Dr Oliver Mathematics
Further Mathematics
Method of Differences
Past Examination Questions

This booklet consists of 23 questions across a variety of examination topics.
The total number of marks available is 185.

1. (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions. (2)

Solution

$$\frac{1}{r(r+2)} = \underline{\underline{\frac{\frac{1}{2}}{r} - \frac{\frac{1}{2}}{r+2}}}.$$

(b) Hence prove, by the method of differences, that (5)

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}.$$

Solution

$$\begin{aligned}
\sum_{r=1}^{n} \frac{4}{r(r+2)} &= 2\sum_{r=1}^{n}\left(\frac{1}{r} + \frac{1}{r+2}\right) \\
&= 2\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right] \\
&= 2\left[1 + \frac{1}{2} - \frac{2}{n+1} - \frac{2}{n+2}\right] \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{(n+1)(n+2)} \\
&= \frac{3(n^2+3n+2) - 2n - 4 - 2n - 2}{(n+1)(n+2)} \\
&= \frac{3n^2+5n}{(n+1)(n+2)} \\
&= \underline{\underline{\frac{n(3n+5)}{(n+1)(n+2)}}}.
\end{aligned}$$

(c) Find the value of $\displaystyle\sum_{r=50}^{100} \frac{4}{r(r+2)}$, to 4 decimal places. (3)

Solution

$$\begin{aligned}\sum_{r=50}^{100} \frac{4}{r(r+2)} &= \sum_{r=1}^{100} \frac{4}{r(r+2)} - \sum_{r=1}^{49} \frac{4}{r(r+2)} \\ &= \frac{100(3\times 100+5)}{101\times 102} - \frac{49(3\times 49+5)}{50\times 51} \\ &= \frac{30\,500}{10\,302} - \frac{7\,448}{2\,550} \\ &= 0.039\,805\,862\,94 \text{ (FCD)} \\ &= \underline{\underline{0.039\,8 \text{ (4 dp)}}}.\end{aligned}$$

2. (a) By expressing (2)

$$\frac{2}{4r^2-1}$$

in partial fractions, or otherwise, prove that

$$\sum_{r=1}^{n} \frac{2}{4r^2-1} = 1 - \frac{1}{2n+1}.$$

Solution

$$\frac{2}{4r^2-1} = \frac{2}{(2r-1)(2r+1)} = \frac{1}{2r-1} - \frac{1}{2r+1},$$

and so

$$\begin{aligned}\sum_{r=1}^{n} \frac{2}{4r^2-1} &= \sum_{r=1}^{n} \left[\frac{1}{2r-1} - \frac{1}{2r+1}\right] \\ &= \left(1-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \underline{\underline{1 - \frac{1}{2n+1}}}.\end{aligned}$$

(b) Hence find the exact value of $\displaystyle\sum_{r=1}^{20} \frac{2}{4r^2-1}$. (3)

Solution

$$\begin{aligned}\sum_{r=11}^{20} \frac{2}{4r^2-1} &= \sum_{r=1}^{20} \frac{2}{4r^2-1} - \sum_{r=1}^{10} \frac{2}{4r^2-1} \\ &= \left(1-\frac{1}{41}\right) - \left(1-\frac{1}{21}\right) \\ &= \frac{1}{21} - \frac{1}{41} \\ &= \underline{\underline{\frac{20}{861}}}.\end{aligned}$$

3. Given that for all real values of r,

$$(2r+1)^3 - (2r-1)^3 \equiv Ar^2 + B,$$

where A and B are constants,

(a) find the value of A and find the value of B. (2)

Solution

$(2r+1)^3 - (2r-1)^3 \equiv (8r^3 + 12r^2 + 6r + 1) - (8r^3 - 12r^2 + 6r - 1) \equiv \underline{\underline{24r^2 + 2}}$.

(b) Hence, or otherwise, prove that (5)

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Solution

$$\begin{aligned}(2\times 1+1)^3 - (2\times 1-1)^3 &= 24\times 1^2 + 2 \\ (2\times 2+1)^3 - (2\times 2-1)^3 &= 24\times 2^2 + 2 \\ &\vdots \\ (2\times n+1)^3 - (2\times n-1)^3 &= 24\times n^2 + 2;\end{aligned}$$

adding up, we find

$$\begin{aligned}
& 24(1^2 + 2^2 + \ldots + n^2) + 2n = (2n+1)^3 - 1 \\
\Rightarrow\quad & 24 \sum_{r=1}^{n} r^2 = (2n+1)^3 - 2n - 1 \\
\Rightarrow\quad & 24 \sum_{r=1}^{n} r^2 = (8n^3 + 12n^2 + 6n + 1) - 2n - 1 \\
\Rightarrow\quad & 24 \sum_{r=1}^{n} r^2 = 8n^3 + 12n^2 + 4n \\
\Rightarrow\quad & 24 \sum_{r=1}^{n} r^2 = 4n(2n^2 + 3n + 1) \\
\Rightarrow\quad & 24 \sum_{r=1}^{n} r^2 = 4n(n+1)(2n+1) \\
\Rightarrow\quad & \underline{\underline{\sum_{r=1}^{n} r^2 = \tfrac{1}{6} n(n+1)(2n+1)}}.
\end{aligned}$$

