Dr Oliver Mathematics
Further Mathematics
Binomial Distribution
Past Examination Questions

This booklet consists of 36 questions across a variety of examination topics.
The total number of marks available is 282.

	Symbol	Expectation	Variance	Continuity Correction?
Binomial	$\mathrm{B}(n, p)$	np	$np(1-p)$	No
Poisson	$\mathrm{Po}(\lambda)$	$\lambda = np$	$\lambda = np$	No
Normal	$\mathrm{N}(\mu, \sigma^2)$	$\mu = np$	$\sigma^2 = np(1-p)$	Yes

1. A farmer noticed that some of the eggs laid by his hens had double yolks. He estimated the probability of this happening to be 0.05. Eggs are packed in boxes of 12.

   Find the probability that in a box, the number of eggs with double yolks will be

   (a) exactly one, (3)

   Solution
   Let X represent the number of double yolks in a box of eggs $\therefore X \sim \mathrm{B}(12, 0.05)$. Then

$$\begin{aligned} \mathrm{P}(X = 1) &= \mathrm{P}(X \geqslant 1) - \mathrm{P}(X \geqslant 0) \\ &= 0.8816 - 0.05404 \text{ (from the tables)} \\ &= \underline{\underline{0.3412 \text{ (4 dp)}}}. \end{aligned}$$

   (b) more than three. (2)

   Solution

$$\begin{aligned} \mathrm{P}(X > 3) &= 1 - \mathrm{P}(X \geqslant 3) \\ &= 1 - 0.9978 \text{ (from the tables)} \\ &= \underline{\underline{0.0022 \text{ (4 dp)}}}. \end{aligned}$$

   A customer bought three boxes.

(c) Find the probability that only 2 of the boxes contained exactly 1 egg with a double yolk. (3)

Solution

$$\begin{aligned}\text{P(only two)} &= \binom{3}{2}(0.3412)^2(0.6588) \\ &= 0.230\,087\,428\,4 \text{ (FCD)} \\ &= \underline{\underline{0.2301 \text{ (4 dp)}}}.\end{aligned}$$

The farmer delivered 10 boxes to a local shop.

(d) Using a suitable approximation, find the probability that the delivery contained at least 9 eggs with double yolks. (4)

Solution

Let Y represent the number of double yolks in 10 boxes of eggs
$\therefore Y \sim \text{B}(120, 0.05)$.
Now,

$$\text{E}(Y) = 120 \times 0.05 = 6 \text{ and } \text{Var}(Y) = 120 \times 0.05 \times 0.95 = 5.7;$$

the expectation and the variance are almost equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \text{Po}(6)$. Then

$$\begin{aligned}\text{P}(Y \geqslant 9) &= 1 - \text{P}(Y \leqslant 8) \\ &= 1 - 0.8472 \text{ (from the tables)} \\ &= \underline{\underline{0.1528}}.\end{aligned}$$

2. (a) State two conditions under which a random variable can be modelled by a binomial distribution. (2)

Solution

e.g., constant probability of success, fixed number of trials, independent trials, success and failure

In the production of a certain electronic component it is found that 10% are defective. The component is produced in batches of 20.

(b) Write down a suitable model for the distribution of defective components in a batch. (1)

Solution

Let X represent the number of defective components $\therefore X \sim \text{B}(20, 0.1)$.

Find the probability that a batch contains

(c) no defective components, (2)

Solution

$\text{P}(X = 0) = 0.1216$ (from the tables)

(d) more than 6 defective components. (2)

Solution

$$\begin{aligned} \text{P}(X > 6) &= 1 - \text{P}(X \geqslant 6) \\ &= 1 - 0.9976 \text{ (from the tables)} \\ &= 0.0024. \end{aligned}$$

(e) Find the mean and the variance of the defective components in a batch. (2)

Solution

Mean $= 20 \times 0.1 = 2$.

Variance $= 20 \times 0.1 \times 0.9 = 1.8$.

A supplier buys 100 components. The supplier will receive a refund if there are more than 15 defective components.

(f) Using a suitable approximation, find the probability that the supplier will receive a refund. (4)

Solution

Let Y represent the number of defective components of 100 $\therefore Y \sim \text{B}(100, 0.1)$. Now,

$$\text{E}(Y) = 100 \times 0.1 = 10 \text{ and } \text{Var}(Y) = 100 \times 0.1 \times 0.9 = 9;$$

the expectation and the variance are almost equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \text{Po}(10)$. Then

$$\begin{aligned} \text{P}(Y > 15) &= 1 - \text{P}(Y \leqslant 15) \\ &= 1 - 0.9513 \text{ (from the tables)} \\ &= \underline{\underline{0.0497}}. \end{aligned}$$

3. The random variables R are distributed as follows $R \sim \text{B}(15, 0.3)$. Find $\text{P}(R = 5)$. (2)

Solution

$$\begin{aligned} \text{P}(R = 5) &= \text{P}(R \leqslant 5) - \text{P}(R \leqslant 4) \\ &= 0.7216 - 0.5155 \text{ (from the tables)} \\ &= \underline{\underline{0.2061}}. \end{aligned}$$

4. In an experiment, there are 250 trials and each trial results in a success or a failure. Write down two other conditions needed to make this into a binomial experiment. (2)

Solution

e.g., constant probability of success, fixed number of trials, independent trials

5. From company records, a manager knows that the probability that a defective article is produced by a particular production line is 0.032.

A random sample of 10 articles is selected from the production line.

(a) Find the probability that exactly 2 of them are defective. (3)

Solution

Let X represent the number of defective articles $\therefore X \sim \mathrm{B}(10, 0.032)$. Then

$$\begin{aligned} \mathrm{P}(X = 2) &= \binom{10}{2}(0.032)^2(0.968)^8 \\ &= 0.035\,523\,465\,66 \text{ (FCD)} \\ &= \underline{\underline{0.0036 \text{ (4 dp)}}}. \end{aligned}$$

On another occasion, a random sample of 100 articles is taken.

(b) Using a suitable approximation, find the probability that fewer than 4 of them are defective. (4)

Solution

Let Y represent the number of defective articles of 100 $\therefore Y \sim \mathrm{B}(100, 0.032)$. Now,

$$\mathrm{E}(Y) = 100 \times 0.032 = 3.2 \text{ and } \mathrm{Var}(Y) = 100 \times 0.032 \times 0.968 = 3.0976;$$

the expectation and the variance are almost (large n and small p) equal so we try the Poisson distribution $\therefore Y \approx\sim \mathrm{Po}(3.2)$. Then

$$\begin{aligned} \mathrm{P}(Y < 4) &= \mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) + \mathrm{P}(Y = 2) + \mathrm{P}(Y = 3) \\ &= \mathrm{e}^{-3.2}\left[1 + 3.2 + \frac{3.2^2}{2!} + \frac{3.2^3}{3!}\right] \\ &= 0.602\,519\,724\,4 \text{ (FCD)} \\ &= \underline{\underline{0.6025 \text{ (4 dp)}}}. \end{aligned}$$

At a later date, a random sample of 1000 is taken.

