Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 Nov Paper 1 Variant 3: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. The coefficient of x^2 in the expansion of $(2+px)^6$ is 60.

   (a) Find the value of the positive constant p. (3)

Solution
Well,

$$(2+px)^6 = \ldots + \binom{6}{5}(2)^5(px) + \binom{6}{4}(2)^4(px)^2 + \ldots$$
$$= \ldots + 192px + 240p^2x^2 + \ldots$$

and so the coefficient of x^2 is $240p^2$:

$$240p^2 = 60 \Rightarrow p^2 = \tfrac{1}{4}$$
$$\Rightarrow \underline{\underline{p = \tfrac{1}{2}}},$$

because $p > 0$.

   (b) Using your value of p, find the coefficient of x^2 in the expansion of (3)

$$(3-x)(2+px)^6.$$

Solution
Now,

$\times$	$96x$	$+60x^2$
3	$\ldots$	$+180x^2$
$-x$	$-96x^2$	$\ldots$

and so the coefficient of x^2 is

$$180 - 96 = \underline{\underline{84}}.$$

2. Solve (5)

$$2\log_{10} y - \log_{10}(5y + 60) = 1.$$

Solution

Well,

$$\begin{aligned}
2\log_{10} y - \log_{10}(5y + 60) = 1 &\Rightarrow \log_{10} y^2 - \log_{10}(5y + 60) = 1 \\
&\Rightarrow \log_{10}\left(\frac{y^2}{5y + 60}\right) = 1 \\
&\Rightarrow \frac{y^2}{5y + 60} = 10^1 \\
&\Rightarrow y^2 = 10(5y + 60) \\
&\Rightarrow y^2 = 50y + 600 \\
&\Rightarrow y^2 - 50y - 600 = 0
\end{aligned}$$

add to: -50
multiply to: -600 $\Big\}$ $-60, +10$

$$\begin{aligned}
&\Rightarrow (y - 60)(y + 10) = 0 \\
&\Rightarrow y = 60 \text{ or } y = -10.
\end{aligned}$$

But $y > 0$ (why?) so $\underline{\underline{y = 60}}$.

3. Show that (4)

$$\tan^2\theta - \sin^2\theta \equiv \sin^4\theta \sec^2\theta.$$

Solution

$$\begin{aligned}\tan^2\theta - \sin^2\theta &\equiv \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta \\ &\equiv \sin^2\theta\left(\frac{1}{\cos^2\theta} - 1\right) \\ &\equiv \sin^2\theta\left(\frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta}\right) \\ &\equiv \sin^2\theta\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) \\ &\equiv \sin^2\theta\left(\sin^2\theta\sec^2\theta\right) \\ &\equiv \underline{\underline{\sin^4\theta\sec^2\theta}},\end{aligned}$$

as required.

4. A curve has equation

$$y = \frac{\mathrm{e}^{2x}}{(x+3)^2}.$$

(a) Show that (4)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A\mathrm{e}^{2x}(x+2)}{(x+3)^3},$$

where A is a constant to be found.

Solution

Product rule:

$$u = \mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2\mathrm{e}^{2x}$$

$$v = (x+3)^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2(x+3)$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{[(x+3)^2][2]-[\mathrm{e}^{2x}][2(x+3)]}{(x+3)^4}\\ &= \frac{\mathrm{e}^{2x}(x+3)[2(x+3)-2]}{(x+3)^4}\\ &= \frac{\mathrm{e}^{2x}(2x+6-2)}{(x+3)^3}\\ &= \frac{\mathrm{e}^{2x}(2x+4)}{(x+3)^3}\\ &= \underline{\underline{\frac{2\mathrm{e}^{2x}(x+2)}{(x+3)^3}}};\end{aligned}$$

so, $\underline{\underline{A=2}}$.

