Dr Oliver Mathematics
OCR FMSQ Additional Mathematics
2008 Paper
2 hours

The total number of marks available is 100.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.
Final answers should be given correct to three significant figures where appropriate.

Section A

1. A driver of a car, initially moving at 30 ms^{-1}, applies the brakes so that the car comes to rest with constant deceleration in 10 seconds.

   (a) Find the value of the deceleration. (2)

   Solution
   $s =?$, $u = 30$, $v = 0$, $a =?$, and $t = 10$: use $v = u + at$:

   $$0 = 30 + 10a \Rightarrow 10t = -30$$
   $$\Rightarrow a = -3;$$

   hence, the deceleration is $\underline{\underline{3 \text{ ms}^{-2}}}$.

   (b) Find the distance travelled in this time. (2)

   Solution
   $s =?$, $u = 30$, $v = 0$, $a = -3$, and $t = 10$: use $s = ut + \frac{1}{2}at^2$:

   $$s = (30 \times 10) + (\tfrac{1}{2} \times (-3) \times 10^2)$$
   $$= 300 - 150$$
   $$= \underline{\underline{150 \text{ m}}}.$$

2. The points A and B have coordinates $(0, 8)$ and $(6, 0)$ respectively.

   (a) Find the equation of the line AB. (3)

Solution

$$\begin{aligned}\text{Gradient} &= \frac{8-0}{0-6}\\ &= -\tfrac{4}{3}\end{aligned}$$

and the equation of the line AB is

$$y - 0 = -\tfrac{4}{3}(x-6) \Rightarrow \underline{\underline{y = -\tfrac{4}{3}x + 8}}.$$

(b) Find the equation of the line perpendicular to AB through its midpoint. (4)

Solution

The midpoint is

$$\left(\frac{6+0}{2}, \frac{0+8}{2}\right) = (3, 4)$$

and the gradient of the tangent is

$$-\frac{1}{\frac{4}{3}} = \tfrac{3}{4}.$$

Hence, the equation of the line perpendicular to AB through its midpoint is

$$\begin{aligned} y - 4 = \tfrac{3}{4}(x-3) &\Rightarrow y - 4 = \tfrac{3}{4}x - 2\tfrac{1}{4}\\ &\Rightarrow \underline{\underline{y = \tfrac{3}{4}x + 1\tfrac{3}{4}}}.\end{aligned}$$

3. Find the points of intersection of the line (5)

$$y = 5x + 13$$

with the circle

$$x^2 + y^2 = 13.$$

Solution

$$x^2 + y^2 = 13 \Rightarrow x^2 + (5x+13)^2 = 13$$

	$5x$	$+13$
$5x$	$25x^2$	$+65x$
$+13$	$+65x$	$+169$

$$\Rightarrow x^2 + (25x^2 + 130x + 169) = 13$$
$$\Rightarrow 26x^2 + 130x + 156 = 0$$
$$\Rightarrow 26(x^2 + 5x + 6) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & +5 \\ \text{multiply to:} & +6\end{array}\right\} +2,\ +3$$

$$\Rightarrow 26(x + 2)(x + 3) = 0$$
$$\Rightarrow x + 2 = 0 \text{ or } x + 3 = 0$$
$$\Rightarrow x = -2 \text{ or } x = -3$$
$$\Rightarrow y = 3 \text{ or } y = -2;$$

hence, the points are $\underline{\underline{(-2, 3) \text{ and } (-3, -2)}}$.

4. Glass marbles are produced in two colours, red and green, in the proportion 7 : 3 respectively. From a large stock of the marbles, 5 are taken at random.