(c) Calculate $\sum_{r=1}^{40} (3r-1)^2$. (3)

Solution

$$\begin{aligned}
\sum_{r=1}^{40} (3r-1)^2 &= \sum_{r=1}^{40} (9r^2 - 6r + 1) \\
&= 9 \sum_{r=1}^{40} r^2 - 6 \sum_{r=1}^{40} r + \sum_{r=1}^{40} 1 \\
&= \frac{3}{2}(40)(41)(81) - 3(40)(41) + 40 \\
&= 199\,260 - 4\,920 + 40 \\
&= \underline{\underline{194\,380}}.
\end{aligned}$$

4. (a) Show that (3)

$$\frac{r^3 - r + 1}{r(r+1)} \equiv r - 1 + \frac{1}{r} - \frac{1}{r+1}.$$

Solution

$$\begin{aligned}\frac{r^3-r+1}{r(r+1)} &\equiv r-1+\frac{A}{r}+\frac{B}{r+1}\\ &\equiv \frac{(r-1)r(r+1)+A(r+1)+Br}{r(r+1)}\end{aligned}$$

and so

$$r^3-r+1 \equiv (r-1)r(r+1)+A(r+1)+Br.$$

$\underline{r=0}$: $1=A$.
$\underline{r=-1}$: $-1=B$.
Hence

$$\frac{r^3-r+1}{r(r+1)} \equiv \underline{\underline{r-1+\frac{1}{r}-\frac{1}{r+1}}}.$$

(b) Find (6)

$$\sum_{r=1}^{n}\frac{r^3-r+1}{r(r+1)},$$

expressing your answer as a single fraction in its simplest form.

Solution

$$\begin{aligned}\sum_{r=1}^{n}\frac{r^3-r+1}{r(r+1)} &= \sum_{r=1}^{n}\left[r-1+\frac{1}{r}-\frac{1}{r+1}\right]\\ &= \left(1-1+\frac{1}{1}-\frac{1}{2}\right)+\left(2-1+\frac{1}{2}-\frac{1}{3}\right)\\ &\qquad +\ldots+\left(r-1+\frac{1}{r}-\frac{1}{r+1}\right)\\ &= \frac{1}{2}(n-1)n+1-\frac{1}{n+1}\\ &= \frac{n(n-1)(n+1)+2(n+1)-2}{2(n+1)}\\ &= \frac{(n^3-n)+2n+2-2}{2(n+1)}\\ &= \frac{n^3+n}{2(n+1)}\\ &= \underline{\underline{\frac{n(n^2+1)}{2(n+1)}}}.\end{aligned}$$

5. (a) Show that (2)

$$(r+1)^3 - (r-1)^3 \equiv 6r^2 + 2.$$

Solution

$$(r+1)^3 - (r-1)^3 \equiv (r^3 + 3r^2 + 3r + 1) - (r^3 - 3r^2 + 3r - 1) \equiv \underline{\underline{6r^2 + 2}}.$$

(b) Hence show that (5)

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Solution

$$\begin{aligned}(1+1)^3 - (1-1)^3 &= 6 \times 1^2 + 2 \\ (2+1)^3 - (2-1)^3 &= 6 \times 2^2 + 2 \\ &\vdots \\ (n+1)^3 - (n-1)^3 &= 6 \times n^2 + 2;\end{aligned}$$

adding up, we find

$$\begin{aligned}& 6(1^2 + 2^2 + \ldots + n^2) + 2n = (n+1)^3 + n^3 - 1 \\ \Rightarrow\quad & 6\sum_{r=1}^{n} r^2 = (n^3 + 3n^2 + 3n + 1) + n^3 - 1 - 2n \\ \Rightarrow\quad & 6\sum_{r=1}^{n} r^2 = 2n^3 + 3n^2 + n \\ \Rightarrow\quad & 6\sum_{r=1}^{n} r^2 = n(2n^2 + 3n + 1) \\ \Rightarrow\quad & 6\sum_{r=1}^{n} r^2 = n(n+1)(2n+1) \\ \Rightarrow\quad & \underline{\underline{\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).}}\end{aligned}$$

(c) Show that (4)

$$\sum_{r=n}^{2n} r^2 = \tfrac{1}{6}n(n+1)(an+b),$$

where a and b are constants to be found

Solution

$$\begin{aligned}\sum_{r=n}^{2n} r^2 &= \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2 \\ &= \tfrac{1}{6}(2n)(2n+1)(4n+1) - \tfrac{1}{6}(n-1)(n)(2n-1) \\ &= \tfrac{1}{6}n[2(2n+1)(4n+1) - (n-1)(2n-1)] \\ &= \tfrac{1}{6}n[(16n^2+12n+2) - (2n^2-3n+1)] \\ &= \tfrac{1}{6}n(14n^2+15n+1) \\ &= \underline{\underline{\tfrac{1}{6}n(n+1)(14n+1)}}.\end{aligned}$$

6. (a) Express (4)

$$\frac{5r+4}{r(r+1)(r+2)}$$

in partial fractions.

