Dr Oliver Mathematics
Further Mathematics
2×2 Matrices
Past Examination Questions

This booklet consists of 29 questions across a variety of examination topics.
The total number of marks available is 223.

1. $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are 2×2 matrices.

   (a) Given that $\mathbf{AB} = \mathbf{AC}$, and that $\mathbf{A}$ is not singular, prove that $\mathbf{B} = \mathbf{C}$. (2)

   Solution
   If $\mathbf{A}$ is not singular, then

   $$\begin{aligned}\mathbf{AB} = \mathbf{AC} &\Rightarrow \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \\ &\Rightarrow \mathbf{IB} = \mathbf{IC} \\ &\Rightarrow \underline{\underline{\mathbf{B} = \mathbf{C}}},\end{aligned}$$

   as required.

   (b) Given that $\mathbf{AB} = \mathbf{AC}$, where (3)

   $$\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$

   find a matrix $\mathbf{C}$ whose elements are all non-zero.

   Solution

   $$\mathbf{AB} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 21 \\ 1 & 7 \end{pmatrix}$$

   and

   $$\mathbf{AC} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a + 6c & 3b + 6d \\ a + 2c & b + 2d \end{pmatrix};$$

   hence,

   $$\underline{\underline{\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}}}$$

   is a suitable matrix.

2. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

   $$\mathbf{A} = \begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix},$$

where k is a constant. For the case $k = -4$,

(a) find the image under T of the line with equation $y = 2x + 1$. (2)

Solution

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ 2x+1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix};$$

the image is the point $\underline{\underline{(2, -1)}}$.

For the case $k = 2$, find

(b) the two eigenvalues of $\mathbf{A}$, (4)

Solution

$$\begin{aligned} & \det(\mathbf{A} - \lambda\mathbf{I}) = 0 \\ \Rightarrow\ & (2-\lambda)(-1-\lambda) - 4 = 0 \\ \Rightarrow\ & (\lambda^2 - \lambda - 2) - 4 = 0 \\ \Rightarrow\ & \lambda^2 - \lambda - 6 = 0 \\ \Rightarrow\ & (\lambda - 3)(\lambda + 2) = 0 \\ \Rightarrow\ & \underline{\underline{\lambda = -2}} \text{ or } \underline{\underline{\lambda = 3}}. \end{aligned}$$

(c) a cartesian equation for each of the two lines passing through the origin which are invariant under T. (3)

Solution

$\underline{\lambda = -2}$:

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$\underline{\underline{y = -2x}}.$$

$\underline{\lambda = 3}$:

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and so

$$\underline{\underline{y = \tfrac{1}{2}x}}.$$

3.
$$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix},$$

where a is real.

(a) Find $\det \mathbf{A}$ in terms of a. (2)

Solution

$$\det \mathbf{A} = a(a+4) - 2 \times (-5) = \underline{\underline{a^2 + 4a + 10}}.$$

(b) Show that the matrix $\mathbf{A}$ is non-singular for all values of a. (3)

Solution

$$\det \mathbf{A} = (a+2)^2 + 6$$

and so $\det \mathbf{A} \geqslant 6$. Hence, $\underline{\underline{\text{the matrix } \mathbf{A} \text{ is non-singular}}}$.

Given that $a = 0$,

(c) find $\mathbf{A}^{-1}$. (3)

Solution

$$\mathbf{A} = \begin{pmatrix} 0 & -5 \\ 2 & 4 \end{pmatrix} \Rightarrow \mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{10}\begin{pmatrix} 4 & 5 \\ -2 & 0 \end{pmatrix}}}.$$

4.
$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Describe fully the geometrical transformation represented by the matrix $\mathbf{M}$. (2)

Solution

$\mathbf{M}$ is a $\underline{\underline{\text{transformation of } 45^\circ\text{, anticlockwise, about } (0,0)}}$.

The transformation represented by $\mathbf{M}$ maps the point A with coordinates (p, q) onto the point B with coordinates $(3\sqrt{2}, 4\sqrt{2})$.