Find the probability that

(a) all 5 are red, (2)

Solution

$$\begin{aligned} \text{P(all 5 are red)} &= (\tfrac{7}{10})^5 \\ &= \underline{\underline{0.168\,07}}. \end{aligned}$$

(b) exactly 3 are red. (3)

Solution

$$\text{P(exactly 3 are red)} = \binom{5}{3}\left(\tfrac{7}{10}\right)^3\left(\tfrac{3}{10}\right)^2$$
$$= \underline{\underline{0.308\,7}}.$$

5. (a) Use calculus to find the stationary points on the curve (6)

$$y = x^3 - 3x + 1,$$

identifying which is a maximum and which is a minimum.

Solution

$$y = x^3 - 3x + 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 3x^2 - 3 = 0$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = -1 \text{ or } x = 1.$$

Next,

$$x = -1 \Rightarrow y = 3$$

and

$$x = 1 \Rightarrow y = -1.$$

Finally,

$$x = -1 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6 < 0$$

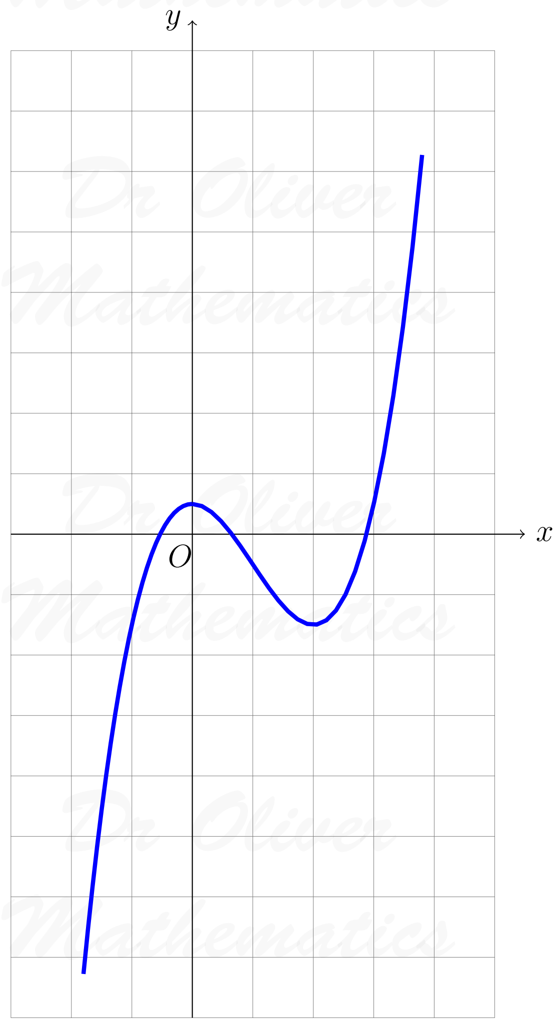
which makes $\underline{\underline{(-1, 3) \text{ a minimum point}}}$ and

$$x = 1 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6 > 0$$

which makes $\underline{\underline{(1, -1) \text{ a maximum point}}}$.

(b) Sketch the curve. (1)

Solution



6. A speedboat accelerates from rest so that t seconds after starting its velocity, in ms^{-1}, is given by the formula

$$v = 0.36t^2 - 0.024t^3.$$

(a) Find the acceleration at time t. (3)

Solution

$$v = 0.36t^2 - 0.024t^3 \Rightarrow a = (0.72t - 0.072t^2) \text{ ms}^{-2}.$$

(b) Find the distance travelled in the first 10 seconds. (4)

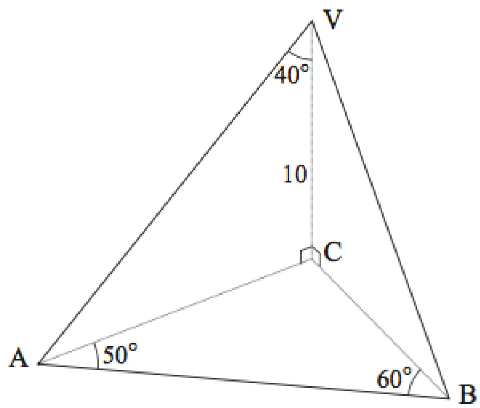
Solution

$$v = 0.36t^2 - 0.024t^3 \Rightarrow s = (0.12t^3 - 0.006t^4 + c) \text{ m}.$$

Finally,

$$\begin{aligned}\text{distance travelled} &= [0.12 \times 10^3 - 0.006 \times 10^4 + c) - 0.12 \times 0 - 0.006 \times 0 + c) \\ &= (60 + c) - c \\ &= \underline{\underline{60 \text{ m}}}.\end{aligned}$$