(b) Find the exact coordinates of the point on the curve where $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$. (2)

Solution

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow \frac{2\mathrm{e}^{2x}(x+2)}{(x+3)^3}=0$$

as $\mathrm{e}^{2x}>0$:

$$\begin{aligned}&\Rightarrow 2(x+2)=0\\ &\Rightarrow x=-2\\ &\Rightarrow y=\mathrm{e}^{-4};\end{aligned}$$

hence, the exact coordinates of the point are $\underline{\underline{(-2,\mathrm{e}^{-4})}}$.

5. For $x\in\mathbb{R}$, the functions f and g are defined by

$$\begin{aligned}\mathrm{f}(x)&=2x^3,\\ \mathrm{g}(x)&=4x-5x^2.\end{aligned}$$

(a) Express $\mathrm{f}^2(\frac{1}{2})$ as a power of 2. (2)

Solution

Well,

$$\begin{aligned} \mathrm{f}^2(\tfrac{1}{2}) &= \mathrm{f}(\mathrm{f}(\tfrac{1}{2})) \\ &= \mathrm{f}(\tfrac{1}{4}) \\ &= 2(\tfrac{1}{4})^3 \\ &= \tfrac{1}{32} \\ &= \underline{\underline{2^{-5}}}. \end{aligned}$$

(b) Find the values of x for which f and g are increasing at the same rate with respect to x. (4)

Solution

Now,

$$\mathrm{f}(x) = 2x^3 \Rightarrow \mathrm{f}'(x) = 6x^2$$

and

$$\mathrm{g}(x) = 4x - 5x^2 \Rightarrow \mathrm{g}'(x) = 4 - 10x.$$

Next,

$$\begin{aligned} \mathrm{f}'(x) = \mathrm{g}'(x) &\Rightarrow 6x^2 = 4 - 10x \\ &\Rightarrow 6x^2 + 10x - 4 = 0 \\ &\Rightarrow 2(3x^2 + 5x - 2) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +5 \\ \text{multiply to:} & (+3) \times (-2) = -6 \end{array}\right\} +6, -1$$

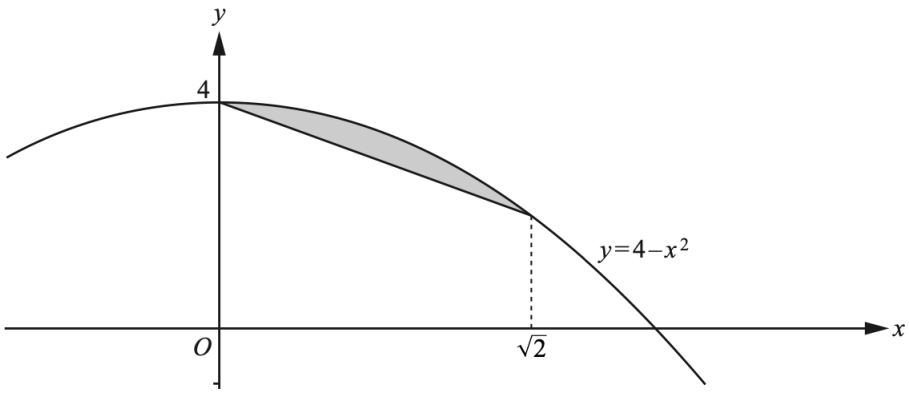
e.g.,

$$\begin{aligned} &\Rightarrow 2[3x^2 + 6x - x - 2] = 0 \\ &\Rightarrow 2[3x(x+2) - 1(x+2)] = 0 \\ &\Rightarrow 2(3x-1)(x+2) = 0 \\ &\Rightarrow 3x - 1 = 0 \text{ or } x + 2 = 0 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{3} \text{ or } x = -2}}. \end{aligned}$$

6. **Do not use a calculator in this question.** (6)

The diagram shows part of the curve

$$y = 4 - x^2.$$



Show that the area of the shaded region can be written in the form

$$\frac{\sqrt{2}}{p},$$

where p is an integer to be found.