(b) Find the value of p and the value of q. (4)

Solution

$$\det \mathbf{M} = \tfrac{1}{\sqrt{2}} \times \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} \times (-\tfrac{1}{\sqrt{2}}) = 1$$

and

$$\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Now,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix} \Rightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix};$$

hence, $\underline{\underline{p = 7}}$ and $\underline{\underline{q = 1}}$.

(c) Find, in its simplest surd form, the length OA, where O is the origin. (2)

Solution

$$OA = \sqrt{(3\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{50} = \underline{\underline{5\sqrt{2}}}.$$

(d) Find $\mathbf{M}^2$. (2)

Solution

$$\mathbf{M}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}.$$

The point B is mapped onto the point C by the transformation represented by $\mathbf{M}^2$.

(e) Find the coordinates of C. (2)

Solution

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3\sqrt{2} \\ 4\sqrt{2} \end{pmatrix} = \begin{pmatrix} -4\sqrt{2} \\ 3\sqrt{2} \end{pmatrix};$$

hence, $\underline{\underline{C(-4\sqrt{2}, 3\sqrt{2})}}$.

5.

$$\mathbf{M} = \begin{pmatrix} 2a & 3 \\ 6 & a \end{pmatrix},$$

where a is a real constant.

(a) Given that $a = 2$, find $\mathbf{M}^{-1}$. (3)

Solution

$$\det \mathbf{M} = 4 \times 2 - 6 \times 3 = -10$$

and

$$\mathbf{M} = \begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix} \Rightarrow \mathbf{M}^{-1} = \underline{\underline{-\tfrac{1}{10}\begin{pmatrix} 2 & -3 \\ -6 & 4 \end{pmatrix}}}.$$

(b) Find the values of a for which $\mathbf{M}$ is singular. (2)

Solution

$$2a^2 - 18 = 0 \Rightarrow a^2 = 9 \Rightarrow \underline{\underline{a = \pm 3}}.$$

6. Write down the 2×2 matrix that represents

(a) an enlargement with centre $(0, 0)$ and scale factor 8, (1)

Solution

$$\underline{\underline{\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}}}.$$

(b) a reflection in the x-axis. (1)

Solution

$$\underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}.$$

Hence, or otherwise,

(c) find the matrix $\mathbf{T}$ that represents an enlargement with centre $(0, 0)$ and scale factor 8, followed by a reflection in the x-axis. (2)

Solution

$$\mathbf{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix}}}.$$

$$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix},$$

where k and c are constants.

(d) Find **AB** (3)

Solution

$$\mathbf{AB} = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 6k + c & 0 \\ 4k + 2c & -8 \end{pmatrix}}}.$$

Given that **AB** represents the same transformation as **T**,

(e) find the value of k and the value of c. (2)

Solution

$$\begin{aligned} 6k + c = 8 &\Rightarrow c = 8 - 6k \\ &\Rightarrow 4k + 2(8 - 6k) = 0 \\ &\Rightarrow 8k = 16 \\ &\Rightarrow \underline{\underline{k = 2}} \\ &\Rightarrow c = 8 - 6 \times 2 = \underline{\underline{-4}}. \end{aligned}$$

7.

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}.$$

(a) Find **AB**. (3)

Solution

$$\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -6 & -2 \\ 0 & 1 \end{pmatrix}}}.$$

Given that

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

(b) describe fully the geometrical transformation represented by $\mathbf{C}$, (2)

Solution

It is a reflection in the y-axis.

(c) write down $\mathbf{C}^{100}$. (1)

Solution

$$\mathbf{C}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \mathbf{C}^{100} = (\mathbf{C}^2)^{50} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

8.

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}.$$

(a) Find $\det \mathbf{A}$. (1)

Solution

$$\det \mathbf{A} = 2 \times 3 - (-2) \times (-1) = 4.$$

(b) Find $\mathbf{A}^{-1}$. (2)

Solution

$$\mathbf{A}^{-1} = \tfrac{1}{4} \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$$

The triangle R is transformed to the triangle S by the matrix $\mathbf{A}$. Given that the area of triangle S is 72 square units,

(c) find the area of triangle R. (2)

Solution

$$4 \times \text{area of triangle } R = 72 \Rightarrow \text{area of triangle } R = 18 \text{ square units}.$$

The triangle S has vertices at the points $(0, 4)$, $(8, 16)$, and $(12, 4)$.