7. A pyramid stands on a horizontal triangular base, ABC, as shown in the figure.



The angles CAB and ABC are $50°$ and $60°$ respectively.
The vertex, V, is directly above C with $VC = 10$ m.
The angle which the edge VA makes with the vertical is $40°$.

(a) Calculate AC. (2)

Solution

$$\begin{aligned}AC &= 10 \tan 40° \\ &= 8.390\,996\,312 \text{ (FCD)} \\ &= \underline{\underline{8.39 \text{ m (3 sf)}}}.\end{aligned}$$

(b) Hence calculate AB. (4)

Solution

$$\angle ACB = 180 - 50 - 60 = 70^\circ$$

and

$$\begin{aligned} \frac{AB}{\sin 70^\circ} = \frac{8.390\ldots}{\sin 60^\circ} &\Rightarrow AB = \frac{8.390\ldots \times \sin 70^\circ}{\sin 60^\circ} \\ &\Rightarrow AB = 9.104\,764\,457 \text{ (FCD)} \\ &= \underline{\underline{9.10 \text{ m (3 sf)}}}. \end{aligned}$$

8. It is required to solve the equation

$$2\cos^2 x = 5\sin x - 1.$$

(a) Show that this equation may be written as (2)

$$2\sin^2 x + 5\sin x - 3 = 0.$$

Solution

$$\begin{aligned} 2\cos^2 x = 5\sin x - 1 &\Rightarrow 2(1 - \sin^2 x) = 5\sin x - 1 \\ &\Rightarrow 2 - 2\sin^2 x = 5\sin x - 1 \\ &\Rightarrow \underline{\underline{2\sin^2 x + 5\sin x - 3 = 0}}, \end{aligned}$$

as required.

(b) Hence solve the equation (4)

$$2\cos^2 x = 5\sin x - 1$$

for values of x in the range $0^\circ \leqslant x \leqslant 360^\circ$.

Solution

$$2\cos^2 x = 5\sin x - 1 \Rightarrow 2\sin^2 x + 5\sin x - 3 = 0$$

$$\left.\begin{array}{ll}\text{add to:} & +5 \\ \text{multiply to:} & (+2)\times(-3) = -6\end{array}\right\} -1,\ +6$$

$$\begin{aligned}
&\Rightarrow 2\sin^2 x + 6\sin x - \sin x - 3 = 0 \\
&\Rightarrow 2\sin x(\sin x + 3) - (\sin x + 3) = 0 \\
&\Rightarrow (2\sin x - 1)(\sin x + 3) = 0 \\
&\Rightarrow \sin x = \tfrac{1}{2} \text{ (because } \sin x \neq -3) \\
&\Rightarrow \underline{\underline{x = 30^\circ,\ 150^\circ}}.
\end{aligned}$$

9. The cubic equation (5)

$$x^3 + ax^2 + bx - 26 = 0$$

has 3 positive, distinct, integer roots.

Find the values of a and b.

