of the plants? (2)

Solution

$$t \to \infty \Rightarrow G \to 3.309\,993\,34 \text{ (FCD)}$$

so the likely long-term height of the plants is

$$0.3 + 3.309\ldots = \underline{\underline{3.6 \text{ m}}}.$$

15. Lines L_1 and L_2 are given by the parametric equations

$$\begin{aligned} L_1: &\quad x = 2 + s,\ y = -s,\ z = 2 - s \\ L_2: &\quad x = -1 - 2t,\ y = t,\ z = 2 + 3t. \end{aligned}$$

(a) Show that L_1 and L_2 do not intersect. (3)

Solution

In L_2, replace t with $-s$ and then the z-component is $z = 2 - 3s$. Now,

$$2 - s = 2 - 3s \Rightarrow s = 0$$

and now apply that to the x-component: L_1's component is $x = 2$ and L_2's component is $x = -1$.
So L_1 and L_2 $\underline{\underline{\text{do not intersect}}}$.

The line L_3 passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both L_1 and L_2.

(b) Obtain parametric equations for L_3. (3)

Solution

In L_1,

$$\frac{x-2}{1} = \frac{y}{-1} = \frac{z-2}{-1}$$

and, in L_2,

$$\frac{x+1}{-2} = \frac{y}{1} = \frac{z-2}{3}.$$

Now,

$$\begin{aligned} (\mathbf{i}-\mathbf{j}-\mathbf{k}) \times (-2\mathbf{i}+\mathbf{j}+3\mathbf{k}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} \\ &= -2\mathbf{i}-\mathbf{j}-\mathbf{k}. \end{aligned}$$

Next, the equation for L_3 is

$$\begin{aligned}\mathbf{r} &= \mathbf{i}+\mathbf{j}+3\mathbf{k} + u(-2\mathbf{i}-\mathbf{j}-\mathbf{k}) \\ &= (1-2u)\mathbf{i}+(1-u)\mathbf{j}+(3-u)\mathbf{k}\end{aligned}$$

and so the parametric equations for L_3 are

$$\underline{\underline{x = 1-2u,\ y = 1-u,\ z = 3-u}}.$$

(c) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1. (3)

Solution

$$\begin{aligned}-1-2t = 1-2u &\Rightarrow 2u-2u = 2 \Rightarrow u-t = 1, \\ t = 1-u &\Rightarrow u+t = 1,\end{aligned}$$

and simplify:

$$u = 1 \text{ and } t = 0.$$

Therefore, $\underline{\underline{Q(-1,0,2)}}$.

Does P lies on L_1?

$$2+s = 1 \Rightarrow s = -1$$

and check in the y- and z-component:

$$y = -(-1) = 1 \text{ and } z = 2-(-1) = 3.$$

Hence $\underline{\underline{P \text{ lies on } L_1}}$.

PQ is the shortest distance between the lines L_1 and L_2.

(d) Calculate PQ. (1)

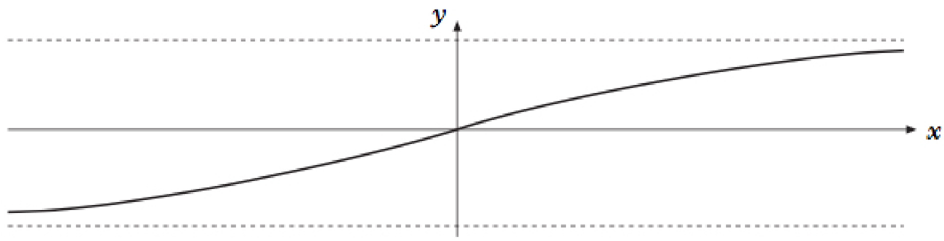
Solution

$$\begin{aligned}PQ &= \sqrt{2^2+1^2+1^2} \\ &= \underline{\underline{\sqrt{6}}}.\end{aligned}$$

16. (a) The diagram shows part of the graph of (2)

$$f(x) = \tan^{-1} 2x$$

and its asymptotes.



State the equations of these asymptotes.

Solution

$\underline{\underline{y = \frac{\pi}{2}}}$ and $\underline{\underline{y = -\frac{\pi}{2}}}$.

(b) Use integration by parts to find the area between $f(x)$, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$. (5)

Solution

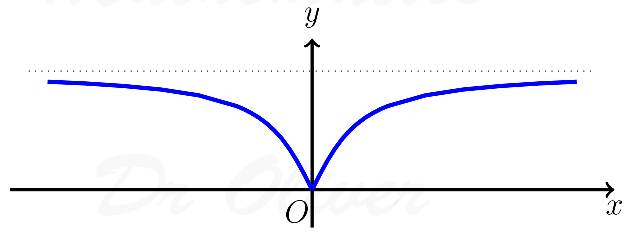
$$u = \tan^{-1} 2x \Rightarrow \frac{du}{dx} = \frac{2}{(2x)^2 + 1} \text{ and } \frac{dv}{dx} = 1 \Rightarrow v = x.$$

Now,

$$\begin{aligned}
\int_0^{\frac{1}{2}} \tan^{-1} 2x \, dx &= \left[x \tan^{-1} 2x\right]_{x=0}^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{2x}{4x^2 + 1} \, dx \\
&= \left(\tfrac{\pi}{8} - 0\right) - \tfrac{1}{4} \int_0^{\frac{1}{2}} \frac{8x}{4x^2 + 1} \, dx \\
&= \tfrac{\pi}{8} - \tfrac{1}{4} \left[\ln \left|4x^2 + 1\right|\right]_{x=0}^{\frac{1}{2}} \\
&= \tfrac{\pi}{8} - \tfrac{1}{4} (\ln 2 - 0) \\
&= \underline{\underline{\tfrac{\pi}{8} - \tfrac{1}{4} \ln 2}}.
\end{aligned}$$

(c) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x axis, and the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. (3)

Solution



$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \tan^{-1} 2x \, \mathrm{d}x = 2\int_{0}^{\frac{1}{2}} \tan^{-1} 2x \, \mathrm{d}x$$

$$= \underline{\underline{\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2}}.$$