(c) Using a suitable approximation, find the probability that more than 42 are defective. (6)

Solution

Let A represent the number of defective articles of 1000 $\therefore A \sim \mathrm{B}(1000, 0.032)$. Now,

$$\mathrm{E}(A) = 1000 \times 0.032 = 32 \text{ and } \mathrm{Var}(A) = 1000 \times 0.032 \times 0.968 = 30.976;$$

$np = 32$ and $nq = 968$ so we use a normal approximation: $A \approx\sim \mathrm{N}(32, 30.976)$. Then

$$\begin{aligned} \mathrm{P}(A > 42) &= \mathrm{P}(A \geqslant 42.5) \\ &= 1 - \mathrm{P}(A \leqslant 42.5) \\ &= 1 - \Phi\left(\frac{42.5 - 32}{\sqrt{30.976}}\right) \\ &= 1 - \Phi(1.89) \\ &= 1 - 0.9706 \text{ (from the tables)} \\ &= \underline{\underline{0.0294}} \end{aligned}$$

6. It is estimated that 4% of people have green eyes. In a random sample of size n, the expected number of people with green eyes is 5.

(a) Calculate the value of n. (3)

Solution

Let X represent the number of green-eyed people $\therefore X \sim \mathrm{B}(n, 0.04)$. Then

$$np = \mathrm{E}(X) \Rightarrow 0.04n = 5 \Rightarrow \underline{\underline{n = 125}}.$$

The expected number of people with green eyes in a second random sample is 3.

(b) Find the standard deviation of the number of people with green eyes in this second sample. (4)

Solution

Let Y represent the number of green-eyed people $\therefore Y \sim \mathrm{B}(3, 0.04)$. Then

$$\begin{aligned} \mathrm{Var}(Y) = 3 \times 0.96 &\Rightarrow \mathrm{Var}(Y) = 2.88 \\ &\Rightarrow \sigma_Y = 1.697\,056\,275 \text{ (FCD)} \\ &\Rightarrow \sigma_Y = \underline{\underline{1.70 \text{ (3 sf)}}}. \end{aligned}$$

7. In a manufacturing process, 2% of the articles produced are defective. A batch of 200 articles is selected.

(a) Giving a justification for your choice, use a suitable approximation to estimate the probability that there are exactly 5 defective articles. (5)

Solution

Let X represent the number of defective articles $\therefore X \sim \mathrm{B}(200, 0.02)$. Now,

$$\mathrm{E}(X) = 200 \times 0.02 = 4 \text{ and } \mathrm{Var}(X) = 200 \times 0.02 \times 0.98 = 3.92;$$

the expectation and the variance are almost equal (large n and small p) so we try the Poisson distribution $\therefore X \approx\sim \mathrm{Po}(4)$. Now,

$$\begin{aligned} \mathrm{P}(X = 5) &= \frac{\mathrm{e}^{-4}\, 4^5}{5!} \\ &= 0.156\,293\,345\,19 \text{ (FCD)} \\ &= \underline{\underline{0.1563 \text{ (4 dp)}}}. \end{aligned}$$

(b) Estimate the probability there are less than 5 defective articles. (2)

Solution

$$\begin{aligned} \mathrm{P}(X < 5) &= \mathrm{P}(X \leqslant 4) \\ &= \underline{\underline{0.6288}} \text{ (from the tables)}. \end{aligned}$$

8. A drugs company claims that 75% of patients suffering from depression recover when treated with a new drug.

A random sample of 10 patients with depression is taken from a doctor's records.

(a) Write down a suitable distribution to model the number of patients in this sample who recover when treated with the new drug. (2)

Solution

Let X represent of patients suffering from depression $\therefore \underline{\underline{X \sim \mathrm{B}(10, 0.75)}}$.

Given that the claim is correct,

(b) find the probability that the treatment will be successful for exactly 6 patients. (2)

Solution

We need to 'reverse' the table: for example, $P(X = 1)$ we take to be $P(X = 9)$.

$$\begin{aligned} P(X = 6) &= P(X \leqslant 6) - P(X \leqslant 5) \\ &= 0.9219 - 0.7759 \text{ (from the tables)} \\ &= \underline{\underline{0.1460}}. \end{aligned}$$

[We do $P(X \leqslant 4) - P(X \leqslant 3)$ instead of $P(X \leqslant 6) - P(X \leqslant 5)$ because we pair $0, 10$ and $1, 9$ and $2, 8$ and $3, 8$ and $4, 6$ and $5, 5$.]

9. A fair coin is tossed 4 times.

Find the probability that

(a) an equal number of heads and tails occur, (2)

Solution
Let X represent the number of heads $\therefore X \sim B(4, 0.5)$. Then

$$P(X = 2) = \binom{4}{2}(0.5)^2(0.5)^2 = \underline{\underline{0.375}}.$$

(b) all the outcomes are the same, (3)

Solution

$$\begin{aligned} P(\text{all the outcomes are the same}) &= P(X = 0) + P(X = 4) \\ &= 0.0625 + 0.0625 \\ &= \underline{\underline{0.125}}. \end{aligned}$$

(c) the first tail occurs on the third throw. (2)

Solution
$P(HHT) = 0.5 \times 0.5 \times 0.5 = \underline{\underline{0.125}}$.

10. The random variable $X \sim B(150, 0.02)$. (4)

Use a suitable approximation to estimate $P(X > 7)$.

Solution

Now,

$$\mathrm{E}(X) = 150 \times 0.02 = 3 \text{ and } \mathrm{Var}(X) = 150 \times 0.02 \times 0.98 = 2.94;$$

the expectation and the variance are almost equal (large n and small p) so we try the Poisson distribution $\therefore X \approx\sim \mathrm{Po}(3)$.

$$\begin{aligned} \mathrm{P}(X > 7) &= 1 - \mathrm{P}(X \leqslant 6) \\ &= 1 - 0.9881 \text{ (from the tables)} \\ &= \underline{\underline{0.0119}}. \end{aligned}$$

11. A manufacturer produces large quantities of coloured mugs. It is known from previous records that 6% of the production will be green. A random sample of 10 mugs was taken from the production line.

(a) Define a suitable distribution to model the number of green mugs in this sample. (1)

Solution

Let X represent the number of green mugs $\therefore X \sim \mathrm{B}(10, 0.06)$.

(b) Find the probability that there were exactly 3 green mugs in the sample. (3)

Solution

$$\begin{aligned} \mathrm{P}(X = 3) &= \binom{10}{3}(0.06)^3(0.94)^7 \\ &= 0.016\,808\,539\,24 \text{ (FCD)} \\ &= \underline{\underline{0.0168 \text{ (4 dp)}}}. \end{aligned}$$

A random sample of 125 mugs was taken.