(d) Find the coordinates of the vertices of R. (4)

Solution

$$\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 8 & 56 & 44 \\ 8 & 40 & 20 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} 2 & 14 & 11 \\ 2 & 10 & 5 \end{pmatrix};$$

hence, the coordinates are $(2, 2)$, $(14, 10)$, and $(7, 5)$.

9. Given that

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix},$$

(a) (i) find $\mathbf{A}^2$, (4)

Solution

$$\mathbf{A}^2 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$.

Solution

It is an enlargement, with centre $(0, 0)$, and scale factor 3.

(b) Given that (2)

$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

describe fully the geometrical transformation represented by $\mathbf{B}$.

Solution

It is a reflection in the line $y = -x$.

(c) Given that (3)

$$\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix},$$

where k is a constant, find the value of k for which the matrix $\mathbf{C}$ is singular.

Solution

$$\begin{aligned}\det \mathbf{C} = 0 &\Rightarrow 9(k+1) - 12k = 0\\ &\Rightarrow 3k = 9\\ &\Rightarrow \underline{\underline{k = 3}}.\end{aligned}$$

10.

$$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix},$$

where a and b are constants. Given that the matrix $\mathbf{A}$ maps the point with coordinates $(4, 6)$ onto the point with coordinates $(2, -8)$,

(a) find the value of a and the value of b. (4)

Solution

$$\begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \Rightarrow \begin{pmatrix} -16+6a \\ 4b-12 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}.$$

Now,

$$-16 + 6a = 2 \Rightarrow 6a = 18 \Rightarrow \underline{\underline{a = 3}}$$

and

$$4b - 12 = -8 \Rightarrow 4b = 4 \Rightarrow \underline{\underline{b = 1}}.$$

A quadrilateral R has area 30 square units. It is transformed into another quadrilateral S by the matrix $\mathbf{A}$. Using your values of a and b,

(b) find the area of quadrilateral S. (4)

Solution

$$\det \mathbf{A} = (-4) \times (-2) - 3 \times 1 = 5$$

and

$$\text{area of triangle } S = 5 \times 30 = \underline{\underline{150 \text{ square units}}}.$$

11. A right angled triangle T has vertices $A(1, 1)$, $B(2, 1)$, and $C(2, 4)$. When T is transformed by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the image is T'.

(a) Find the coordinates of the vertices of T'. (2)

Solution

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \end{pmatrix};$$

hence, $\underline{\underline{A'(1,1)}}$, $\underline{\underline{B'(1,2)}}$, and $\underline{\underline{C'(4,2)}}$.

(b) Describe fully the transformation represented by $\mathbf{P}$. (2)

Solution

It is $\underline{\underline{\text{a reflection in the line } y = x}}$.

The matrices

$$\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represent two transformations. When T is transformed by the matrix $\mathbf{QR}$, the image is T''.

(c) Find $\mathbf{QR}$. (2)

Solution

$$\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}}}.$$

(d) Find the determinant of $\mathbf{QR}$. (2)

Solution

$$\det \mathbf{QR} = (-2) \times 2 - 0 \times 0 = \underline{\underline{-4}}.$$

(e) Using your answer to part (d), find the area of T''. (3)

Solution

$$\text{area of } T'' = |-4| \times \tfrac{1}{2}(1)(3) = \underline{\underline{6 \text{ square units}}}.$$

12.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix},$$

(a) Show that $\mathbf{A}$ is non-singular. (2)

Solution

$$\det \mathbf{A} = 0 \times 3 - 1 \times 2 = -2$$

so $\underline{\underline{\mathbf{A} \text{ is non-singular}}}$.