Solution

$$\begin{aligned}\frac{5r+4}{r(r+1)(r+2)} &\equiv \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2} \\ &\equiv \frac{A(r+1)(r+2) + Br(r+2) + Cr(r+1)}{r(r+1)(r+2)}\end{aligned}$$

and so

$$5r+4 \equiv A(r+1)(r+2) + Br(r+2) + Cr(r+1).$$

$\underline{r=0}$: $4 = 2A \Rightarrow A = 2$.
$\underline{r=-1}$: $-1 = -B \Rightarrow B = 1$.
$\underline{r=-2}$: $-6 = 2C \Rightarrow C = -3$.
Hence,

$$\frac{5r+4}{r(r+1)(r+2)} \equiv \underline{\underline{\frac{2}{r} + \frac{1}{r+1} - \frac{3}{r+2}}}.$$

(b) Hence, or otherwise, show that (5)

$$\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} = \frac{7n^2+11n}{2(n+1)(n+2)}.$$

Solution

$$
\begin{aligned}
\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} &= \sum_{r=1}^{n} \left(\frac{2}{r} + \frac{1}{r+1} - \frac{3}{r+2} \right) \\
&= \left(\frac{2}{1} + \frac{1}{2} - \frac{3}{3} \right) + \left(\frac{2}{2} + \frac{1}{3} - \frac{3}{4} \right) \\
&\qquad + \ldots + \left(\frac{2}{n} + \frac{1}{n+1} - \frac{3}{n+2} \right) \\
&= \frac{2}{1} + \frac{1}{2} + \frac{2}{2} - \frac{1}{n+1} - \frac{1}{n+1} - \frac{3}{n+2} \\
&= \frac{7}{2} - \frac{2}{n+1} - \frac{3}{n+2} \\
&= \frac{7(n+1)(n+2) - 4(n+2) - 6(n+1)}{2(n+1)(n+2)} \\
&= \frac{7(n^2+3n+2) - 4n - 8 - 6n - 6}{2(n+1)(n+2)} \\
&= \underline{\underline{\frac{7n^2+11n}{2(n+1)(n+2)}}}.
\end{aligned}
$$

7. (a) Express (2)

$$\frac{2}{(r+1)(r+3)}$$

in partial fractions.

Solution

$$\frac{2}{(r+1)(r+3)} = \underline{\underline{\frac{1}{r+1} - \frac{1}{r+3}}}.$$

(b) Hence prove, by the method of differences, that (6)

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(an+b)}{6(n+2)(n+3)},$$

where a and b are constants to be found.

Solution

$$\begin{aligned}
\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} &= \sum_{r=1}^{n} \left(\frac{1}{r+1} - \frac{1}{r+3} \right) \\
&= \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+3} \right) \\
&= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \\
&= \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3} \\
&= \frac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)} \\
&= \frac{5(n^2+5n+6) - 6n - 18 - 6n - 12}{6(n+2)(n+3)} \\
&= \frac{5n^2 + 13n}{6(n+2)(n+3)} \\
&= \underline{\underline{\frac{n(5n+13)}{6(n+2)(n+3)}}}.
\end{aligned}$$

(c) Find the value of (3)

$$\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)},$$

to 5 decimal places.

Solution

$$\begin{aligned}
\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)} &= \sum_{r=1}^{30} \frac{2}{(r+1)(r+3)} - \sum_{r=1}^{20} \frac{2}{(r+1)(r+3)} \\
&= \frac{30 \times 163}{6 \times 32 \times 33} - \frac{20 \times 113}{6 \times 22 \times 23} \\
&= \frac{815}{1056} - \frac{565}{759} \\
&= 0.027\,379\,776\,02 \text{ (FCD)} \\
&= \underline{\underline{0.027\,38 \text{ (5 dp)}}}.
\end{aligned}$$

8. (a) Show, using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, that (5)

$$\sum_{r=1}^{n}(6r^2 + 4r - 1) = n(n+2)(2n+1).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(6r^2 + 4r - 1) &= 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= n(n+1)(2n+1) + 2n(n+1) - n \\ &= n[(n+1)(2n+1) + 2(n+1) - 1] \\ &= n(2n^2 + 3n + 1 + 2n + 2 - 1) \\ &= n(2n^2 + 5n + 2) \\ &= \underline{\underline{n(n+2)(2n+1)}}.\end{aligned}$$

(b) Hence, or otherwise, find the value of (2)

$$\sum_{r=11}^{20}(6r^2 + 4r - 1).$$

Solution

$$\begin{aligned}\sum_{r=11}^{20}(6r^2 + 4r - 1) &= \sum_{r=1}^{20}(6r^2 + 4r - 1) - \sum_{r=1}^{10}(6r^2 + 4r - 1) \\ &= 20 \times 22 \times 41 - 10 \times 12 \times 21 \\ &= 18\,040 - 2\,520 \\ &= \underline{\underline{15\,520}}.\end{aligned}$$