(c) Find the probability that there were between 10 and 13 (inclusive) green mugs in this sample, using

(i) a Poisson approximation, (3)

Solution

Let Y represent the number of green mugs $\therefore Y \sim \mathrm{B}(125, 0.06)$. Now,

$$\mathrm{E}(Y) = 125 \times 0.06 = 7.5 \text{ and } \mathrm{Var}(Y) = 125 \times 0.06 \times 0.94 = 7.05;$$

the expectation and the variance are almost equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \text{Po}(7.5)$.

$$\begin{aligned} \text{P}(10 \leqslant Y \leqslant 13) &= \text{P}(Y \leqslant 13) - \text{P}(Y \leqslant 9) \\ &= 0.9784 - 0.7764 \text{ (from the tables)} \\ &= \underline{\underline{0.2020}}. \end{aligned}$$

(ii) a Normal approximation. (6)

Solution
Also, $Y \approx\sim \text{N}(7.5, 7.05)$.

$$\begin{aligned} \text{P}(10 \leqslant Y \leqslant 13) &= \text{P}(9.5 \leqslant Y \leqslant 13.5) \\ &= \Phi\left(\frac{9.5 - 7.5}{\sqrt{7.05}} \leqslant Z \leqslant \frac{13.5 - 7.5}{\sqrt{7.05}}\right) \\ &= \Phi\left(0.75 \leqslant Z \leqslant 2.26\right) \\ &= 0.9881 - 0.7734 \text{ (from the tables)} \\ &= \underline{\underline{0.2147}}. \end{aligned}$$

12. The random variable K has a binomial distribution with parameters $n = 25$ and $p = 0.27$.

Find $\text{P}(K \leqslant 1)$.

Solution

$$\begin{aligned} \text{P}(K \leqslant 1) &= \text{P}(K = 0) + \text{P}(K = 1) \\ &= (0.73)^{25} + 25(0.73)^{24}(0.27) \\ &= 0.003\,923\,288\,459 \text{ (FCD)} \\ &= \underline{\underline{0.0039 \text{ (4 dp)}}}. \end{aligned}$$

13. For a particular type of plant 45% have white flowers and the remainder have coloured flowers. Gardenmania sells plants in batches of 12. A batch is selected at random.

Calculate the probability that this batch contains

(a) exactly 5 plants with white flowers, (3)

Solution
Let X represent the number of white flowers $\therefore X \sim \mathrm{B}(12, 0.45)$.

$$\begin{aligned}\mathrm{P}(X = 5) &= \mathrm{P}(X \leqslant 5) - \mathrm{P}(X \leqslant 4) \\ &= 0.5269 - 0.3044 \text{ (from the tables)} \\ &= \underline{\underline{0.2225}}.\end{aligned}$$

(b) more plants with white flowers than coloured ones. (2)

Solution

$$\begin{aligned}\mathrm{P}(X \geqslant 7) &= 1 - \mathrm{P}(X \leqslant 6) \\ &= 1 - 0.7393 \text{ (from the tables)} \\ &= \underline{\underline{0.2607}}.\end{aligned}$$

Gardenmania takes a random sample of 10 batches of plants.

(c) Find the probability that exactly 3 of these batches contain more plants with white flowers than coloured ones. (3)

Solution

$$\begin{aligned}\mathrm{P}(\text{exactly three}) &= \binom{10}{3}(0.2607)^3(0.7393)^7 \\ &= 0.256\,654\,736\,2 \text{ (FCD)} \\ &= \underline{\underline{0.2567 \text{ (4 dp)}}}.\end{aligned}$$

Due to an increasing demand for these plants by large companies, Gardenmania decides to sell them in batches of 50.

(d) Use a suitable approximation to calculate the probability that a batch of 50 plants contains more than 25 plants with white flowers. (7)

Solution
Let Y represent the number of white flowers $\therefore Y \sim \mathrm{B}(50, 0.45)$. Now,

$$\mathrm{E}(Y) = 50 \times 0.45 = 22.5 \text{ and } \mathrm{Var}(Y) = 50 \times 0.45 \times 0.55 = 12.375;$$

we have large n and p close to 0.5 so we try the Normal distribution
$\therefore Y \approx\sim \text{N}(22.5, 12.375)$.

$$\begin{aligned} \text{P}(Y > 25) &= \text{P}(Y \geqslant 26) \\ &= \text{P}(Y \geqslant 25.5) \\ &= 1 - \text{P}(Y \leqslant 25.5) \\ &= 1 - \text{P}\left(Z \leqslant \frac{25.5 - 22.5}{\sqrt{12.375}}\right) \\ &= 1 - \Phi(0.85) \\ &= 1 - 0.8023 \text{ (from the tables)} \\ &= \underline{\underline{0.1977}}. \end{aligned}$$

14. (a) (i) Write down two conditions for $X \sim \text{B}(n, p)$ to be approximated by a normal distribution $Y \sim \text{N}(\mu, \sigma^2)$. (2)

Solution
$\underline{\underline{\text{large } n}}$, $\underline{\underline{n > 10}}$, or $\underline{\underline{np > 5}}$ **and** $\underline{\underline{nq > 5}}$

$\underline{\underline{p \text{ close to } 0.5}}$

(ii) Write down the mean and variance of this normal approximation in terms of n and p. (2)

Solution
$\text{E}(X) = \underline{\underline{np}}$

$\text{Var}(X) = \underline{\underline{npq}}$ or $\underline{\underline{np(1-p)}}$

A factory manufactures 2000 DVDs every day. It is known that 3% of DVDs are faulty.

(b) Using a normal approximation, estimate the probability that at least 40 faulty DVDs are produced in one day. (5)

Solution

Using $Y \sim \mathrm{N}(60, 58.2)$,

$$\begin{aligned}
\mathrm{P}(Y \geqslant 40) &= \mathrm{P}(Y \geqslant 39.5) \\
&= \mathrm{P}\left(Z \geqslant \frac{39.5 - 60}{\sqrt{58.2}}\right) \\
&= 1 - \mathrm{P}\left(Z \leqslant \frac{39.5 - 60}{\sqrt{58.2}}\right) \\
&= 1 - \Phi(-2.70) \text{ (it is the closest we can get to } -2.687\ldots) \\
&= 1 - (1 - \Phi(2.70)) \\
&= \Phi(2.70) \\
&= \underline{\underline{0.9965}} \text{ (from the tables).}
\end{aligned}$$

The quality control system in the factory identifies and destroys every faulty DVD at the end of the manufacturing process. It costs £0.70 to manufacture a DVD and the factory sells non-faulty DVDs for £11.

