Dr Oliver Mathematics
Applied Mathematics: Matrices

The total number of marks available is 57.
You must write down all the stages in your working.

1. Given that (3)

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix},$$

show that

$$\mathbf{A}^2 - \mathbf{A} = k\mathbf{I}$$

for a suitable value of k, where $\mathbf{I}$ is the 2×2 unit matrix.

Solution

$$\begin{aligned} \mathbf{A}^2 - \mathbf{A} &= \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\ &= 2\mathbf{I}; \end{aligned}$$

hence, $\underline{\underline{k = 2}}$.

2. (a) Calculate $\mathbf{A}^{-1}$ where (3)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Solution

$$\begin{aligned} \det \mathbf{A} &= 1(3-2) - 1(2-2) + 0 \\ &= 1. \end{aligned}$$

Matrix of minors:

$$\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}$$

Inverse:

$$\mathbf{A}^{-1} = \underline{\underline{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}}}.$$

(b) Hence solve the system of equations (2)

$$\begin{aligned} x + y \quad\quad &= 1 \\ 2x + 3y + z &= 2 \\ 2x + 2y + z &= 1. \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix};$$

hence, $\underline{\underline{x = 0,\ y = 1,\ z = -1}}$.

3. (a) For the matrix (3)

$$\mathbf{A} = \begin{pmatrix} \lambda & 2 \\ 2 & \lambda - 3 \end{pmatrix},$$

find the values of λ such that the matrix is singular.

Solution

$$\begin{aligned}\det \mathbf{A} = 0 &\Rightarrow \lambda(\lambda - 3) - 4 = 0\\ &\Rightarrow \lambda^2 - 3\lambda - 4 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -3\\ \text{multiply to:} & -4\end{array}\right\} -4,\ +1$$

$$\begin{aligned}&\Rightarrow (\lambda - 4)(\lambda + 1) = 0\\ &\Rightarrow \underline{\underline{\lambda = 4 \text{ or } \lambda = -1}}.\end{aligned}$$

(b) Write down the matrix $\mathbf{A}^{-1}$ when $\lambda = 3$. (1)

Solution

$$\lambda = 3 \Rightarrow \det \mathbf{A} = 9 - 9 - 4 = -4$$

and

$$\mathbf{A}^{-1} = \underline{\underline{-\tfrac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 3 \end{pmatrix}}}.$$

4. Given that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are square matrices where:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix}.$$

(a) Find $\mathbf{AB}$. (1)

Solution

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 0 & -3 \end{pmatrix}\\ &= \underline{\underline{\begin{pmatrix} 8 & 15 \\ 12 & 3 \end{pmatrix}}}.\end{aligned}$$

(b) Express $4\mathbf{C} + \mathbf{D}$ as a single matrix. (2)

Solution

$$\begin{aligned}4\mathbf{C}+\mathbf{D} &= 4\begin{pmatrix} x & 2 \\ 0 & y \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 12 & -1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 4x+2 & 15 \\ 12 & 4y-1 \end{pmatrix}}}.\end{aligned}$$

(c) Given that (2)

$$\mathbf{AB} = 4\mathbf{C}+\mathbf{D},$$

find the values of x and y.

Solution

$$\begin{aligned}4x+2=8 &\Rightarrow 4x=6 \\ &\Rightarrow \underline{\underline{x=1\tfrac{1}{2}}} \\ 4y-1=3 &\Rightarrow 4y=4 \\ &\Rightarrow \underline{\underline{y=1}}.\end{aligned}$$

5. Determine k such that the matrix (4)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & k-2 & -1 \\ 1 & 2 & k \end{pmatrix}$$

does not have an inverse.

Solution

$$\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & k-2 & -1 \\ 1 & 2 & k \end{pmatrix} = 0$$
$$\Rightarrow \quad 1[k(k-2)+2] - 1[0+1] + 0 = 0$$
$$\Rightarrow \quad k^2 - 2k + 2 - 1 = 0$$
$$\Rightarrow \quad k^2 - 2k + 1 = 0$$
$$\Rightarrow \quad (k-1)^2 = 0$$
$$\Rightarrow \quad k - 1 = 0$$
$$\Rightarrow \quad \underline{\underline{k = 1}}.$$

6. (a) Find the value(s) of m for which the matrix (3)

$$\begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

is singular.

Solution

$$\det\begin{pmatrix} m & 1 & 1 \\ 0 & m & -2 \\ 1 & 0 & 1 \end{pmatrix} = 0 \Rightarrow m(m-0) - 1(0+2) + 1(0-m) = 0$$
$$\Rightarrow m^2 - m - 2 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & -2 \end{array}\right\} -2,\ +1$$

$$\Rightarrow (m-2)(m+1) = 0$$
$$\Rightarrow m - 2 = 0 \text{ or } m + 1 = 0$$
$$\Rightarrow \underline{\underline{m = 2 \text{ or } m = -1}}.$$

The matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}.$$

(b) Use elementary row operations to obtain $\mathbf{B}^{-1}$. (4)

Solution

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right)$$

Do $R_3 - R_1$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right)$$

Do $R_3 + R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Do $R_1 + R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Do $R_1 - 2R_3$ and $R_2 + R_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right)$$

Do $-R_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right)$$

Hence,

$$\mathbf{B}^{-1} = \underline{\underline{\begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}}}.$$

(c) Hence, or otherwise, solve the system of equations (2)

$$\begin{aligned} x + y - z &= 3 \\ y + z &= -2 \\ x - 3z &= 7. \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -3 & -2 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix};$$

hence,

$$\underline{\underline{x = 1, \, y = 0, \, z = -2.}}$$

7. (a) Given (2)

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix},$$

obtain $\mathbf{A}^{-1}$.