(b) Find $\mathbf{B}$ such that $\mathbf{BA}^2 = \mathbf{A}$. (4)

Solution

$$\begin{aligned} \mathbf{BA}^2 = \mathbf{A} &\Rightarrow \mathbf{BA} = \mathbf{I} \\ &\Rightarrow \mathbf{B} = \mathbf{A}^{-1} \\ &\Rightarrow \underline{\underline{\mathbf{B} = -\tfrac{1}{2}\begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix}}}. \end{aligned}$$

13. Given that (4)

$$\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix},$$

where k is a constant and

$$\mathbf{E} = \mathbf{C} + \mathbf{D},$$

find the value of k for which $\mathbf{E}$ has no inverse.

Solution

$$\begin{aligned} \det \begin{pmatrix} 8 & 2+2k \\ 12 & 6+k \end{pmatrix} = 0 &\Rightarrow 8(6+k) - 12(2+2k) = 0 \\ &\Rightarrow 48 + 8k = 24 + 24k \\ &\Rightarrow 16k = 24 \\ &\Rightarrow \underline{\underline{k = 1\tfrac{1}{2}}}. \end{aligned}$$

14.

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}.$$

(a) Find $\det \mathbf{M}$. (1)

Solution

$$\det \mathbf{M} = 3 \times (-5) - 4 \times 2 = \underline{\underline{-23}}.$$

The transformation represented by $\mathbf{M}$ maps the point $S(2a-7, a-1)$, where a is a constant, onto the point $S'(25, -14)$.

(b) Find the value of a. (3)

Solution
$$\begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2a-7 \\ a-1 \end{pmatrix} = \begin{pmatrix} 25 \\ -14 \end{pmatrix} \Rightarrow \begin{pmatrix} 3(2a-7)+4(a-1) \\ 2(2a-7)-5(a-1) \end{pmatrix} = \begin{pmatrix} 25 \\ -14 \end{pmatrix}.$$
Now,
$$6a - 21 + 4a - 4 = 25 \Rightarrow 10a = 50 \Rightarrow \underline{\underline{a = 5}}.$$

The point R has coordinates $(6, 0)$. Given that O is the origin,

(c) find the area of triangle ORS. (2)

Solution

$$\text{area of the triangle} = \tfrac{1}{2} \times 6 \times 4 = \underline{\underline{12 \text{ square units}}}.$$

Triangle ORS is mapped onto triangle $OR'S'$ by the transformation represented by $\mathbf{M}$.

(d) Find the area of triangle $OR'S'$. (2)

Solution

$$-23 \times 12 = -276$$

and

$$\text{area of the triangle} = \underline{\underline{276 \text{ square units}}}.$$

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

(e) describe fully the single geometrical transformation represented by $\mathbf{A}$. (2)

Solution

It is a rotation, 90° anti-clockwise, about $(0, 0)$.

The transformation represented by $\mathbf{A}$ followed by the transformation represented by $\mathbf{B}$ is equivalent to the transformation represented by $\mathbf{M}$.

(f) Find $\mathbf{B}$ (4)

Solution

$$\begin{aligned}\mathbf{BA} = \mathbf{M} \Rightarrow \mathbf{B} &= \mathbf{MA}^{-1} \\ \Rightarrow \mathbf{B} &= \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \Rightarrow \mathbf{B} &= \underline{\underline{\begin{pmatrix} -4 & 3 \\ 5 & 2 \end{pmatrix}}}.\end{aligned}$$

15. The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation through 90° anticlockwise about the origin.

(a) Write down the matrix $\mathbf{P}$. (1)

Solution

$$\mathbf{P} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}.$$

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y = -x$.

(b) Write down the matrix $\mathbf{Q}$. (1)

Solution

$$\mathbf{Q} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}}.$$

Given that U followed by V is transformation T, which is represented by the matrix $\mathbf{R}$,

(c) express $\mathbf{R}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$, (1)

Solution

$\mathbf{R} = \underline{\underline{\mathbf{QP}}}$.

(d) find the matrix $\mathbf{R}$, (2)

Solution

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}}.$$

(e) give a full geometrical description of T as a single transformation. (2)

Solution

It is a reflection in the y-axis.