Solution

Well,

$$\begin{aligned}\int_0^{\sqrt{2}} (4 - x^2)\,\mathrm{d}x &= \left[4x - \tfrac{1}{3}x^3\right]_{x=0}^{\sqrt{2}} \\ &= \left(4\sqrt{2} - \tfrac{2}{3}\sqrt{2}\right) - (0 - 0) \\ &= \tfrac{10}{3}\sqrt{2}.\end{aligned}$$

Now,

$$x = \sqrt{2} \Rightarrow y = 2$$

and

$$\begin{aligned}\text{trapezium} &= \tfrac{1}{2} \times \sqrt{2} \times (4 + 2) \\ &= 3\sqrt{2}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{area} &= \text{integral} - \text{trapezium}\\ &= \tfrac{10}{3}\sqrt{2} - 3\sqrt{2}\\ &= \underline{\underline{\frac{\sqrt{2}}{3}}};\end{aligned}$$

so, $\underline{\underline{p = 3}}$.

7. It is given that

$$\mathbf{A} = \begin{pmatrix} 2t & 2 \\ t^2 - t + 1 & t \end{pmatrix}.$$

(a) Find the value of t for which $\det \mathbf{A} = 1$. (3)

Solution

Well,

$$\begin{aligned}\det \mathbf{A} = 1 &\Rightarrow (2t)(t) - (2)(t^2 - t + 1) = 1\\ &\Rightarrow 2t^2 - (2t^2 - 2t + 2) = 1\\ &\Rightarrow 2t - 2 = 1\\ &\Rightarrow 2t = 3\\ &\Rightarrow \underline{\underline{t = 1\tfrac{1}{2}}}.\end{aligned}$$

(b) In the case when $t = 3$, find $\mathbf{A}^{-1}$ and hence solve (5)

$$\begin{aligned}3x + y &= 5,\\ 7x + 3y &= 11.\end{aligned}$$

Solution

Now,

$$t = 3 \Rightarrow \mathbf{A} = \begin{pmatrix} 6 & 2 \\ 7 & 3 \end{pmatrix}.$$

Next,

$$\det \mathbf{A} = 18 - 14 = 4$$

and

$$\mathbf{A}^{-1} = \tfrac{1}{4}\begin{pmatrix} 3 & -2 \\ -7 & 6 \end{pmatrix}.$$

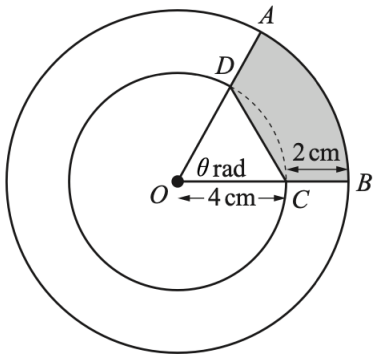
Then,

$$\begin{pmatrix} 3 & 1 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 3 & -2 \\ -7 & 6 \end{pmatrix}\begin{pmatrix} 10 \\ 11 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} 8 \\ -4 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix};$$

hence,

$$\underline{\underline{x = 2,\, y = -1}}.$$

8. The diagram shows two concentric circles, centre O, radii 4 cm and 6 cm.



The points A and B lie on the larger circle and the points C and D lie on the smaller circle such that ODA and OCB are straight lines.

(a) Given that the area of triangle OCD is 7.5 cm^2, show that $\theta = 1.215$ radians, to 3 decimal places. (2)

Solution

Well,

$$\begin{aligned}\tfrac{1}{2} \times 4^2 \times \sin\theta = 7.5 &\Rightarrow \sin\theta = \tfrac{15}{16} \\ &\Rightarrow \theta = 1.215\,375\,125 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 1.215 \text{ radians (3 dp)}}}.\end{aligned}$$

(b) Find the perimeter of the shaded region. (4)