9. (a) Show that (4)

$$\sum_{r=1}^{n}(r^2 - r - 1) = \tfrac{1}{3}n(n^2 - 4).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^2-r-1) &= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \\ &= \tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}n(n+1) - n \\ &= \tfrac{1}{6}n[(n+1)(2n+1) - 3(n+1) - 6] \\ &= \tfrac{1}{6}n(2n^2+3n+1-3n-3-6) \\ &= \tfrac{1}{6}n(2n^2-8) \\ &= \underline{\underline{\tfrac{1}{3}n(n^2-4)}}.\end{aligned}$$

(b) Hence, or otherwise, find the value of $\sum_{r=10}^{20}(r^2-r-1)$. (2)

Solution

$$\begin{aligned}\sum_{r=10}^{20}(r^2-r-1) &= \sum_{r=1}^{20}(r^2-r-1) - \sum_{r=1}^{9}(r^2-r-1) \\ &= \tfrac{1}{3}\times 20 \times 396 - \tfrac{1}{3}\times 9 \times 77 \\ &= 2640 - 231 \\ &= \underline{\underline{2409}}.\end{aligned}$$

10. (a) Show that (5)

$$\sum_{r=1}^{n} r(r+2)(r+4) = \frac{1}{4}n(n+1)(n+4)(n+5).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} r(r+2)(r+4) &= \sum_{r=1}^{n} r(r^2+6r+8) \\ &= \sum_{r=1}^{n} (r^3+6r^2+8r) \\ &= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 8\sum_{r=1}^{n} r \\ &= \frac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + 4n(n+1) \\ &= \frac{1}{4}n(n+1)[n(n+1)+4(2n+1)+16] \\ &= \frac{1}{4}n(n+1)(n^2+9n+20) \\ &= \underline{\underline{\frac{1}{4}n(n+1)(n+4)(n+5)}}.\end{aligned}$$

(b) Hence evaluate (2)

$$\sum_{r=21}^{30} r(r+2)(r+4).$$

Solution

$$\begin{aligned}\sum_{r=21}^{30} r(r+2)(r+4) &= \sum_{r=1}^{30} r(r+2)(r+4) - \sum_{r=1}^{20} r(r+2)(r+4) \\ &= \frac{1}{4}(30)(31)(34)(35) - \frac{1}{4}(20)(21)(24)(25) \\ &= 276\,675 - 63\,000 \\ &= \underline{\underline{213\,675}}.\end{aligned}$$

11. (a) Using the formulae for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$, and $\sum_{r=1}^{n} r^3$, show that (7)

$$\sum_{r=1}^{n} r(r+1)(r+3) = \frac{1}{12}n(n+1)(n+2)(3n+k),$$

where k is a constant to be found.

Solution

$$\begin{aligned}\sum_{r=1}^{n} r(r+1)(r+3) &= \sum_{r=1}^{n} r(r^2+4r+3)\\ &= \sum_{r=1}^{n}(r^3+4r^2+3r)\\ &= \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r\\ &= \frac{1}{4}n^2(n+1)^2 + \frac{2}{3}n(n+1)(2n+1) + \frac{3}{2}n(n+1)\\ &= \frac{1}{12}n(n+1)\left[3n(n+1)+8(2n+1)+18\right]\\ &= \frac{1}{12}n(n+1)(3n^2+3n+16n+8+18)\\ &= \frac{1}{12}n(n+1)(3n^2+19n+26)\\ &= \underline{\underline{\frac{1}{12}n(n+1)(n+2)(3n+13)}}.\end{aligned}$$

(b) Hence evaluate $\sum_{r=21}^{40} r(r+1)(r+3)$. (2)

Solution

$$\begin{aligned}\sum_{r=21}^{40} r(r+1)(r+3) &= \sum_{r=1}^{40} r(r+1)(r+3) - \sum_{r=1}^{20} r(r+1)(r+3)\\ &= \frac{1}{12}(40)(41)(42)(133) - \frac{1}{12}(20)(21)(22)(73)\\ &= 763\,420 - 56\,210\\ &= \underline{\underline{707\,210}}.\end{aligned}$$

12. (a) Express (2)

$$\frac{1}{(2r+1)(2r+5)}$$

in partial fractions.