16.

$$\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix},$$

where a is a constant.

(a) Find the value of a for which the matrix $\mathbf{X}$ is singular. (2)

Solution

$$\begin{aligned} \det \mathbf{X} = 0 &\Rightarrow 2 - 3a = 0 \\ &\Rightarrow \underline{\underline{a = \tfrac{2}{3}}}. \end{aligned}$$

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

(b) Find $\mathbf{Y}^{-1}$. (2)

Solution

$$\det \mathbf{Y}^{-1} = 2 - (-3) = 5$$

and

$$\mathbf{Y}^{-1} = \underline{\underline{\tfrac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}}}.$$

The transformation represented by $\mathbf{Y}$ maps the point A onto the point B. Given that B has coordinates $(1-\lambda, 7\lambda-2)$, where λ is a constant,

(c) find, in terms of λ, the coordinates of point A. (4)

Solution

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1-\lambda \\ 7\lambda-2 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1-\lambda \\ 7\lambda-2 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 2(1-\lambda)+(7\lambda-2) \\ -3(1-\lambda)+(7\lambda-2) \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 5\lambda \\ 10\lambda-5 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda-1 \end{pmatrix};$$

hence, $\underline{\underline{A(\lambda, 2\lambda-1)}}$.

17. (4)

$$\mathbf{M} = \begin{pmatrix} x & x-2 \\ 3x-6 & 4x-11 \end{pmatrix}.$$

Given that the matrix $\mathbf{M}$ is singular, find the possible values of x.

Solution

$$\begin{aligned} \det \mathbf{M} = 0 &\Rightarrow x(4x-11) - (x-2)(3x-6) = 0 \\ &\Rightarrow (4x^2-11x) - (3x^2-12x+12) = 0 \\ &\Rightarrow x^2 + x - 12 = 0 \\ &\Rightarrow (x+4)(x-3) = 0 \\ &\Rightarrow \underline{\underline{x=-4}} \text{ or } \underline{\underline{x=3}}. \end{aligned}$$

18.

$$\mathbf{A} = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}$$

and $\mathbf{I}$ is the 2×2 identity matrix.

(a) Prove that (2)

$$\mathbf{A}^2 = 7\mathbf{A} + 2\mathbf{I}.$$

Solution

$$\text{LHS} = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ -28 & 9 \end{pmatrix}$$

and

$$\text{RHS} = 7\mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 42 & -14 \\ -28 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ -28 & 9 \end{pmatrix};$$

hence, the LHS and RHS agree.

(b) Hence show that (2)

$$\mathbf{A}^{-1} = \tfrac{1}{2}(\mathbf{A} - 7\mathbf{I}).$$

Solution

Post-multiply by $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A}^2 = 7\mathbf{A} + 2\mathbf{I} &\Rightarrow \mathbf{A}^2\mathbf{A}^{-1} = 7\mathbf{A}\mathbf{A}^{-1} + 2\mathbf{I}\mathbf{A}^{-1} \\ &\Rightarrow \mathbf{A} = 7\mathbf{I} + 2\mathbf{A}^{-1} \\ &\Rightarrow 2\mathbf{A}^{-1} = \mathbf{A} - 7\mathbf{I} \\ &\Rightarrow \mathbf{A}^{-1} = \tfrac{1}{2}(\mathbf{A} - 7\mathbf{I}). \end{aligned}$$

The transformation represented by $\mathbf{A}$ maps the point P onto the point Q. Given that Q has coordinates $(2k + 8, -2k - 5)$, where k is a constant,

(c) find, in terms of k, the coordinates of P. (4)

Solution

$$\begin{aligned} \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2k+8 \\ -2k-5 \end{pmatrix} &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{2}(\mathbf{A} - 7\mathbf{I})\begin{pmatrix} 2k+8 \\ -2k-5 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 2k+8 \\ -2k-5 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k+1 \\ 2k-1 \end{pmatrix}; \end{aligned}$$

hence, $P(k + 1, 2k - 1)$.