(c) Find the expected profit made by the factory per day. (3)

Solution
Using $\mathrm{E}(Y) = 60$,

$$\begin{aligned}
\text{expected profit} &= 1940 \times 11 - 2000 \times 0.70 \\
&= 21\,340 - 1\,400 \\
&= \underline{\underline{£19\,940}}.
\end{aligned}$$

15. The probability of a bolt being faulty is 0.3. Find the probability that in a random sample of 20 bolts there are

(a) exactly 2 faulty bolts, (2)

Solution
Let X represent the number of faulty bolts $\therefore X \sim \mathrm{B}(20, 0.3)$.

$$\begin{aligned}
\mathrm{P}(X = 2) &= \mathrm{P}(X \leqslant 2) - \mathrm{P}(X \leqslant 1) \\
&= 0.0355 - 0.0076 \text{ (from the tables)} \\
&= \underline{\underline{0.0279}}.
\end{aligned}$$

(b) more than 3 faulty bolts. (2)

Solution

$$\begin{aligned} \mathrm{P}(X > 3) &= \mathrm{P}(X \geqslant 4) \\ &= 1 - \mathrm{P}(X \leqslant 3) \\ &= 1 - 0.1071 \text{ (from the tables)} \\ &= \underline{\underline{0.8929}}. \end{aligned}$$

These bolts are sold in bags of 20. John buys 10 bags.

(c) Find the probability that exactly 6 of these bags contain more than 3 faulty bolts. (3)

Solution

Let Y represent the number of of faulty bolts $\therefore Y \sim \mathrm{B}(200, 0.3)$.

$$\begin{aligned} \mathrm{P}(Y = 6) &= \binom{10}{6}(0.8929)^6(0.1071)^4 \\ &= 0.014\,002\,130\,34 \text{ (FCD)} \\ &= \underline{\underline{0.0140 \text{ (4 dp)}}}. \end{aligned}$$

16. Each cell of a certain animal contains 11000 genes. It is known that each gene has a probability 0.0005 of being damaged.

A cell is chosen at random.

(a) Suggest a suitable model for the distribution of the number of damaged genes in the cell. (2)

Solution

Let X represent the number of faulty bolts
$\therefore \underline{\underline{X \sim \mathrm{B}(11000, 0.005)}}$.

(b) Find the mean and variance of the number of damaged genes in the cell. (2)

Solution

$E(X) = 11000 \times 0.0005 = \underline{\underline{5.5}}$

$\text{Var}(X) = 11000 \times 0.0005 \times 0.995 = \underline{\underline{5.4725}}$

(c) Using a suitable approximation, find the probability that there are at most 2 damaged genes in the cell. (4)

Solution

We have large n and small p so we try the Poisson distribution

$\therefore X \approx\sim \text{Po}(5.5)$. Now,

$$P(X \leqslant 2) = \underline{\underline{0.0884}} \text{ (from the tables)}$$

17. Sue throws a fair coin 15 times and records the number of times it shows a head.

(a) State the distribution to model the number of times the coin shows a head. (2)

Solution

Let X represent the number of heads

$\therefore \underline{\underline{X \sim B(15, 0.5)}}$.

Find the probability that Sue records

(b) exactly 8 heads, (2)

Solution

$$\begin{aligned} P(X = 8) &= P(X \leqslant 8) - P(X \leqslant 7) \\ &= 0.6964 - 0.5000 \text{ (from the tables)} \\ &= \underline{\underline{0.1964}}. \end{aligned}$$

(c) at least 4 heads. (2)

Solution

$$\begin{aligned} P(X \geqslant 4) &= 1 - P(X \leqslant 3) \\ &= 1 - 0.0176 \text{ (from the tables)} \\ &= \underline{\underline{0.9824}}. \end{aligned}$$

18. A factory produces components of which 1% are defective. The components are packed in boxes of 10. A box is selected at random.

(a) Find the probability that the box contains exactly one defective component. (2)

Solution

Let X represent the number of defective components $\therefore X \sim \mathrm{B}(10, 0.01)$.

$$\begin{aligned}\mathrm{P}(X = 1) &= \binom{10}{1}(0.01)^1(0.99)^9 \\ &= 0.091\,351\,772\,475 \text{ (FCD)} \\ &= \underline{\underline{0.0914 \text{ (4 dp)}}}.\end{aligned}$$

(b) Find the probability that there are at least 2 defective components in the box. (3)

Solution

$$\begin{aligned}\mathrm{P}(X \geqslant 2) &= 1 - \mathrm{P}(X \leqslant 1) \\ &= 1 - (0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 \\ &= 0.004\,266\,200\,243 \text{ (FCD)} \\ &= \underline{\underline{0.0043 \text{ (4 dp)}}}.\end{aligned}$$

(c) Using a suitable approximation, find the probability that a batch of 250 components contains between 1 and 4 (inclusive) defective components. (4)

Solution

Let Y represent the number of defective components of 250 $\therefore Y \sim \mathrm{B}(250, 0.01)$. Now,

$$\mathrm{E}(Y) = 250 \times 0.01 = 2.5 \text{ and } \mathrm{Var}(Y) = 250 \times 0.01 \times 0.99 = 2.475;$$

the expectation and the variance are almost equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \mathrm{Po}(2.5)$. Then

$$\begin{aligned}\mathrm{P}(1 \leqslant Y \leqslant 4) &= \mathrm{P}(Y \leqslant 4) - \mathrm{P}(Y = 0) \\ &= 0.8912 - 0.0821 \text{ (from the tables)} \\ &= \underline{\underline{0.8091}}.\end{aligned}$$

19. A bag contains a large number of counters of which 15% are coloured red. A random sample of 30 counters is selected and the number of red counters is recorded.

(a) Find the probability of no more than 6 red counters in this sample. (2)

Solution

Let X represent the number of red counters $\therefore X \sim \mathrm{B}(30, 0.15)$. Then

$$\mathrm{P}(X \leqslant 6) = \underline{\underline{0.8474}} \text{ (from the tables).}$$

A second random sample of 30 counters is selected and the number of red counters is recorded.

(b) Using a Poisson approximation, estimate the probability that the total number of red counters in the combined sample of size 60 is less than 13. (3)

Solution

Let Y represent the number of red counters $\therefore Y \sim \mathrm{B}(60, 0.15)$. Now,

$$\mathrm{E}(Y) = 60 \times 0.15 = 9 \text{ and } \mathrm{Var}(Y) = 60 \times 0.15 \times 0.85 = 7.65;$$

the expectation and the variance are roughly equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \mathrm{Po}(9)$. Then

$$\begin{aligned}\mathrm{P}(Y < 13) &= \mathrm{P}(Y \leqslant 12) \\ &= \underline{\underline{0.8758 \text{ (from the tables)}}}.\end{aligned}$$

20. A manufacturer supplies DVD players to retailers in batches of 20. It has 5% of the players returned because they are faulty.

