Dr Oliver Mathematics
Mathematics: Higher
2012 Paper 1: Non-Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

Section A

1. A sequence is defined by the recurrence relation (2)

$$u_{n+1} = 3u_n + 4, \text{ with } u_0 = 1.$$

Find the value of u_2.

A. 7

B. 10

C. 25

D. 35

Solution

C

$u_1 = 3 \times 1 + 4 = 7.$
$u_2 = 3 \times 7 + 4 = 25.$

2. What is the gradient of the tangent to the curve with equation (2)

$$y = x^3 - 6x + 1$$

at the point where $x = -2$?

A. -24

B. 3

C. 5

D. 6

Solution

D

$$y = x^3 - 6x + 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6$$

and

$$\begin{aligned} \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-2} &= 3(-2)^2 - 6 \\ &= 12 - 6 \\ &= 6. \end{aligned}$$

3. If (2)

$$x^2 - 6x + 14$$

is written in the form

$$(x - p)^2 + q,$$

what is the value of q?

A. -22

B. 5

C. 14

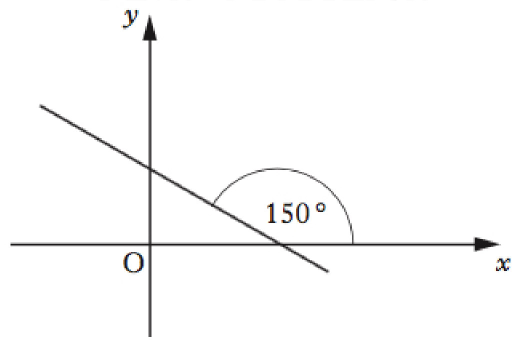
D. 50

Solution

B

$$\begin{aligned} x^2 - 6x + 14 &= (x^2 - 6x + 9) + 5 \\ &= (x - 3)^2 + 5. \end{aligned}$$

4. What is the gradient of the line shown in the diagram? (2)



A. $-\sqrt{3}$

B. $-\dfrac{1}{\sqrt{3}}$

C. $-\frac{1}{2}$

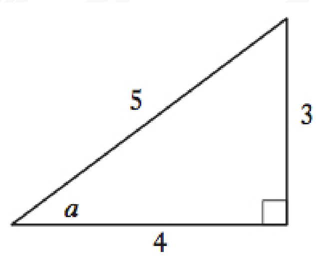
D. $-\dfrac{\sqrt{3}}{2}$

Solution

B

$$\tan 150^\circ = -\tan 30^\circ = -\frac{1}{\sqrt{3}}.$$

5. The diagram shows a right-angled triangle with sides and angles as marked. (2)



What is the value of $\cos 2a$?

A. $\frac{7}{25}$

B. $\frac{3}{5}$

C. $\frac{24}{25}$

D. $\frac{6}{5}$

Solution

A

$$\begin{aligned}\cos 2a &= 2\cos^2 a - 1\\ &= 2(\tfrac{4}{5})^2 - 1\\ &= 2(\tfrac{16}{25}) - 1\\ &= \tfrac{32}{25} - 1\\ &= \tfrac{7}{25}.\end{aligned}$$

6. If (2)

$$y = 3x^{-2} + 2x^{\frac{3}{2}},\ x > 0,$$

determine $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

A. $-6x^{-3} + \frac{4}{5}x^{\frac{5}{2}}$

B. $-3x^{-1} + 3x^{\frac{1}{2}}$

C. $-6x^{-3} + 3x^{\frac{1}{2}}$

D. $-3x^{-1} + \frac{4}{5}x^{\frac{5}{2}}$

Solution

C

$$y = 3x^{-2} + 2x^{\frac{3}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -6x^{-3} + 3x^{\frac{1}{2}}.$$

7. If (2)

$$\mathbf{u} = \begin{pmatrix} -3 \\ 1 \\ 2t \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} 1 \\ t \\ -1 \end{pmatrix}$$

are perpendicular, what is the value of t?

A. -3

B. -2

C. $\frac{2}{3}$

D. 1

Solution

A

$$\begin{aligned}\mathbf{u}.\mathbf{v} = 0 &\Rightarrow -3 + t - 2t = 0\\ &\Rightarrow -3 = t.\end{aligned}$$

8. The volume of a sphere is given by the formula (2)

$$V = \tfrac{4}{3}\pi r^3.$$

What is the rate of change of V with respect to r, at $r = 2$?