19.

$$\mathbf{A} = \begin{pmatrix} 2k+1 & k \\ -3 & -5 \end{pmatrix},$$

where k is a constant. Given that

$$\mathbf{B} = \mathbf{A} + 3\mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix, find

(a) $\mathbf{B}$ in terms of k, (2)

Solution

$$\mathbf{B} = \begin{pmatrix} 2k+1 & k \\ -3 & -5 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2k+4 & k \\ -3 & -2 \end{pmatrix}}}.$$

(b) the value of k for which $\mathbf{B}$ is singular. (2)

Solution

$$\begin{aligned} \det \mathbf{B} = 0 &\Rightarrow -2(2k+4) - (-3k) = 0 \\ &\Rightarrow -4k - 8 + 3k = 0 \\ &\Rightarrow \underline{\underline{k = -8}}. \end{aligned}$$

20.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

The transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$ is equivalent to the transformation represented by $\mathbf{P}$.

(a) Find the matrix $\mathbf{P}$. (2)

Solution

$$\mathbf{P} = \mathbf{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix}}}.$$

Triangle T is transformed to the triangle T' by the transformation represented by $\mathbf{P}$. Given that the area of triangle T' is 24 square units,

(b) find the area of triangle T. (3)

Solution

$$\det \mathbf{P} = 1 \times (-3) - 4 \times (-2) = 5$$

and

$$\text{area of triangle } T = \frac{\text{area of triangle } T'}{5} = \frac{24}{5} = \underline{\underline{4\tfrac{4}{5} \text{ square units}}}.$$

Triangle T' is transformed to the original triangle T by the the matrix represented by $\mathbf{Q}$.

(c) Find the matrix $\mathbf{Q}$. (2)

Solution

$$\mathbf{Q} = \mathbf{P}^{-1} = \underline{\underline{\tfrac{1}{5}\begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix}}}.$$

21. Given that (3)

$$\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix},$$

where k is a real number, find $\mathbf{C}^{-1}$, giving your answer in terms of k.

Solution

$$\det \mathbf{C} = 2k \times k - 3 \times (-2) = 2k^2 + 6$$

and

$$\mathbf{C}^{-1} = \underline{\underline{\tfrac{1}{2k^2+6}\begin{pmatrix} k & 2 \\ -3 & 2k \end{pmatrix}}}.$$

22. In each of the following cases, find a 2×2 matrix that represents

(a) (i) a reflection in the line $y = -x$, (4)

Solution

$$\underline{\underline{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}}}.$$

(ii) a rotation of 135° anticlockwise about $(0, 0)$,

Solution
$\underline{\underline{\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}}$.

(iii) a reflection in the line $y = -x$ followed by a rotation of $135°$ anticlockwise about $(0, 0)$.

Solution

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}}.$$

The triangle T has vertices at the points $(1, k)$, $(3, 0)$, and $(11, 0)$, where k is a constant. Triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}.$$

(b) Given that the area of triangle T' is 364 square units, find the value of k. (6)

Solution

$$\det\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix} = 6 \times 2 - 1 \times (-2) = 14$$

and

$$\text{area of triangle } T = \frac{\text{area of triangle } T'}{14} = \frac{364}{14} = 26.$$

Finally,

$$\tfrac{1}{2} \times 8 \times k = 26 \Rightarrow \underline{\underline{k = 6\tfrac{1}{2}}}.$$

23.

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) (i) Describe fully the single transformation represented by the matrix $\mathbf{A}$. (2)

Solution
$\mathbf{A}$ is a $\underline{\underline{\text{transformation of } 45°\text{, anticlockwise, about } (0, 0)}}$.

The matrix $\mathbf{B}$ represents an enlargement, scale factor -2, with centre the origin.

(ii) Write down the matrix **B**. (1)

Solution
$\underline{\underline{\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}}}.$

$$\mathbf{M} = \begin{pmatrix} 3 & k \\ -2 & 3 \end{pmatrix},$$

where k is a positive constant. Triangle T has an area of 16 square units. Triangle T is transformed onto the triangle T' by the transformation represented by the matrix **M**.