Solution

Well,

$$\begin{aligned} & CD^2 = OC^2 + OD^2 - 2 \times OC \times OD \times \cos COD \\ \Rightarrow \quad & CD^2 = 4^2 + 4^2 - 2 \times 4 \times 4 \times \cos 1.215\ldots \\ \Rightarrow \quad & CD^2 = 20.864\,471\,27 \text{ (FCD)} \\ \Rightarrow \quad & CD^2 = 4.567\,764\,63 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\text{perimeter} &= \text{curve } AB + BC + CD + AD \\ &= (6 \times 1.215\ldots) + 2 + 4.567\ldots + 2 \\ &= 15.860\,015\,11 \text{ (FCD)} \\ &= \underline{\underline{15.9 \text{ cm (3 sf)}}}.\end{aligned}$$

(c) Find the area of the shaded region. (3)

Solution

Now,

$$\begin{aligned}\text{area} &= \text{sector} - \text{triangle} \\ &= (\tfrac{1}{2} \times 6^2 \times 1.215\ldots) - 7.5 \\ &= 14.376\,752\,25 \text{ (FCD)} \\ &= \underline{\underline{14.4 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

9. (a) (i) Solve (4)

$$6\sin^2 x = 5 + \cos x \text{ for } 0^\circ < x < 180^\circ.$$

Solution

$$6\sin^2 x = 5 + \cos x \Rightarrow 6(1 - \cos^2 x) = 5 + \cos x$$
$$\Rightarrow 6 - 6\cos^2 x = 5 + \cos x$$
$$\Rightarrow 6\cos^2 x + \cos x - 1 = 0$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+6) \times (-1) = -6 \end{array}\right\} -2,\ +3$$

e.g.,

$$\Rightarrow 6\cos^2 x - 2\cos x + 3\cos x - 1 = 0$$
$$\Rightarrow 2\cos x(3\cos x - 1) + 1(3\cos x - 1) = 0$$
$$\Rightarrow (2\cos x + 1)(3\cos x - 1) = 0$$
$$\Rightarrow 2\cos x + 1 = 0 \text{ or } 3\cos x - 1 = 0$$
$$\Rightarrow \cos x = -\tfrac{1}{2} \text{ or } \cos x = \tfrac{1}{3}.$$

$\underline{\cos x = -\frac{1}{2}}$:

$$\cos x = -\tfrac{1}{2} \Rightarrow \underline{\underline{x = 120}}.$$

$\underline{\cos x = -\frac{1}{2}}$:

$$\cos x = \tfrac{1}{3} \Rightarrow x = 70.528\,779\,37 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 70.5^\circ \text{ (3 sf)}}}.$$

(ii) Hence, or otherwise, solve (3)

$$6\cos^2 y = 5 + \sin y \text{ for } 0^\circ < y < 180^\circ.$$

Solution
Well,

$$\sin y = \cos(90 - y) \Rightarrow y = 19.471\,220\,63,\ 160.528\,779\,4 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{y = 19.5,\ 161 \text{ (3 sf)}}}.$$

(b) Solve (4)

$$4\cot^2 z - 3\cot z = 0 \text{ for } 0 < z < \pi \text{ radians.}$$

Solution

$$\begin{aligned} 4\cot^2 z - 3\cot x = 0 &\Rightarrow \cot z(4\cot z - 3) = 0 \\ &\Rightarrow \cot z = 0 \text{ or } 4\cot z - 3 = 0 \\ &\Rightarrow \cot z = 0 \text{ or } \cot z = \tfrac{3}{4} \\ &\Rightarrow \tan z = \text{undefined or } \tan z = \tfrac{4}{3}. \end{aligned}$$

$\underline{\tan z = \text{undefined}}$:

$$\tan z = \text{undefined} \Rightarrow \underline{\underline{z = \tfrac{1}{2}\pi}}.$$

$\underline{\tan z = \tfrac{4}{3}}$:

$$\begin{aligned} \tan z = \tfrac{4}{3} &\Rightarrow z = 0.927\,295\,218 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{z = 0.927 \text{ (3 sf)}}}. \end{aligned}$$

10. The variables s and t are related by the equation

$$t = ks^n,$$

where k and n are constants.