Solution

Let us call the solutions α, β, and γ.

	x	$-\beta$
x	x^2	$-\beta x$
$-\gamma$	$-\gamma x$	$+\beta\gamma$

$$(x-\alpha)(x-\beta)(x-\gamma) = (x-\alpha)[x^2 - (\beta+\gamma)x + \beta\gamma].$$

	x^2	$-(\beta+\gamma)x$	$+\beta\gamma$
x	x^3	$-(\beta+\gamma)x^2$	$+\beta\gamma x$
$-\alpha$	$-\alpha x^2$	$-(\alpha+\beta+\gamma)x$	$-\alpha\beta\gamma x$

Hence,

$$(x-\alpha)(x-\beta)(x-\gamma) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma.$$

Then

$$\begin{aligned}\alpha+\beta+\gamma &= -a\\ \alpha\beta+\alpha\gamma+\beta\gamma &= b\\ \alpha\beta\gamma &= 26.\end{aligned}$$

Thus,

$$\alpha\beta\gamma = 1\times 2\times 13.$$

All the roots are positive so, for example, we take $\alpha = 1$, $\beta = 2$, and $\gamma = 13$. Next,

$$\alpha+\beta+\gamma = 16$$

and

$$\alpha\beta+\alpha\gamma+\beta\gamma = 41.$$

So,

$$\underline{\underline{a = -16}} \text{ and } \underline{\underline{b = 41}}.$$

Section B

10. Simon and Gavin each drive a distance of 140 km along a motorway, both at constant speed. Simon drives at 5 km per hour faster than Gavin.

Let Gavin's speed be v km per hour.

(a) Write down expressions in terms of v for the times, in hours, taken by Gavin and Simon. (2)

Solution

$$\text{Gary's rate} = \underline{\underline{\left(\frac{140}{v}\right)\text{ h}}} \text{ and Simon's rate} = \underline{\underline{\left(\frac{140}{v+5}\right)\text{ h}}}.$$

Simon completes the journey in 15 minutes less than Gavin.

(b) Explain why (5)

$$\frac{140}{v} - \frac{140}{v+5} = \tfrac{1}{4}$$

and show that this equation reduces to the equation

$$v^2 + 5v - 2\,800 = 0.$$

Solution

$$\begin{aligned} &\text{Gary's rate} - \text{Simon's rate} = 15 \text{ min} \\ \Rightarrow\quad &\left(\frac{140}{v}\right) \text{ h} - \left(\frac{140}{v+5}\right) \text{ h} = \tfrac{1}{4} \text{ h} \\ \Rightarrow\quad &\underline{\underline{\frac{140}{v} - \frac{140}{v+5} = \tfrac{1}{4}}}, \end{aligned}$$

as required. Multiply by $4v(v+5)$:

$$\begin{aligned} \frac{140}{v} - \frac{140}{v+5} = \tfrac{1}{4} &\Rightarrow 560(v+5) - 560v = v(v+5) \\ &\Rightarrow 2\,800 = v^2 + 5v \\ &\Rightarrow \underline{\underline{v^2 + 5v - 2\,800 = 0}}, \end{aligned}$$

as required.

(c) Solve this equation to find v and hence find the times taken by Simon and Gavin. Give your answers correct to the nearest minute. (5)

Solution

$a = 1$, $b = 5$, $c = -2\,800$:

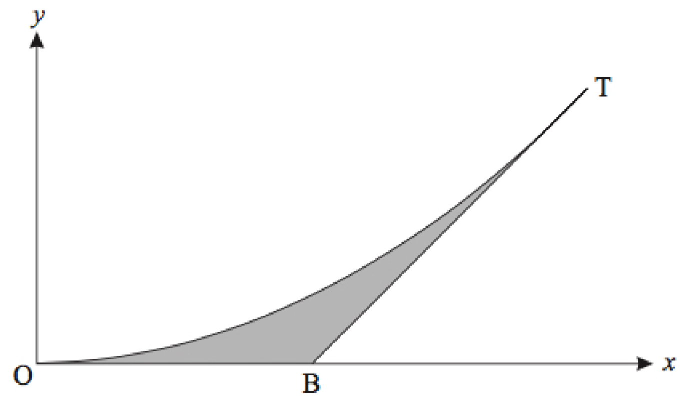
$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times (-2\,800)}}{2 \times 1} \\ &= \frac{-5 \pm \sqrt{11\,225}}{2} \\ &= -55.474\ldots \text{ or } 50.474\ldots; \end{aligned}$$

