

Dr Oliver Mathematics

Pigeonhole Principle

In this note, we will examine the pigeonhole principle. It is due to Johann Peter Gustav Lejeune Dirichlet: it is also known as Dirichlet's Box (or Drawer) Principle, or, as Dirichlet named it, *Schubfachprinzip* (drawer principle or shelf principle).

Theorem 1

If $(n + 1)$ objects are put into n boxes, then at least one box contains two or more objects.

Solution

Suppose it is not true, that is, the i th box contains at most 1 object, for $i = 1, 2, \dots, n$. Then the total number of objects contained in the n boxes can be at most

$$1 + 1 + \dots + 1 = n,$$

which is one less than the number of objects distributed – contradiction. ■

Theorem 2

Let q_1, q_2, \dots, q_n be positive integers. If

$$q_1 + q_2 + \dots + q_n - n + 1$$

objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, \dots , the n th box contains at least q_n objects.

Solution

Suppose it is not true, that is, the i th box contains at most $q_i - 1$ objects, for $i = 1, 2, \dots, n$. Then the total number of objects contained in the n boxes can be at most

$$(q_1 - 1) + (q_2 - 1) + \dots + (q_n - 1) = q_1 + q_2 + \dots + q_n - n,$$

which is one less than the number of objects distributed – contradiction. ■

Example 1

If you pick five cards from a standard deck of 52 cards, then at least two will be of the same suit.

Solution

Each of the five cards can belong to one of four suits. By the pigeonhole principle, two or more must belong to the same suit. ■

Example 2

A bag contains 7 red marbles, 7 white marbles, and 7 blue marbles. What is the minimum numbers of marbles you have to choose randomly from the bag to ensure that we get 5 marbles of same colour?

Solution

We need 4 red marbles, 4 white marbles, 4 blue marbles, and one other marble; so

$$(3 \times 4) + 1 = 13 \text{ marbles. } \blacksquare$$

Example 3

Consider a chess board with two of the diagonally opposite corners removed. Is it possible to cover the board with pieces of domino whose size is exactly two board squares?

Solution

No, it is not possible. Two diagonally opposite squares on a chess board are of the same colour. Therefore, when these are removed, the number of squares of one colour exceeds by 2 the number of squares of another colour. However, every piece of domino covers exactly two squares and these are of different colours. ■

Here are some examples for you to try.

1. If you pick five numbers from the integers 1 to 8, then two of them must add up to nine.
2. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 7 apples or at least 8 bananas or at least 5 oranges?
3. A box contains three pairs of socks coloured red, blue, and green, respectively. If the socks are chosen without looking, how many socks must be drawn to guarantee at least one matching pair?
4. Show that given a set of n positive integers, there exists a non-empty subset whose sum is divisible by n .
5. Consider any five points P_1, P_2, \dots, P_5 in the interior of a square S of side length 1. Show that one can find two of the points at distance at most $\frac{\sqrt{2}}{2}$ apart.