(a) Write down a suitable model for the distribution of the number of faulty DVD players in a batch. (2)

Solution

Let X represent the number of faulty DVD players $\therefore \underline{\underline{X \sim \mathrm{B}(20, 0.05)}}$.

Find the probability that a batch contains

(b) no faulty DVD players, (2)

Solution

$$\begin{aligned}\mathrm{P}(X=0) &= (0.95)^{20} \\ &= 0.358\,485\,922\,4 \text{ (FCD)} \\ &= \underline{\underline{0.3585 \text{ (4 dp)}}}.\end{aligned}$$

(c) more than 4 faulty DVD players. (2)

Solution

$$\begin{aligned}\mathrm{P}(X>4) &= 1-\mathrm{P}(X \leqslant 4) \\ &= 1-0.9974 \text{ (from the tables)} \\ &= \underline{\underline{0.0026}}.\end{aligned}$$

(d) Find the mean and variance of the number of faulty DVD players in a batch. (2)

Solution

$\mathrm{E}(X) = 20 \times 0.05 = \underline{\underline{1}}$

$\mathrm{Var}(X) = 20 \times 0.05 \times 0.95 = \underline{\underline{0.95}}$

21. Bhim and Joe play each other at badminton and for each game, independently of all others, the probability that Bhim loses is 0.2.

Find the probability that, in 9 games, Bhim loses

(a) exactly 3 of the games, (3)

Solution

Let X represent the number of Bhim loses $\therefore X \sim \mathrm{B}(9, 0.2)$. Then

$$\begin{aligned}\mathrm{P}(X=3) &= \mathrm{P}(X \leqslant 3)-\mathrm{P}(X \leqslant 2) \\ &= 0.9144-0.7382 \text{ (from the tables)} \\ &= \underline{\underline{0.1762}}.\end{aligned}$$

(b) fewer than half of the games. (2)

Solution
$\mathrm{P}(X \geqslant 4) = \underline{\underline{0.9804}}$ (from the tables)

Bhim attends coaching sessions for 2 months. After completing the coaching, the probability that he loses each game, independently of all others, is 0.05.

Bhim and Joe agree to play a further 60 games.

(c) Calculate the mean and variance for the number of these 60 games that Bhim loses. (2)

Solution
Let Y represent the number of Bhim loses $\therefore Y \sim \mathrm{B}(60, 0.05)$.
$\mathrm{E}(Y) = 60 \times 0.05 = \underline{\underline{3}}$
$\mathrm{Var}(Y) = 60 \times 0.05 \times 0.95 = \underline{\underline{2.85}}$

(d) Using a suitable approximation calculate the probability that Bhim loses more than 4 games. (3)

Solution
The expectation and the variance are roughly equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \mathrm{Po}(3)$. Then

$$\begin{aligned}\mathrm{P}(Y > 4) &= 1 - \mathrm{P}(Y \geqslant 4) \\ &= 1 - 0.8153 \text{ (from the tables)} \\ &= \underline{\underline{0.1847}}.\end{aligned}$$

22. A disease occurs in 3% of a population.

(a) State any assumptions that are required to model the number of people with the disease in a random sample of size n as a binomial distribution. (2)

Solution
e.g., constant probability of success, fixed number of trials, independent trials, success and failure

(b) Using this model, find the probability of exactly 2 people having the disease in a random sample of 10 people. (3)

Solution

Let X represent the number of the diseased people $\therefore X \sim \text{B}(10, 0.03)$. Then

$$\begin{aligned}\text{P}(X = 2) &= \binom{10}{2}(0.03)^2(0.97)^8 \\ &= 0.031\,741\,606\,06 \text{ (FCD)} \\ &= \underline{\underline{0.0317 \text{ (4 dp)}}}.\end{aligned}$$

(c) Find the mean and variance of the number of people with the disease in a random sample of 100 people. (2)

Solution

$\text{E}(X) = 10 \times 0.03 = \underline{\underline{0.3}}$.

$\text{Var}(X) = 10 \times 0.03 \times 0.97 = \underline{\underline{0.291}}$.

A doctor tests a random sample of 100 patients for the disease. He decides to offer all patients a vaccination to protect them from the disease if more than 5 of the sample have the disease.

(d) Using a suitable approximation, find the probability that the doctor will offer all patients a vaccination. (3)

Solution

Let Y represent the number of diseased people $\therefore Y \sim \text{B}(100, 0.03)$. The expectation and the variance are roughly equal (large n and small p) so we try the Poisson distribution $\therefore Y \approx\sim \text{Po}(3)$. Then

$$\begin{aligned}\text{P}(Y > 5) &= 1 - \text{P}(Y \leqslant 5) \\ &= 1 - 0.9161 \text{ (from the tables)} \\ &= \underline{\underline{0.0839}}.\end{aligned}$$

23. The probability of an electrical component being defective is 0.075.
The component is supplied in boxes of 120.

(a) Using a suitable approximation, estimate the probability that there are more than 3 defective components in a box. (5)

Solution

Let X represent the number of defective components $\therefore X \sim \mathrm{B}(120, 0.075)$. Now,

$$\mathrm{E}(X) = 120 \times 0.075 = 9 \text{ and } \mathrm{Var}(X) = 120 \times 0.075 \times 0.925 = 8.325;$$

the expectation and the variance are roughly equal (large n and small p) so we try the Poisson distribution $\therefore X \approx\sim \mathrm{Po}(9)$. Now,

$$\begin{aligned} \mathrm{P}(X > 3) &= 1 - \mathrm{P}(X \leqslant 3) \\ &= 1 - 0.0212 \text{ (from the tables)} \\ &= \underline{\underline{0.9788}}. \end{aligned}$$

A retailer buys 2 boxes of components.

(b) Estimate the probability that there are at least 4 defective components in each box. (2)

Solution

$\mathrm{P}(\text{two boxes}) = 0.9788^2 = \underline{\underline{0.958\,049\,44}}$.

24. Write down two conditions needed to approximate the binomial distribution by the Poisson distribution. (2)

Solution

The expectation and the variance are almost equal ($\underline{\underline{\text{large } n}}$ and $\underline{\underline{\text{small } p}}$)

25. In a large restaurant an average of 3 out of every 5 customers ask for water with their meal.

A random sample of 10 customers is selected.

(a) Find the probability that

(i) exactly 6 ask for water with their meal, (2)

Solution

Let X represent the number of who want water $\therefore X \sim \mathrm{B}(10, 0.6)$

Let Y represent the number of who do not want water $\therefore Y \sim \mathrm{B}(10, 0.4)$

Now,

$$\begin{aligned}
P(X = 6) &= P(Y = 4) \\
&= P(Y \leqslant 4) - P(Y \leqslant 3) \\
&= 0.6331 - 0.3823 \text{ (from the tables)} \\
&= \underline{\underline{0.2508}},
\end{aligned}$$

'reversing' the table.