so, Gary takes

$$\begin{aligned} \frac{140}{50.474\ldots} &= 2.773\,702\,513 \text{ hours (FCD)} \\ &= 2 \text{ hours } 46.422\,150\,75 \text{ min (FCD)} \\ &= \underline{\underline{2 \text{ hours } 46 \text{ min (to the nearest minute)}}} \end{aligned}$$

and Simon takes $\underline{\underline{\text{2 hours 31 min (to the nearest minute)}}}$.

11. The side of a fairground slide is in the shaded shape as shown in the figure below. Units are metres.



The curve has equation $y = \lambda x^2$.
T has coordinates $(4, 2)$.
The line BT is a tangent to the curve at T.
It meets the x-axis at the point B.

(a) Find the value of λ. (1)

Solution

$$2 = \lambda \times 4^2 \Rightarrow 16\lambda = 2$$
$$\Rightarrow \underline{\underline{\lambda = \tfrac{1}{8}}}.$$

(b) Find the equation of the tangent BT and hence find the coordinates of the point B. (6)

Solution

$$y = \tfrac{1}{8}x^2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{4}x$$

and

$$x = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

Now,

$$y - 2 = 1(x - 4) \Rightarrow \underline{\underline{y = x - 2}}$$

and the coordinates of the point B are $\underline{\underline{(2, 0)}}$.

(c) Find the area of the shaded portion of the graph. (5)

Solution

$$\begin{aligned}\text{Area} &= \text{area under the graph} - \text{area of the triangle} \\ &= \int_0^4 \tfrac{1}{8}x^2 \,\mathrm{d}x - (\tfrac{1}{2} \times 2 \times 2) \\ &= \left[\tfrac{1}{24}x^3\right]_{x=0}^4 - 2 \\ &= \tfrac{1}{24}(64 - 0) - 2 \\ &= \underline{\underline{\tfrac{2}{3}}}.\end{aligned}$$

12. A furniture manufacturer produces tables and chairs.

In each week the following constraints apply.

- There are 24 workers, each working for 40 hours (i.e., there are 960 worker-hours available).
- There is a maximum of £1 800 available for the purchase of materials.
- Each table requires £30 worth of materials and 12 worker-hours.
- Each chair requires £10 worth of materials and 6 worker-hours.
- It is necessary to make at least 3 times as many chairs as tables.

Let x be the number of tables produced each week and y be the number of chairs produced each week.

(a) Show that the worker-hour constraint reduces to the inequality $2x + y \leqslant 160$. (2)

Solution

Now,

$$\text{worker-hours for the tables} = 12x$$

and

$$\text{worker-hours for the chairs} = 6y.$$

There are 960 worker-hours so

$$\begin{aligned}12x + 6y \leqslant 960 &\Rightarrow 6(2x + y) \leqslant 6 \times 160 \\ &\Rightarrow \underline{\underline{2x + y \leqslant 160}},\end{aligned}$$

as required.

(b) Find the inequality relating to the cost of materials constraint and the inequality relating to the numbers of tables and chairs. (3)