Solution

$$\det \mathbf{A} = 0 - (-6) = 6$$

and so

$$\underline{\underline{\mathbf{A}^{-1} = \tfrac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}}}.$$

(b) Given (3)

$$\mathbf{AB} = \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix},$$

find the matrix $\mathbf{B}$.

Solution

$$
\begin{aligned}
\mathbf{AB} &= \begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \\
\Rightarrow \quad \mathbf{A}^{-1}\mathbf{AB} &= \mathbf{A}^{-1}\begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \\
\Rightarrow \quad \mathbf{B} &= \tfrac{1}{6}\begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} -4 & -3 \\ 6 & -3 \end{pmatrix} \\
\Rightarrow \quad \mathbf{B} &= \tfrac{1}{6}\begin{pmatrix} 12 & -6 \\ 18 & 6 \end{pmatrix} \\
\Rightarrow \quad \mathbf{B} &= \underline{\underline{\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}}}.
\end{aligned}
$$

8. Given

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}:$$

(a) Calculate $\mathbf{M}^2$. (2)

Solution

$$
\begin{aligned}
\mathbf{M}^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}}}.
\end{aligned}
$$

(b) Calculate $\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3$. (2)

Solution

$$\begin{aligned}\mathbf{M}^3 &= \mathbf{M}^2\mathbf{M} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 0 & 0 & \lambda^3 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 3 & 0 & 0 \\ 18 & 3 & 0 \\ 0 & 0 & \lambda + \lambda^2 + \lambda^3 \end{pmatrix}}}.\end{aligned}$$

(c) For what values of λ does $\mathbf{M}$ have an inverse? (2)

Solution

$$\begin{aligned}\det \mathbf{M} &= 1(\lambda - 0) - 0 + 0 \\ &= \lambda;\end{aligned}$$

hence, $\mathbf{M}$ has an inverse if $\underline{\underline{\lambda \neq 0}}$.

9. Matrices are given as

$$\mathbf{A} = \begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} y & 3 \\ -1 & 2 \end{pmatrix}.$$

(a) Write (2)

$$\mathbf{A}^2 - 3\mathbf{B}$$

as a single matrix.

Solution

$$\begin{aligned}\mathbf{A}^2 - 3\mathbf{B} &= \begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix} - 3\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 6x \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 15 & 3 \\ 0 & 3 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 & 6x-3 \\ 0 & 1 \end{pmatrix}}}.\end{aligned}$$

(b) (i) Given that $\mathbf{C}$ is non-singular, find $\mathbf{C}^{-1}$, the inverse of $\mathbf{C}$. (2)

Solution

$$\begin{aligned}\det\mathbf{C} &= 2y - (-3) \\ &= 2y + 3\end{aligned}$$

and so

$$\mathbf{C}^{-1} = \underline{\underline{\tfrac{1}{2y+3}\begin{pmatrix} 2 & -3 \\ 1 & y \end{pmatrix}}}.$$

(ii) For what value of y would matrix $\mathbf{C}$ be singular? (1)

Solution

$$\begin{aligned}\det\mathbf{C} = 0 &\Rightarrow 2y + 3 = 0 \\ &\Rightarrow 2y = -3 \\ &\Rightarrow \underline{\underline{y = -1\tfrac{1}{2}}}.\end{aligned}$$

10. Matrices are given as

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ k & 0 & -1 \\ 5 & 3 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}.$$

(a) Calculate $\mathbf{A} + \mathbf{B}$. (1)

Solution

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 & 3 & 4 \\ k & 0 & -1 \\ 5 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 & -7 & 6 \\ k-3 & 9 & -1 \\ 5 & 1 & 1 \end{pmatrix}}}.$$

(b) Find the determinant of $\mathbf{A}$. (2)

Solution

$$\begin{aligned} \det \mathbf{A} &= 1(0+3) - 3(0+5) + 4(3k-0) \\ &= 3 - 15 + 12k \\ &= \underline{\underline{12k - 12}}. \end{aligned}$$

(c) Calculate $\mathbf{BC}$. (1)

Solution

$$\mathbf{BC} = \begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}}}.$$

(d) Describe the relationship between $\mathbf{B}$ and $\mathbf{C}$. (2)

Solution

$$\mathbf{BC} = 3\mathbf{I} \Rightarrow \underline{\underline{\mathbf{B} = 3\mathbf{C}^{-1}}}.$$

11. (a) Given matrix (2)

$$\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix},$$

find $\mathbf{A}^2$ and show that the inverse of $\mathbf{A}^2$ exists.

Solution

$$\mathbf{A}^2 = \begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ 1 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}}}$$

and

$$\det \mathbf{A}^2 = -16 + 20 = 4$$

so $\mathbf{A}^2$ $\underline{\underline{\text{does exist}}}$.

(b) Hence, or otherwise, find matrix $\mathbf{B}$ such that (3)

$$\mathbf{A}^2\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix}.$$

Solution

$$\begin{aligned} \mathbf{A}^2\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} &\Rightarrow \begin{pmatrix} 4 & -10 \\ 2 & -4 \end{pmatrix}\mathbf{B} = \begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} \\ &\Rightarrow \mathbf{B} = \tfrac{1}{4}\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 2 & -2 \end{pmatrix} \\ &\Rightarrow \mathbf{B} = \tfrac{1}{4}\begin{pmatrix} 4 & -44 \\ 0 & -20 \end{pmatrix} \\ &\Rightarrow \mathbf{B} = \underline{\underline{\begin{pmatrix} 1 & -11 \\ 0 & -5 \end{pmatrix}}}. \end{aligned}$$