(ii) less than 9 ask for water with their meal (3)

Solution

$$\begin{aligned}
P(X < 9) &= P(X \leqslant 8) \\
&= P(Y \geqslant 2) \\
&= 1 - P(Y \leqslant 1) \\
&= 1 - 0.0464 \text{ (from the tables)} \\
&= \underline{\underline{0.9536}}.
\end{aligned}$$

A second random sample of 50 customers is selected.

(b) Find the smallest value of n such that (3)

$$P(X < n) \geqslant 0.9,$$

where the random variable X represents the number of these customers who ask for water.

Solution

Let X represent the number of who want water $\therefore X \sim B(50, 0.6)$
Let Y represent the number of who do not want water $\therefore Y \sim B(50, 0.4)$

$$\begin{aligned}
P(X < n) \geqslant 0.9 &\Rightarrow P(Y > 50 - n) \geqslant 0.9 \\
&\Rightarrow P(Y \leqslant 50 - n) \leqslant 0.1 \\
&\Rightarrow 50 - n \leqslant 15 \\
&\Rightarrow n \geqslant 35;
\end{aligned}$$

$\underline{\underline{n = 35}}$ because $P(Y = 14) = 0.0955$ and $P(Y = 15) = 0.1561$.

26. (a) Write down the conditions under which the Poisson distribution can be used as an approximation to the binomial distribution. (2)

Solution

The expectation and the variance are almost equal (large n and small p)

The probability of any one letter being delivered to the wrong house is 0.01.
On a randomly selected day Peter delivers 1000 letters.

(b) Using a Poisson approximation, find the probability that Peter delivers at least 4 letters to the wrong house. (3)

Give your answer to 4 decimal places.

Solution

$1000 \times 0.01 = 10$ and so we have $X \sim \text{Po}(10)$.

$$\begin{aligned} \text{P}(X \geqslant 4) &= 1 - \text{P}(X \leqslant 3) \\ &= 1 - 0.0103 \text{ (from the tables)} \\ &= \underline{\underline{0.9897}}. \end{aligned}$$

27. A random variable X has the distribution $\text{B}(12, p)$.

(a) Given that $p = 0.25$, find

(i) $\text{P}(X < 5)$, (1)

Solution

$$\begin{aligned} \text{P}(X < 5) &= \text{P}(X \leqslant 4) \\ &= \underline{\underline{0.8424}} \text{ (from the tables).} \end{aligned}$$

(ii) $\text{P}(X \geqslant 7)$. (2)

Solution

$$\begin{aligned} \text{P}(X \geqslant 7) &= 1 - \text{P}(X \leqslant 6) \\ &= 1 - 0.9857 \text{ (from the tables)} \\ &= \underline{\underline{0.0143}}. \end{aligned}$$

(b) Given that $\text{P}(X = 0) = 0.05$, find the value of p to 3 decimal places. (3)

Solution

$$\begin{aligned} P(X = 0) = 0.05 &\Rightarrow (1-p)^{12} = 0.05 \\ &\Rightarrow 1 - p = \sqrt[12]{0.05} \\ &\Rightarrow p = 1 - \sqrt[12]{0.05} \\ &\Rightarrow p = 0.220\,922\,191\,9 \text{ (FCD)} \\ &\Rightarrow p = \underline{\underline{0.221 \text{ (3 dp)}}}. \end{aligned}$$

(c) Given that the variance of X is 1.92, find the possible values of p. (4)

Solution

$$\begin{aligned} 12p(1-p) = 1.92 &\Rightarrow 12p - 12p^2 = 1.92 \\ &\Rightarrow 12p^2 - 12p + 1.92 = 0 \\ &\Rightarrow 25p^2 - 25p + 4 = 0 \\ &\Rightarrow (5p-1)(5p-4) = 0 \\ &\Rightarrow 5p = 1 \text{ or } 5p = 4 \\ &\Rightarrow \underline{\underline{p = 0.2}} \text{ or } \underline{\underline{p = 0.8}}. \end{aligned}$$

28. A telesales operator is selling a magazine. Each day he chooses a number of people to telephone. The probability that each person he telephones buys the magazine is 0.1.

(a) Suggest a suitable distribution to model the number of people who buy the magazine from the telesales operator each day. (1)

Solution

Let X represent the number of who want to subscribe $\therefore$ $\underline{\underline{X \sim \mathrm{B}(n, 0.1)}}$

(b) On Monday, the telesales operator telephones 10 people. Find the probability that he sells at least 4 magazines. (3)

Solution

$X \sim \text{B}(10, 0.1)$:

$$\begin{aligned} \text{P}(X \geqslant 4) &= 1 - \text{P}(X \leqslant 3) \\ &= 1 - 0.9872 \text{ (from the tables)} \\ &= \underline{\underline{0.0128}}. \end{aligned}$$

(c) Calculate the least number of people he needs to telephone on Tuesday, so that the probability of selling at least 1 magazine, on that day, is greater than 0.95. (3)

Solution

$X \sim \text{B}(n, 0.1)$:

$$\begin{aligned} \text{P(at least one)} \geqslant 0.95 &\Rightarrow 1 - 0.9^n \geqslant 0.95 \\ &\Rightarrow 0.9^n < 0.05 \\ &\Rightarrow n \log 0.9 < \log 0.05 \\ &\Rightarrow n > \frac{\log 0.05}{\log 0.9} \\ &\Rightarrow n > 28.433\,158 \text{ (FCD)} \end{aligned}$$

and it is $\underline{\underline{n = 29}}$.

A call centre also sells the magazine. The probability that a telephone call made by the call centre sells a magazine is 0.05. The call centre telephones 100 people every hour.

(d) Using a suitable approximation, find the probability that more than 10 people telephoned by the call centre buy a magazine in a randomly chosen hour. (3)

Solution

Let Y represent the number of who want to subscribe by telephone $\therefore Y \sim \text{B}(100, 0.05)$. Now,

$$\text{E}(Y) = 100 \times 0.05 = 5 \text{ and } \text{Var}(X) = 100 \times 0.05 \times 0.95 = 4.75;$$

the expectation and the variance are roughly equal (large n and small p) so we try the Poisson distribution:

$$\begin{aligned} \text{P}(Y > 10) &= 1 - \text{P}(Y \leqslant 10) \\ &= 1 - 0.9863 \text{ (from the tables)} \\ &= \underline{\underline{0.0137}}. \end{aligned}$$

29. As part of a selection procedure for a company, applicants have to answer all 20 questions of a multiple choice test. If an applicant chooses answers at random the probability of choosing a correct answer is 0.2 and the number of correct answers is represented by the random variable X.

(a) Suggest a suitable distribution for X. (2)

Solution

Let X represent the number who guessed $\therefore \underline{\underline{X \sim \mathrm{B}(20, 0.2)}}$

Each applicant gains 4 points for each correct answer but loses 1 point for each incorrect answer. The random variable S represents the final score, in points, for an applicant who chooses answers to this test at random.