(b) Given that the area of the triangle T' is 224 square units, find the value of k. (3)

Solution

$$\det \mathbf{M} = 3 \times 3 - k \times (-2) = 2k + 9$$

and

$$|2k + 9| = \frac{224}{16} \Rightarrow |2k + 9| = 14 \Rightarrow \underline{\underline{k = 2\tfrac{1}{2}}},$$

as $k > 0$.

24.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Given that $\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$,

(a) calculate the matrix **M**, (6)

Solution

$$\begin{aligned} \mathbf{M} &= (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B}) \\ &= \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 1 & -1 \\ -7 & -2 \end{pmatrix}}}. \end{aligned}$$

(b) find the matrix **C** such that **MC** = **A**. (4)

Solution

$$\mathbf{MC} = \mathbf{A} \Rightarrow \mathbf{C} = \mathbf{M}^{-1}\mathbf{A}$$
$$\Rightarrow \mathbf{C} = -\tfrac{1}{9}\begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\Rightarrow \mathbf{C} = \underline{\underline{-\tfrac{1}{9}\begin{pmatrix} -5 & -2 \\ 13 & 7 \end{pmatrix}}}.$$

25. (a) (4)

$$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix},$$

where k is a real constant. Given that $\mathbf{A}$ is a singular matrix, find the possible values of k.

Solution

$$\det \mathbf{A} = 0 \Rightarrow 5k(k+1) - (-3)(3k-1) = 0$$
$$\Rightarrow 5k^2 + 14k - 3 = 0$$
$$\Rightarrow (5k-1)(k+3) = 0$$
$$\Rightarrow \underline{\underline{k = -3}} \text{ or } \underline{\underline{k = \tfrac{1}{5}}}.$$

(b)

$$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}.$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix $\mathbf{B}$. The vertices of triangle T' have coordinates $(0,0)$, $(-20,6)$, and $(10c, 6c)$, where c is a positive constant. The area of triangle T' is 135 square units.

(i) Find the matrix $\mathbf{B}^{-1}$. (2)

Solution

$$\det \mathbf{B} = 10 \times 3 - (-3) \times 5 = 45$$

and

$$\mathbf{B}^{-1} = \underline{\underline{\tfrac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}}}.$$

(ii) Find the coordinates of the vertices of the triangle T, in terms of c where necessary. (3)

Solution

$$\begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \tfrac{1}{45}\begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}\begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \tfrac{1}{45}\begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix};$$

the coordinates of are $\underline{\underline{(0, 0)}}$, $\underline{\underline{(-2, 0)}}$, and $\underline{\underline{(0, 2c)}}$.

(iii) Find the value of c. (3)

Solution

$$\text{area of triangle } T = \frac{\text{area of triangle } T'}{|\det \mathbf{B}|} = \frac{135}{45} = 3$$

and

$$\tfrac{1}{2} \times 2 \times 2c = 3 \Rightarrow \underline{\underline{c = 1\tfrac{1}{2}}}.$$

26. Given that k is a real number and that (3)

$$\mathbf{A} = \begin{pmatrix} 1+k & k \\ k & 1-k \end{pmatrix},$$

find the exact values of k for which $\mathbf{A}$ is a singular matrix. Give your answers in their simplest form.

Solution

$$\begin{aligned} \det \mathbf{A} = 0 &\Rightarrow (1+k)(1-k) - k^2 = 0 \\ &\Rightarrow 1 - 2k^2 = 0 \\ &\Rightarrow \underline{\underline{k = \pm\tfrac{\sqrt{2}}{2}}}. \end{aligned}$$

27.

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Describe fully the single geometrical transformation U represented by the matrix $\mathbf{P}$. (2)

Solution

It is a rotation of 135° anticlockwise about $(0,0)$.

The transformation U maps the point A, with coordinates (p,q), onto the point B, with $(6\sqrt{2}, 3\sqrt{2})$.