The table below shows values of variables s and t.

s	2	4	6	8
t	25.00	6.25	2.78	1.56

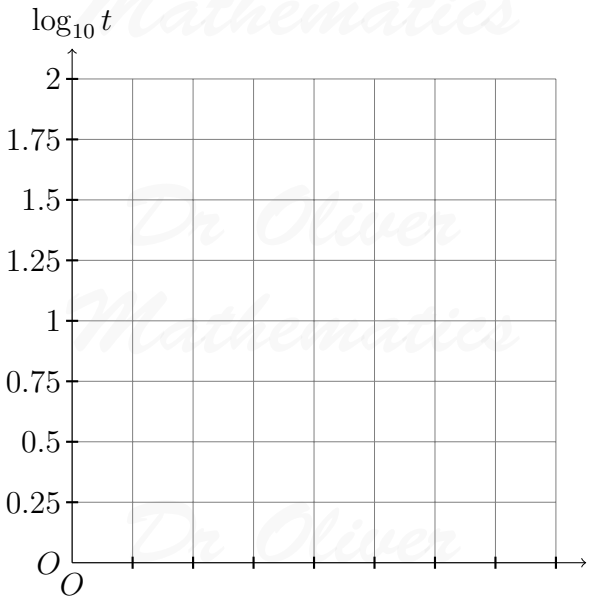
(a) A straight line graph is to be drawn for this information with $\log_{10} t$ plotted on the vertical axis. (1)
State the variable which must be plotted on the horizontal axis.

Solution
Well,

$$\begin{aligned} t = ks^n &\Rightarrow \log_{10} t = \log_{10}(ks^n) \\ &\Rightarrow \log_{10} t = \log_{10} k + \log_{10} s^n \\ &\Rightarrow \log_{10} t = \log_{10} k + n\log_{10} s \end{aligned}$$

so $\underline{\underline{\log_{10} s}}$ must be plotted on the horizontal axis.

(b) Draw this straight line graph on the grid below. (3)



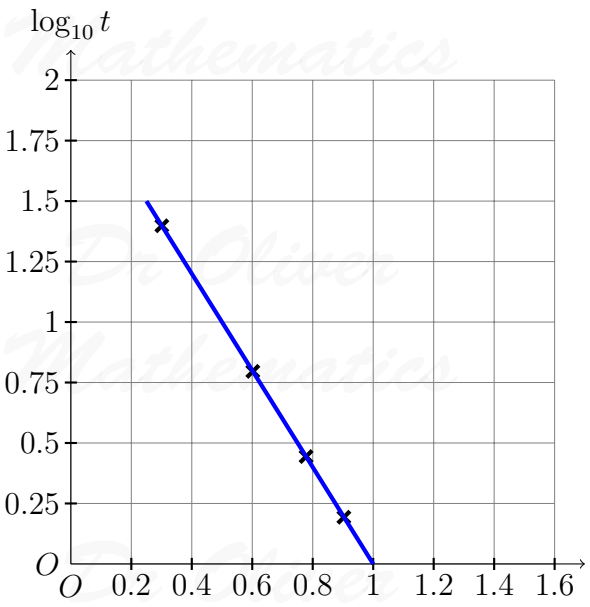
Solution

We round to 3 decimal places:

$\log_{10} s$	0.301	0.602	0.778	0.903
$\log_{10} t$	1.398	0.796	0.444	0.193

and

Dr Oliver Mathematics



(c) Use your graph to find the value of k and of n. (4)

Solution

Well, the line of best fit goes through $(0.25, 1.5)$ and $(1, 0)$:

$$\begin{aligned} m &= \frac{1.5 - 0}{0.25 - 1} \\ &= -2 \end{aligned}$$

and

$$\log_{10} t = \log_{10} k - 2 \log_{10} s$$

$$\begin{aligned} \log_{10} s = 1, \log_{10} t = 0 &\Rightarrow 0 = \log_{10} k - 2 \\ &\Rightarrow \log_{10} k = 2. \end{aligned}$$