(b) Show that $S = 5X - 20$. (2)

Solution

$S = 4 \times X - 1 \times (20 - X) = \underline{\underline{5X - 20}}$.

(c) Find $\mathrm{E}(S)$ and $\mathrm{Var}(S)$. (4)

Solution

$$
\begin{aligned}
\mathrm{E}(X) = 20 \times 0.2 &\Rightarrow \mathrm{E}(X) = 4 \\
&\Rightarrow \mathrm{E}(S) = \mathrm{E}(5X - 20) \\
&\Rightarrow \mathrm{E}(S) = 5\,\mathrm{E}(X) - 20 \\
&\Rightarrow \mathrm{E}(S) = 5 \times 4 - 20 \\
&\Rightarrow \mathrm{E}(S) = \underline{\underline{0}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Var}(X) = 20 \times 0.2 \times 0.8 &\Rightarrow \mathrm{Var}(X) = 3.2 \\
&\Rightarrow \mathrm{Var}(S) = \mathrm{Var}(5X - 20) \\
&\Rightarrow \mathrm{Var}(S) = \mathrm{Var}(5X) \\
&\Rightarrow \mathrm{Var}(S) = 5^2\,\mathrm{Var}(X) \\
&\Rightarrow \mathrm{Var}(S) = 25 \times 3.2 \\
&\Rightarrow \mathrm{Var}(S) = \underline{\underline{80}}.
\end{aligned}
$$

An applicant who achieves a score of at least 20 points is invited to take part in the final stage of the selection process.

(d) Find $\mathrm{P}(S \geqslant 20)$. (4)

Solution
Now,

$$\begin{aligned} S \geqslant 20 &\Rightarrow 5X - 20 \geqslant 20 \\ &\Rightarrow 5X \geqslant 40 \\ &\Rightarrow X \geqslant 8 \end{aligned}$$

and

$$\begin{aligned} \mathrm{P}(X \geqslant 8) &= 1 - \mathrm{P}(X \leqslant 7) \\ &= 1 - 0.9679 \text{ (from the tables)} \\ &= \underline{\underline{0.0321}}. \end{aligned}$$

Cameron is taking the final stage of the selection process which is a multiple choice test consisting of 100 questions. He has been preparing for this test and believes that his chance of answering each question correctly is 0.4.

(e) Using a suitable approximation, estimate the probability that Cameron answers more than half of the questions correctly. (5)

Solution
Let Y represent the number who guessed $\therefore Y \sim \mathrm{B}(100, 0.4)$. Now,

$$\mathrm{E}(Y) = 100 \times 0.4 = 40 \text{ and } \mathrm{Var}(X) = 100 \times 0.4 \times 0.6 = 24;$$

$np = 40$ and $nq = 24$ so we use a normal approximation: $Y \approx\sim \mathrm{N}(40, 24)$. Then

$$\begin{aligned} \mathrm{P}(Y > 50) &= \mathrm{P}(Y > 50.5) \\ &= 1 - \mathrm{P}(Y \leqslant 50.5) \\ &= 1 - \mathrm{P}\left(Z \leqslant \frac{50.5 - 40}{\sqrt{24}}\right) \\ &= 1 - \Phi(2.14) \\ &= 1 - 0.9838 \text{ (from the tables)} \\ &= \underline{\underline{0.0162}}. \end{aligned}$$

30. A cadet fires shots at a target at distances ranging from 25 m to 90 m. The probability

of hitting the target with a single shot is p. When firing from a distance d m,

$$p = \tfrac{3}{200}(90 - d).$$

Each shot is fired independently.

The cadet fires 10 shots from a distance of 40 m.

(a) (i) Find the probability that exactly 6 shots hit the target. (3)

Solution

Let X represent the number of shots which hit the target $\therefore X \sim \mathrm{B}(40, 0.75)$
Let Y represent the number of shots which miss the target $\therefore Y \sim \mathrm{B}(40, 0.25)$
Then

$$\begin{aligned} \mathrm{P}(X = 6) &= \mathrm{P}(Y = 4) \\ &= \mathrm{P}(Y \leqslant 4) - \mathrm{P}(Y \leqslant 3) \\ &= 0.9219 - 0.7759 \text{ (from the tables)} \\ &= \underline{\underline{0.1460}}. \end{aligned}$$

(ii) Find the probability that at least 8 shots hit the target. (2)

Solution

$$\begin{aligned} \mathrm{P}(X \geqslant 8) &= \mathrm{P}(Y \leqslant 2) \\ &= \underline{\underline{0.5256}} \text{ (from the tables)} \end{aligned}$$

The cadet fires 20 shots from a distance of x m.

(b) Find, to the nearest integer, the value of x if the cadet has an 80% chance of hitting the target at least once. (4)

Solution

$$\begin{aligned}
\text{P(at least one)} = 0.80 \Rightarrow \text{P(miss every time)} &= 0.20 \\
\Rightarrow \left[1 - \tfrac{3}{200}(90 - x)\right]^{20} &= 0.2 \\
\Rightarrow 1 - \tfrac{3}{200}(90 - x) &= \sqrt[20]{0.2} \\
\Rightarrow \tfrac{3}{200}(90 - x) &= 1 - \sqrt[20]{0.2} \\
\Rightarrow 90 - x &= \tfrac{200}{3}\left(1 - \sqrt[20]{0.2}\right) \\
\Rightarrow x &= 90 - \tfrac{200}{3}\left(1 - \sqrt[20]{0.2}\right) \\
\Rightarrow x &= 84.845\,388\,97 \text{ (FCD)} \\
\Rightarrow \underline{\underline{x = 85 \text{ (2 sf)}}}.
\end{aligned}$$

The cadet fires 100 shots from 25 m.

(c) Using a suitable approximation, estimate the probability that at least 95 of these shots hit the target. (5)

Solution

Let A represent the number of shots which hit the target $\therefore A \sim \text{B}(100, 0.975)$
Let B represent the number of shots which do not hit the target
$\therefore B \sim \text{B}(100, 0.025)$
Now,

$$\text{E}(B) = 100 \times 0.025 = 2.5 \text{ and } \text{Var}(X) = 100 \times 0.025 \times 0.975 = 2.4375;$$

the expectation and the variance are roughly equal (large n and small p) so we try the Poisson distribution. Then

$$\begin{aligned}
\text{P}(A \geqslant 95) &= \text{P}(B \leqslant 5) \\
&= \underline{\underline{0.9580}} \text{ (from the tables).}
\end{aligned}$$

31. State the conditions under which the normal distribution may be used as an approximation to the binomial distribution. (2)

Solution

$\underline{\underline{n \text{ is large}}}$
$\underline{\underline{p \text{ is close to } 0.5}}$

32. A bag contains a large number of counters. Each counter has a single digit number on it and the mean of all the numbers in the bag is the unknown parameter μ. The number 2 is on 40% of the counters and the number 5 is on 25% of the counters. All the remaining counters have numbers greater than 5 on them.