(b) Find the value of p and the value of q. (3)

Solution

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix} \Rightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix};$$

hence, $p = -3$ and $q = -9$.

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line with equation $y = x$.

(c) Write down the matrix $\mathbf{Q}$. (1)

Solution

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The transformation U followed by the transformation V is the transformation T. The transformation T is represented by the matrix $\mathbf{R}$.

(d) Find the matrix $\mathbf{R}$. (3)

Solution

$$\mathbf{R} = \mathbf{QP} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(e) Deduce that the transformation T is self-inverse. (1)

Solution

$$\det \mathbf{R} = \tfrac{1}{\sqrt{2}} \times (-\tfrac{1}{\sqrt{2}}) - (-\tfrac{1}{\sqrt{2}}) \times (-\tfrac{1}{\sqrt{2}}) = -1$$

and

$$\mathbf{R}^{-1} = -\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{R};$$

so, the matrix is self-inverse and so transformation is self-inverse.

28.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}.$$

(a) Find $\mathbf{A}^{-1}$. (2)

Solution

$$\det \mathbf{A} = 2 \times 3 - 4 \times (-1) = 10$$

and

$$\mathbf{A}^{-1} = \tfrac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}.$$

The transformation represented by the matrix **B** followed by the transformation represented by the matrix **A** is equivalent to the transformation represented by the matrix **P**.

(b) Find **B**, giving your answer in its simplest form. (3)

Solution

$$\begin{aligned}
\mathbf{P} = \mathbf{AB} &\Rightarrow \mathbf{A}^{-1}\mathbf{P} = \mathbf{A}^{-1}\mathbf{AB} \\
&\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{P} \\
&\Rightarrow \mathbf{B} = \tfrac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix} \\
&\Rightarrow \mathbf{B} = \tfrac{1}{10}\begin{pmatrix} 20 & 10 \\ 10 & -40 \end{pmatrix} \\
&\Rightarrow \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}.
\end{aligned}$$

29.

$$\mathbf{A} = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix},$$

where p is a constant.

(a) (i) Find, in terms of p, the matrix $\mathbf{AB}$. (2)

> **Solution**
>
> $$\begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}\begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 12-5p & 4p-10 \\ 6p-15 & 12-5p \end{pmatrix}}}.$$

Given that

$$\mathbf{AB} + 2\mathbf{A} = k\mathbf{I},$$

where k is a constant and $\mathbf{I}$ is the 2×2 identity matrix,

(ii) find the value of p and the value of k. (4)

> **Solution**
>
> $$\begin{aligned} \mathbf{AB} + 2\mathbf{A} = k\mathbf{I} &\Rightarrow \begin{pmatrix} 12-5p & 4p-10 \\ 6p-15 & 12-5p \end{pmatrix} + 2\begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix} = k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} 12-3p-k & 4p-6 \\ 6p-9 & 12-3p-k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \end{aligned}$$
>
> now,
>
> $$4p - 6 = 0 \Rightarrow \underline{\underline{p = 1\tfrac{1}{2}}}$$
>
> and
>
> $$12 - 3p - k = 0 \Rightarrow k = 12 - 3 \times (1\tfrac{1}{2}) = \underline{\underline{7\tfrac{1}{2}}}.$$

$$\mathbf{M} = \begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix},$$

where a is a real constant. Triangle T has an area of 15 square units. Triangle T is transformed to the triangle T' by the transformation represented by the matrix $\mathbf{M}$.

(b) Given that the area of triangle T' is 270 square units, find the possible values of a. (5)

> **Solution**
>
> $$\det \mathbf{M} = a \times 2 - 1 \times (-9) = 2a + 9$$

and

$$
\begin{aligned}
270 = |\det \mathbf{M}| \times 15 &\Rightarrow |\det \mathbf{M}| = 18 \\
&\Rightarrow 2a + 9 = 18 \text{ or } 2a + 9 = -18 \\
&\Rightarrow \underline{\underline{a = 4\tfrac{1}{2}}} \text{ or } \underline{\underline{a = -13\tfrac{1}{2}}}.
\end{aligned}
$$