Solution

$$\frac{1}{(2r+1)(2r+5)} = \underline{\underline{\frac{\frac{1}{4}}{2r+1} - \frac{\frac{1}{4}}{2r+5}}}.$$

(b) Hence show, using the method of differences, that (6)

$$\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+5)} = \frac{n(8n+17)}{15(2n+3)(2n+5)}.$$

Solution

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+5)} &= \frac{1}{4}\sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+5}\right) \\
&= \frac{1}{4}\left[\left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{2n+3} - \frac{1}{2n+5}\right)\right] \\
&= \frac{1}{4}\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2n+3} - \frac{1}{2n+5}\right) \\
&= \frac{1}{4}\left(\frac{8}{15} - \frac{1}{2n+3} - \frac{1}{2n+5}\right) \\
&= \frac{8(2n+3)(2n+5) - 15(2n+5) - 15(2n+3)}{60(2n+3)(2n+5)} \\
&= \frac{8(4n^2+16n+15) - 30n - 75 - 30n - 45}{60(2n+3)(2n+5)} \\
&= \frac{32n^2 + 128n + 120 - 60n - 120}{60(2n+3)(2n+5)} \\
&= \frac{32n^2 + 68n}{60(2n+3)(2n+5)} \\
&= \frac{4n(8n+17)}{60(2n+3)(2n+5)} \\
&= \underline{\underline{\frac{n(8n+17)}{15(2n+3)(2n+5)}}}.
\end{aligned}$$

13. (a) Prove by induction that, for any positive integer n, (5)

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2.$$

Solution

$\underline{n = 1}$: $\sum_{r=1}^{1} r^3 = 1$ and $\frac{1}{4} \times 1^2 \times 2^2 = 1$. Hence, $n = 1$ is correct.

We assume that this is true for $n = k$, i.e.,

$$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2.$$

Now,

$$\begin{aligned}\sum_{r=1}^{k+1} r^3 &= \sum_{r=1}^{k} r^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2[k^2 + 4(k+4)] \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \frac{1}{4}(k+1)^2(k+2)^2,\end{aligned}$$

and it is true for $n = k + 1$.

Hence, by mathematical induction, it is true for any positive number n.

(b) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$, show that (5)

$$\sum_{r=1}^{n} (r^3 + 3r + 2) = \frac{1}{4}n(n+2)(n^2+7).$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}(r^3+3r+2) &= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r + 2\sum_{r=1}^{n} 1\\ &= \frac{1}{4}n^2(n+1)^2 + \frac{3}{2}n(n+1) + 2n\\ &= \frac{1}{4}n[n(n+1)^2 + 6(n+1) + 8]\\ &= \frac{1}{4}n[n(n^2+2n+1) + 6n + 6 + 8]\\ &= \frac{1}{4}n(n^3+2n^2+n+6n+14)\\ &= \frac{1}{4}n(n^3+2n^2+7n+14)\\ &= \underline{\underline{\frac{1}{4}n(n+2)(n^2+7)}}.\end{aligned}$$

(c) Hence evaluate (2)

$$\sum_{r=15}^{25}(r^3+3r+2).$$

Solution

$$\begin{aligned}\sum_{r=15}^{25}(r^3+3r+2) &= \sum_{r=1}^{25}(r^3+3r+2) - \sum_{r=1}^{14}(r^3+3r+2)\\ &= \frac{25\times 27\times 632}{4} - \frac{14\times 16\times 203}{4}\\ &= 106\,650 - 11\,368\\ &= \underline{\underline{95\,282}}.\end{aligned}$$

14. (a) Express (2)

$$\frac{3}{(3r-1)(3r+2)}$$

in partial fractions.

Solution

$$\frac{3}{(3r-1)(3r+2)} = \underline{\underline{\frac{1}{3r-1} - \frac{1}{3r+2}}}.$$

(b) Using the result in part (a) and the method of differences, show that (3)

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}.$$

Solution

$$\begin{aligned}
\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} &= \sum_{r=1}^{n} \left[\frac{1}{3r-1} - \frac{1}{3r+2}\right] \\
&= \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \ldots + \left(\frac{1}{3n-1} - \frac{1}{3n+2}\right) \\
&= \frac{1}{2} - \frac{1}{3n+2} \\
&= \underline{\underline{\frac{3n}{2(3n+2)}}}.
\end{aligned}$$

(c) Evaluate (2)

$$\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)},$$

giving your answer to 3 significant figures.

Solution

$$\begin{aligned}
\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)} &= \sum_{r=1}^{1000} \frac{3}{(3r-1)(3r+2)} - \sum_{r=1}^{99} \frac{3}{(3r-1)(3r+2)} \\
&= \frac{3 \times 1000}{2(3 \times 1000 + 2)} - \frac{3 \times 99}{2(3 \times 99 + 2)} \\
&= \frac{3000}{6004} - \frac{297}{598} \\
&= 0.003\,011\,370\,346 \text{ (FCD)} \\
&= \underline{\underline{0.003\,01 \text{ (3 sf)}}}.
\end{aligned}$$

15. Given that

$$(2r+1)^3 = Ar^3 + Br^2 + Cr + 1,$$

(a) find the values of the constants A, B, and C. (2)

Solution

$$\begin{aligned}(2r+1)^3 &= (2r+1)(2r+1)^2 \\ &= (2r+1)(4r^2+4r+1) \\ &= \underline{\underline{8r^3+12r^2+6r+1}};\end{aligned}$$

hence, $\underline{\underline{A=8}}$, $\underline{\underline{B=12}}$, and $\underline{\underline{C=6}}$.