The random variable T represents the number of counters in a random sample of 10 with the number 2 on them.

(a) Specify the sampling distribution of T. (2)

Solution
$\underline{\underline{T \sim \mathrm{B}(10, 0.4)}}$.

The counters are selected one by one.

(b) Find the probability that the third counter selected is the first counter with the number 2 on it. (2)

Solution

$$\begin{aligned} \mathrm{P}(\overline{2}, \overline{2}, 2) &= \tfrac{3}{5} \times \tfrac{3}{5} \times \tfrac{2}{5} \\ &= \underline{\underline{\tfrac{18}{125}}}. \end{aligned}$$

33. The proportion of houses in Radville which are unable to receive digital radio is 25%. In a survey of a random sample of 30 houses taken from Radville, the number, X, of houses which are unable to receive digital radio is recorded. (3)

Find $\mathrm{P}(5 \leqslant X < 11)$.

Solution

$$\begin{aligned} \mathrm{P}(5 \leqslant X < 11) &= \mathrm{P}(X \leqslant 10) - \mathrm{P}(X \leqslant 4) \\ &= 0.8943 - 0.0979 \text{ (from the tables)} \\ &= \underline{\underline{0.7964}}. \end{aligned}$$

34. In a region of the UK, 5% of people have red hair. In a random sample of size n, taken from this region, the expected number of people with red hair is 3.

(a) Calculate the value of n. (2)

Solution

Let X represent the number of red hair persons $\therefore X \sim \mathrm{B}(n, 0.05)$. Then

$$\mathrm{P}(X = n) = 3 \Rightarrow 0.05n = 3$$
$$\Rightarrow \underline{\underline{n = 60}}.$$

A random sample of 20 people is taken from this region.
Find the probability that

(b) (i) exactly 4 of these people have red hair, (3)

Solution

We use $X \sim \mathrm{B}(20, 0.05)$

$$\begin{aligned}\mathrm{P}(X = 4) &= \mathrm{P}(X \leqslant 4) - \mathrm{P}(X \leqslant 3) \\ &= 0.9974 - 0.9841 \text{ (from the tables)} \\ &= \underline{\underline{0.0133}}.\end{aligned}$$

(ii) at least 4 of these people have red hair. (2)

Solution

$$\begin{aligned}\mathrm{P}(X \geqslant 4) &= 1 - \mathrm{P}(X \leqslant 3) \\ &= 1 - 0.9841 \text{ (from the tables)} \\ &= \underline{\underline{0.0159}}.\end{aligned}$$

35. In a large school, 20% of students own a touch screen laptop. A random sample of n students is chosen from the school. Using a normal approximation, the probability that more than 55 of these n students own a touch screen laptop is 0.0401, correct to 3 significant figures. (8)

Find the value of n.

Solution

Let X represent the number of students own a touch screen laptop $\therefore X \sim \mathrm{B}(n, 0.2)$. We are directed to use the normal approximation:

$$\mathrm{E}(X) = n \times 0.2 = 0.2n \text{ and } \mathrm{Var}(X) = n \times 0.2 \times 0.8 = 0.16n;$$

therefore, $\therefore X \approx\sim \mathrm{N}(0.2n, 0.16n)$. Then

$$\begin{aligned}
\mathrm{P}(X > 55) = 0.0401 &\Rightarrow \mathrm{P}(X > 55.5) = 0.0401 \\
&\Rightarrow 1 - \mathrm{P}(X \leqslant 55.5) = 0.0401 \\
&\Rightarrow \mathrm{P}(X \leqslant 55.5) = 0.9599 \\
&\Rightarrow \mathrm{P}\left(\frac{55.5 - 0.2n}{\sqrt{0.16n}}\right) = 0.9599 \\
&\Rightarrow \frac{55.5 - 0.2n}{\sqrt{0.16n}} = 1.75 \\
&\Rightarrow 55.5 - 0.2n = 0.7\sqrt{n} \\
&\Rightarrow 0.2n + 0.7\sqrt{n} - 55.5 = 0 \\
&\Rightarrow 2n + 7\sqrt{n} - 555 = 0 \\
&\Rightarrow \left(2\sqrt{n} + 37\right)\left(\sqrt{n} - 15\right) = 0 \\
&\Rightarrow \sqrt{n} = 15 \text{ (only)} \\
&\Rightarrow \underline{\underline{n = 225}}.
\end{aligned}$$

36. 10% of people take less than 12 minutes to complete the test.
Graham selects 15 people at random.

(a) Find the probability that fewer than 2 of these people will take less than 12 minutes to complete the test. (3)

Solution

Let X represent the number of students who take less than 12 minutes to complete the test $\therefore X \sim \mathrm{B}(15, 0.1)$.

$$\begin{aligned}
\mathrm{P}(X \leqslant 1) &= (0.9)^{15} + \binom{15}{1}(0.9)^{14}(0.1) \\
&= 0.549\,043\,018\,9 \text{ (FCD)} \\
&= \underline{\underline{0.5490 \text{ (4 dp)}}}.
\end{aligned}$$

Jovanna takes a random sample of n people. Using a normal approximation, the probability that fewer than 9 of these n people will take less than 12 minutes to complete the test is 0.3085 to 4 decimal places.

(b) Find the value of n. (8)

Solution

We are directed to use the normal approximation:

$$\mathrm{E}(X) = n \times 0.1 = 0.1n \text{ and } \mathrm{Var}(X) = n \times 0.1 \times 0.9 = 0.09n;$$

therefore, $\therefore X \approx\sim \mathrm{N}(0.1n, 0.09n)$. Then

$$\begin{aligned}
\mathrm{P}(X < 9) = 0.3085 &\Rightarrow \mathrm{P}(X < 8.5) = 0.3085 \\
&\Rightarrow \mathrm{P}\left(\frac{8.5 - 0.1n}{\sqrt{0.09n}}\right) = 0.3085 \\
&\Rightarrow \frac{8.5 - 0.1n}{0.3\sqrt{n}} = -0.5 \\
&\Rightarrow 8.5 - 0.1n = -0.15\sqrt{n} \\
&\Rightarrow 0.1n - 0.15\sqrt{n} - 8.5 = 0 \\
&\Rightarrow 2n - 3\sqrt{n} - 170 = 0 \\
&\Rightarrow \left(2\sqrt{n} + 17\right)\left(\sqrt{n} - 10\right) = 0 \\
&\Rightarrow \sqrt{n} = 10 \text{ (only)} \\
&\Rightarrow \underline{\underline{n = 100}}.
\end{aligned}$$