Solution

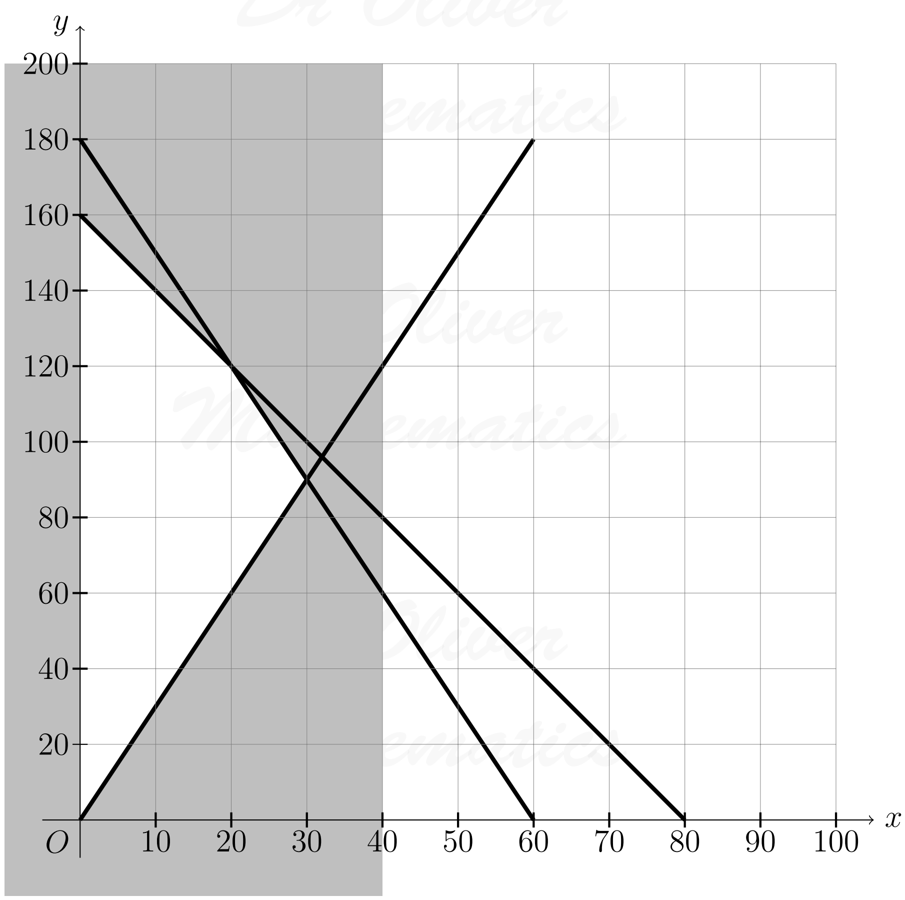
$$30x + 10y \leqslant 1\,800 \Rightarrow \underline{\underline{3x + y \leqslant 180}}$$

and

$$\underline{\underline{y \geqslant 3x}}.$$

(c) Plot these three inequalities on a graph, using 1 cm to represent 10 tables on the x-axis and 1 cm to represent 20 chairs on the y-axis. Indicate the region for which these inequalities hold. You should shade the region which is **not** required. (4)

Solution



When finished, each table is sold for a profit of £20 and each chair is sold for a profit of £5.

(d) The manufacturer wishes to maximise the profit. Explain why the objective function is given by (1)

$$P = 20x + 5y.$$

Solution

We wish to maximise the profit. Well, profit per table is £20 and the profit per chair £5 so that makes

$$\underline{\underline{P = 20x + 5y}}.$$

(e) Find the number of tables and chairs that should be made in order to maximise the profit. (2)

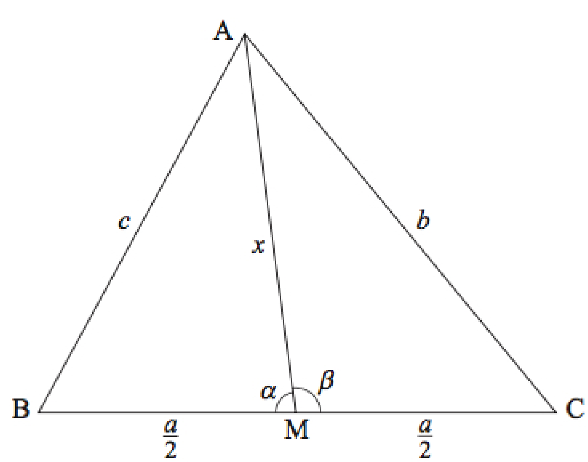
Solution

Two obvious methods. The first is to go around the polygon and determine $P = 20x + 5y$ at its vertices. The second is to go $20x + 5y =?$ and translate this line out and upwards: $20x + 5y = 0$, $20x + 5y = 10$, $20x + 5y = 20$, $\ldots$, until we reach the very last point on its vertices. We take the first one:

$$\begin{aligned} (0, 160): &\quad P = 20(0) + 5(160) = 800 \\ (20, 120): &\quad P = 20(20) + 5(120) = 1\,000 \\ (30, 90): &\quad P = 20(30) + 5(90) = 1\,050; \end{aligned}$$

hence, the number of tables is $\underline{\underline{30}}$ and the number of chairs is $\underline{\underline{90}}$ leading to a profit of $\underline{\underline{£1\,050}}$.

13. In the triangle shown below, M is the midpoint of BC.



(a) Explain why (2)

$$\cos\alpha = -\cos\beta.$$

Solution

$$\begin{aligned}\alpha+\beta = 180 &\Rightarrow \alpha = 180-\beta\\ &\Rightarrow \cos\alpha = \cos(180-\beta)\\ &\Rightarrow \underline{\underline{\cos\alpha = -\cos\beta}},\end{aligned}$$

as required.

(b) Using the cosine rule in the triangle BMA, show that (2)

$$\cos\alpha = \frac{4x^2+a^2-4c^2}{4ax}.$$

Solution

$$\begin{aligned}\cos\alpha &= \frac{x^2+(\frac{1}{2}a)^2-c^2}{2x(\frac{1}{2}a)}\\ &= \frac{x^2+\frac{1}{4}a^2-c^2}{ax}\\ &= \underline{\underline{\frac{4x^2+a^2-4c^2}{4ax}}},\end{aligned}$$

as required.

(c) Find a similar expression for $\cos\beta$. (1)

Solution

$$\begin{aligned}\cos\beta &= \frac{x^2+(\frac{1}{2}a)^2-b^2}{2x(\frac{1}{2}a)}\\ &= \frac{x^2+\frac{1}{4}a^2-b^2}{ax}\\ &= \underline{\underline{\frac{4x^2+a^2-4b^2}{4ax}}}.\end{aligned}$$

(d) Using the results in parts (a), (b), and c), show that (5)

$$4x^2 + a^2 = 2(c^2 + b^2).$$

Solution

$$\begin{aligned}\cos\alpha = -\cos\beta &\Rightarrow \frac{4x^2 + a^2 - 4c^2}{4ax} = -\left(\frac{4x^2 + a^2 - 4b^2}{4ax}\right)\\ &\Rightarrow \frac{4x^2 + a^2 - 4c^2}{4ax} = \frac{4b^2 - 4x^2 - a^2}{4ax}\\ &\Rightarrow 4x^2 + a^2 - 4c^2 = 4b^2 - 4x^2 - a^2\\ &\Rightarrow 8x^2 + 2a^2 = 4b^2 + 4c^2\\ &\Rightarrow 2(4x^2 + a^2) = 4(b^2 + c^2)\\ &\Rightarrow \underline{\underline{4x^2 + a^2 = 2(b^2 + c^2)}},\end{aligned}$$

as required.

A triangular lawn has sides 46 m, 29 m and 27 m.

(e) Find the distance from the midpoint of the longest side to the opposite corner. (2)

Solution

Take $a = 46$, $b = 29$, $c = 27$, and insert them into the equation for x:

$$\begin{aligned}4x^2 + a^2 = 2(b^2 + c^2) &\Rightarrow 4x^2 + 46^2 = 2(29^2 + 27^2)\\ &\Rightarrow 4x^2 = 1\,024\\ &\Rightarrow x^2 = 256\\ &\Rightarrow \underline{\underline{x = 16}}.\end{aligned}$$