(b) Show that (2)

$$(2r+1)^3-(2r-1)^3 = 24r^2+2.$$

Solution

$$\begin{aligned}(2r+1)^3-(2r-1)^3 &= (8r^3+12r^2+6r+1)-(8r^3-12r^2+6r-1) \\ &= \underline{\underline{24r^2+2}}.\end{aligned}$$

(c) Using the result in part (b) and the method of differences, show that (5)

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Solution

$$\begin{aligned}(2\times 1+1)^3-(2\times 1-1)^3 &= 24\times 1^2+2 \\ (2\times 2+1)^3-(2\times 2-1)^3 &= 24\times 2^2+2 \\ &\vdots \\ (2\times n+1)^3-(2\times n-1)^3 &= 24\times n^2+2;\end{aligned}$$

adding up, we find

$$\begin{aligned}
& 24(1^2 + 2^2 + \ldots + n^2) + 2n = (2n+1)^3 - 1 \\
\Rightarrow \quad & 24 \sum_{r=1}^{n} r^2 = (2n+1)^3 - 2n - 1 \\
\Rightarrow \quad & 24 \sum_{r=1}^{n} r^2 = (8n^3 + 12n^2 + 6n + 1) - 2n - 1 \\
\Rightarrow \quad & 24 \sum_{r=1}^{n} r^2 = 8n^3 + 12n^2 + 4n \\
\Rightarrow \quad & 24 \sum_{r=1}^{n} r^2 = 4n(2n^2 + 3n + 1) \\
\Rightarrow \quad & 24 \sum_{r=1}^{n} r^2 = 4n(n+1)(2n+1) \\
\Rightarrow \quad & \underline{\underline{\sum_{r=1}^{n} r^2 = \tfrac{1}{6} n(n+1)(2n+1).}}
\end{aligned}$$

16. (a) Express (2)

$$\frac{1}{r(r+2)}$$

in partial fractions.

Solution

$$\frac{1}{r(r+2)} = \underline{\underline{\frac{\frac{1}{2}}{r} - \frac{\frac{1}{2}}{r+2}}}.$$

(b) Hence prove, by the method of differences, that (6)

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(an+b)}{4(n+1)(n+2)},$$

where a and b are constants to be found.

Solution

$$
\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+2)} &= \sum_{r=1}^{n} \left[\frac{\frac{1}{2}}{r} - \frac{\frac{1}{2}}{r+2} \right] \\
&= \frac{1}{2} \sum_{r=1}^{n} \left[\frac{1}{r} - \frac{1}{r+2} \right] \\
&= \frac{1}{2} \left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+2}\right) \right] \\
&= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\
&= \frac{1}{2} \left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\
&= \frac{3(n^2+3n+2) - 2n - 4 - 2n - 2}{4(n+1)(n+2)} \\
&= \frac{3n^2+5n}{4(n+1)(n+2)} \\
&= \underline{\underline{\frac{n(3n+5)}{4(n+1)(n+2)}}}.
\end{aligned}
$$

(c) Hence show that (3)

$$
\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}.
$$

Solution

$$\begin{aligned}
\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} &= \sum_{r=1}^{2n} \frac{1}{r(r+2)} - \sum_{r=1}^{n} \frac{1}{r(r+2)} \\
&= \frac{2n(6n+5)}{4(2n+1)(2n+2)} - \frac{n(3n+5)}{4(n+1)(n+2)} \\
&= \frac{n(6n+5)}{4(2n+1)(n+1)} - \frac{n(3n+5)}{4(n+1)(n+2)} \\
&= \frac{n(6n+5)(n+2) - n(3n+5)(2n+1)}{4(n+1)(n+2)(2n+1)} \\
&= \frac{n(6n^2+17n+10) - n(6n^2+13n+5)}{4(n+1)(n+2)(2n+1)} \\
&= \frac{4n^2+5n}{4(n+1)(n+2)(2n+1)} \\
&= \underline{\underline{\frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}}}.
\end{aligned}$$

17. (a) Express (2)

$$\frac{2}{(2r+1)(2r+3)}$$

in partial fractions.

Solution

$$\frac{2}{(2r+1)(2r+3)} = \underline{\underline{\frac{1}{2r+1} - \frac{1}{2r+3}}}.$$

(b) Using your answer to part (a), find, in terms of n, (3)

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)}.$$

Give your answer as a single fraction in its simplest form.

Solution

$$
\begin{aligned}
\sum_{r=1}^{n} \frac{2}{(2r+1)(2r+3)} &= \sum_{r=1}^{n} \left(\frac{1}{2r+1} - \frac{1}{2r+3} \right) \\
&= \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \ldots + \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right) \\
&= \frac{1}{3} - \frac{1}{2n+3} \\
&= \frac{(2n+3)-3}{3(2n+3)} \\
&= \frac{2n}{3(2n+3)},
\end{aligned}
$$

and hence

$$\sum_{r=1}^{n} \frac{3}{(2r+1)(2r+3)} = \underline{\underline{\frac{n}{2n+3}}}.$$

18. (a) Express (2)

$$\frac{2}{(r+1)(r+3)}$$

in partial fractions.