Finally,

$$\begin{aligned} \log_{10} t = 2 - 2 \log_{10} s &\Rightarrow \log_{10} t + 2 \log_{10} s = 2 \\ &\Rightarrow \log_{10} t + \log_{10} s^2 = 2 \\ &\Rightarrow \log_{10}(ts^2) = 2 \\ &\Rightarrow ts^2 = 10^2 \\ &\Rightarrow \underline{\underline{t = 100s^{-2}}}. \end{aligned}$$

(d) Estimate the value of s when $t = 4$. (2)

Solution

$$\begin{aligned} t = 4 &\Rightarrow 100s^{-2} = 4 \\ &\Rightarrow s^{-2} = \tfrac{1}{25} \\ &\Rightarrow s^2 = 5 \\ &\Rightarrow \underline{\underline{s = 5}}. \end{aligned}$$

11. (a) Given that (5)

$$\int_0^k \left(2\mathrm{e}^{2x} - \tfrac{5}{2}\mathrm{e}^{-2x}\right) \mathrm{d}x = \tfrac{3}{4},$$

where k is a constant, show that

$$4\mathrm{e}^{4k} - 12\mathrm{e}^{2k} + 5 = 0.$$

Solution

Well,

$$\begin{aligned} \int_0^k \left(2\mathrm{e}^{2x} - \tfrac{5}{2}\mathrm{e}^{-2x}\right) \mathrm{d}x = \tfrac{3}{4} &\Rightarrow \left[\mathrm{e}^{2x} + \tfrac{5}{4}\mathrm{e}^{-2x}\right]_{x=0}^{k} = \tfrac{3}{4} \\ &\Rightarrow \left(\mathrm{e}^{2k} + \tfrac{5}{4}\mathrm{e}^{-2k}\right) - \left(1 + \tfrac{5}{4}\right) = \tfrac{3}{4} \\ &\Rightarrow \mathrm{e}^{2k} + \tfrac{5}{4}\mathrm{e}^{-2k} - 3 = 0 \end{aligned}$$

multiply by $4\mathrm{e}^{2k}$:

$$\begin{aligned} &\Rightarrow 4\mathrm{e}^{4k} + 5 - 12\mathrm{e}^{2k} = 0 \\ &\Rightarrow \underline{\underline{4\mathrm{e}^{4k} - 12\mathrm{e}^{2k} + 5 = 0}}, \end{aligned}$$

as required.

(b) Using a substitution of $y = \mathrm{e}^{2k}$, or otherwise, find the possible values of k. (4)

Solution

$$\left.\begin{array}{ll} \text{add to:} & -12 \\ \text{multiply to:} & (+4) \times (+5) = +20 \end{array}\right\} -10,\ -2$$

and so

$$\begin{aligned}
4e^{4k} - 12e^{2k} + 5 = 0 &\Rightarrow 4(e^{2k})^2 - 12e^{2k} + 5 = 0 \\
&\Rightarrow 4(e^{2k})^2 - 10e^{2k} - 2e^{2k} + 5 = 0 \\
&\Rightarrow 2e^{2k}(2e^{2k} - 5) - 1(2e^{2k} - 5) = 0 \\
&\Rightarrow (2e^{2k} - 1)(2e^{2k} - 5) = 0 \\
&\Rightarrow 2e^{2k} - 1 = 0 \text{ or } 2e^{2k} - 5 = 0 \\
&\Rightarrow 2e^{2k} = 1 \text{ or } 2e^{2k} = 5 \\
&\Rightarrow e^{2k} = \tfrac{1}{2} \text{ or } e^{2k} = \tfrac{5}{2} \\
&\Rightarrow 2k = \ln \tfrac{1}{2} \text{ or } 2k = \ln \tfrac{5}{2} \\
&\Rightarrow \underline{\underline{k = \tfrac{1}{2} \ln \tfrac{1}{2} \text{ or } k = \tfrac{1}{2} \ln \tfrac{5}{2}}}.
\end{aligned}$$

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