Solution

$$\frac{2}{(r+1)(r+3)} = \underline{\underline{\frac{1}{r+1} - \frac{1}{r+3}}}.$$

(b) Hence show that (4)

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n}\frac{2}{(r+1)(r+3)} &= \sum_{r=1}^{n}\left(\frac{1}{r+1}-\frac{1}{r+3}\right)\\ &= \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\ldots+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\\ &= \frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\\ &= \frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}\\ &= \frac{(n+2)(n+3)-6(n+3)-6(n+2)}{6(n+2)(n+3)}\\ &= \frac{5(n^2+5n+6)-6n-18-6n-12}{6(n+2)(n+3)}\\ &= \frac{5n^2+13n}{6(n+2)(n+3)}\\ &= \underline{\underline{\frac{n(5n+13)}{6(n+2)(n+3)}}}.\end{aligned}$$

(c) Evaluate (2)

$$\sum_{r=10}^{100}\frac{2}{(r+1)(r+3)},$$

giving your answer to 3 significant figures.

Solution

$$\begin{aligned}\sum_{r=10}^{100}\frac{2}{(r+1)(r+3)} &= \sum_{r=1}^{100}\frac{2}{(r+1)(r+3)}-\sum_{r=1}^{9}\frac{2}{(r+1)(r+3)}\\ &= \frac{100(5\times100+13)}{6(100+2)(100+3)}-\frac{9(5\times9+13)}{6(9+2)(9+3)}\\ &= \frac{51\,300}{63\,036}-\frac{522}{792}\\ &= 0.154\,729\,764\,8 \text{ (FCD)}\\ &= \underline{\underline{0.155 \text{ (3 sf)}}}.\end{aligned}$$

19. (a) Express (2)

$$\frac{2}{4r^2-1}$$

in partial fractions.

Solution

$$\frac{2}{4r^2-1} = \underline{\underline{\frac{1}{2r-1} - \frac{1}{2r+1}}}.$$

(b) Hence use the method of difference to show that (3)

$$\sum_{r=1}^{n} \frac{2}{4r^2-1} = \frac{n}{2n+1}.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} \frac{2}{4r^2-1} &= \sum_{r=1}^{n} \left(\frac{1}{2r-1} - \frac{1}{2r+1}\right) \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= 1 - \frac{1}{2n+1} \\ &= \frac{2n+1}{2n+1} - \frac{1}{2n+1} \\ &= \frac{2n}{2n+1},\end{aligned}$$

and so

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \underline{\underline{\frac{n}{2n+1}}}.$$

20. (a) Express (1)

$$\frac{2}{(r+2)(r+4)}$$

in partial fractions.

Solution

$$\frac{2}{(r+2)(r+4)} = \underline{\underline{\frac{1}{r+2} - \frac{1}{r+4}}}.$$

(b) Hence show that (4)

$$\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}.$$

Solution

$$\begin{aligned}
\sum_{r=1}^{n} \frac{2}{(r+2)(r+4)} &= \sum_{r=1}^{n} \left(\frac{1}{r+2} - \frac{1}{r+4}\right) \\
&= \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots + \left(\frac{1}{n+2} - \frac{1}{n+4}\right) \\
&= \frac{1}{3} + \frac{1}{4} - \frac{1}{n+3} - \frac{1}{n+4} \\
&= \frac{7}{12} - \frac{1}{n+3} - \frac{1}{n+4} \\
&= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)} \\
&= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{12(n+3)(n+4)} \\
&= \frac{7n^2 + 25n}{12(n+3)(n+4)} \\
&= \underline{\underline{\frac{n(7n+25)}{12(n+3)(n+4)}}}.
\end{aligned}$$

21. (a) Show that (3)

$$r^2(r+1)^2 - (r-1)^2r^2 \equiv 4r^3.$$

Solution

$$\begin{aligned}
r^2(r+1)^2 - (r-1)^2r^2 &\equiv r^2(r^2+2r+1) - r^2(r^2-2r+1) \\
&\equiv (r^4 + 2r^3 + r^2) - (r^4 - 2r^3 + r^2) \\
&\equiv \underline{\underline{4r^3}}.
\end{aligned}$$

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$,

(b) use the identity in part (a) and the method of differences to show that (4)

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} 4r^3 &= \sum_{r=1}^{n} [r^2(r+1)^2 - (r-1)^2 r^2] \\ &= (1^2 \times 2^2 - 0^2 \times 1^2) + (2^2 \times 3^2 - 1^2 \times 2^2) \\ &\qquad + \ldots + [n^2(n+1)^2 - (n-1)^2 n^2] \\ &= n^2(n+1)^2\end{aligned}$$

so

$$\begin{aligned}\sum_{r=1}^{n} r^3 &= \tfrac{1}{4} n^2 (n+1)^2 \\ &= \left[\tfrac{1}{2} n(n+1)\right]^2 \\ &= \left[\sum_{r=1}^{n} r\right]^2;\end{aligned}$$

or

$$\underline{\underline{1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2}}.$$

22. (a) Show that, for $r > 0$, (2)

$$r - 3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{r^3 - 7r - 5}{(r+1)(r+2)}.$$

Solution

$$
\begin{aligned}
r-3+\frac{1}{r+1}-\frac{1}{r+2} &= \frac{(r-3)(r+1)(r+2)+(r+2)-(r+1)}{(r+1)(r+2)} \\
&= \frac{(r-3)(r^2+3r+2)+1}{(r+1)(r+2)} \\
&= \frac{(r^3-7r-6)+1}{(r+1)(r+2)} \\
&= \underline{\underline{\frac{r^3-7r-5}{(r+1)(r+2)}}}.
\end{aligned}
$$

(b) Hence prove, using the method of differences, that (5)

$$
\sum_{r=1}^{n} \frac{r^3-7r-5}{(r+1)(r+2)} = \frac{n(n^2+an+b)}{2(n+2)},
$$

where a and b are constants to be found.

Solution

$$
\sum_{r=1}^{n}(r-3) = \sum_{r=1}^{n} r - \sum_{r=1}^{n} 3 = \tfrac{1}{2}n(n+1) - 3n
$$

and

$$
\begin{aligned}
&\sum_{r=1}^{n} \frac{r^3-7r-5}{(r+1)(r+2)} \\
&= \sum_{r=1}^{n}\left[r-3+\frac{1}{r+1}-\frac{1}{r+2}\right] \\
&= \tfrac{1}{2}n(n+1) - 3n + (\tfrac{1}{2}-\tfrac{1}{3}) + (\tfrac{1}{3}-\tfrac{1}{4}) + \ldots + (\tfrac{1}{n+1}-\tfrac{1}{n+2}) \\
&= \tfrac{1}{2}n(n+1) - 3n + \tfrac{1}{2} - \tfrac{1}{n+2} \\
&= \frac{n(n+1)(n+2)-6n(n+2)+(n+2)-2}{2(n+2)} \\
&= \frac{n(n^2+3n+2)-6n(n+2)+n}{2(n+2)} \\
&= \frac{n(n^2+3n+2-6n-12+1)}{2(n+2)} \\
&= \underline{\underline{\frac{n(n^2-3n-9)}{2(n+2)}}}.
\end{aligned}
$$

23. (a) Show that, for $r > 0$, (1)

$$\frac{1}{r^2} - \frac{1}{(r+1)^2} \equiv \frac{2r+1}{r^2(r+1)^2}.$$

Solution

$$\begin{aligned}\frac{1}{r^2} - \frac{1}{(r+1)^2} &\equiv \frac{(r+1)^2 - r^2}{r^2(r+1)^2} \\ &\equiv \frac{(r^2+2r+1) - r^2}{r^2(r+1)^2} \\ &\equiv \underline{\underline{\frac{2r+1}{r^2(r+1)^2}}}.\end{aligned}$$

(b) Hence prove that, for $n \in \mathbb{N}$, (3)

$$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = \frac{n(n+2)}{(n+1)^2}.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} &= \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \ldots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \\ &= 1 - \frac{1}{(n+1)^2} \\ &= \frac{(n+1)^2 - 1}{(n+1)^2} \\ &= \frac{(n^2+2n+1) - 1}{(n+1)^2} \\ &= \frac{n^2+2n}{(n+1)^2} \\ &= \underline{\underline{\frac{n(n+2)}{(n+1)^2}}}.\end{aligned}$$

(c) Show that, $n \in \mathbb{N}$, $n > 1$, (3)

$$\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} = \frac{an^2+bn+c}{n^2(3n+1)^2},$$

where a, b, and c are constants to be found.

Solution

$$\begin{aligned}
\sum_{r=n}^{3n} \frac{6r+3}{r^2(r+1)^2} &= \sum_{r=1}^{3n} \frac{3(2r+1)}{r^2(r+1)^2} - \sum_{r=1}^{n-1} \frac{3(2r+1)}{r^2(r+1)^2} \\
&= \frac{9n(3n+2)}{(3n+1)^2} - \frac{3(n-1)[(n-1)+2]}{n^2} \\
&= \frac{9n(3n+2)}{(3n+1)^2} - \frac{3(n-1)(n+1)}{n^2} \\
&= \frac{9n^3(3n+2) - 3(3n+1)^2(n^2-1)}{n^2(3n+1)^2} \\
&= \frac{(27n^4+18n^3) - 3(9n^2+6n+1)(n^2-1)}{n^2(3n+1)^2} \\
&= \frac{(27n^4+18n^3) - 3(9n^4+6n^3+n^2-9n^2-6n-1)}{n^2(3n+1)^2} \\
&= \frac{(27n^4+18n^3) - 3(9n^4+6n^3-8n^2-6n-1)}{n^2(3n+1)^2} \\
&= \frac{(27n^4+18n^3) - (27n^4+18n^3-24n^2-18n-3)}{n^2(3n+1)^2} \\
&= \underline{\underline{\frac{24n^2+18n+3}{n^2(3n+1)^2}}}.
\end{aligned}$$