A. $\frac{16}{3}\pi$

B. $\frac{32}{3}\pi$

C. 16π

D. 32π

Solution

C

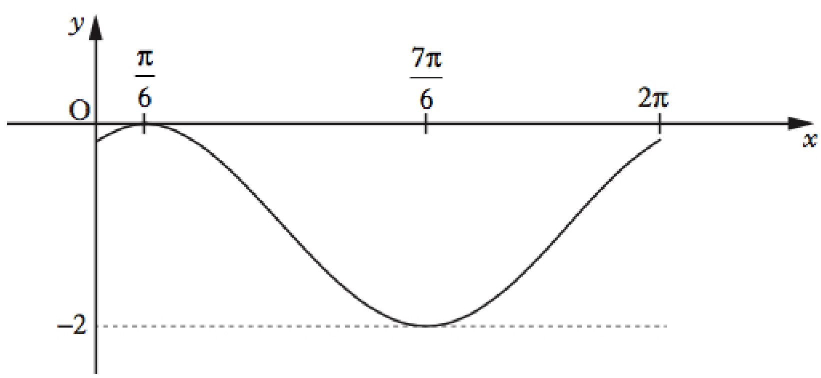
$$V = \tfrac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

and

$$r = 2 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi \times 2^2 = 16\pi.$$

9. The diagram shows the curve with equation of the form (2)

$$y = \cos(x + a) + b, \text{ for } 0 \leqslant x \leqslant 2\pi.$$



What is the equation of this curve?

A. $y = \cos(x - \frac{1}{6}\pi) - 1$

B. $y = \cos(x - \frac{1}{6}\pi) + 1$

C. $y = \cos(x + \frac{1}{6}\pi) - 1$

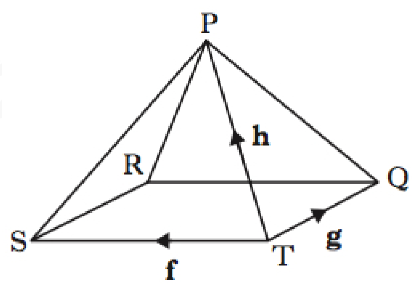
D. $y = \cos(x + \frac{1}{6}\pi) + 1$

Solution

A

The curve is a (a) translation by $\frac{1}{6}\pi$ in the x-direction and then (b) translation by -1 in the y-direction.

10. The diagram shows a square-based pyramid $PQRST$. (2)



$\overrightarrow{TS}$, $\overrightarrow{TQ}$, and $\overrightarrow{TP}$ represent $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ respectively.
Express $\overrightarrow{RP}$ in terms of $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$.

A. $-\mathbf{f} + \mathbf{g} - \mathbf{h}$

B. $-\mathbf{f} - \mathbf{g} + \mathbf{h}$

C. $\mathbf{f} - \mathbf{g} - \mathbf{h}$

D. $\mathbf{f} + \mathbf{g} + \mathbf{h}$

Solution

B

$$\begin{aligned}\overrightarrow{RP} &= \overrightarrow{RQ} + \overrightarrow{QT} + \overrightarrow{TP} \\ &= -\overrightarrow{ST} - \overrightarrow{TQ} + \overrightarrow{TP} \\ &= -\mathbf{f} - \mathbf{g} + \mathbf{h}.\end{aligned}$$

11. Find (2)

$$\int \frac{1}{6x^2}\,\mathrm{d}x,\ x \neq 0.$$

A. $-12x^{-3} + c$

B. $-6x^{-1} + c$

C. $-\frac{1}{3}x^{-3} + c$

D. $-\frac{1}{6}x^{-1} + c$

Solution

D

$$\begin{aligned}\int \frac{1}{6x^2}\,\mathrm{d}x &= \int \tfrac{1}{6}x^{-2}\,\mathrm{d}x\\ &= -\tfrac{1}{6}x^{-1} + c.\end{aligned}$$

12. Find the maximum value of (2)

$$2 - 3\sin(x - \tfrac{1}{3}\pi)$$

and the value of x where this occurs in the interval $0 \leqslant x \leqslant 2\pi$.

A. Maximum value is -1 when $x = \frac{11}{6}\pi$

B. Maximum value is 5 when $x = \frac{11}{6}\pi$

C. Maximum value is -1 when $x = \frac{5}{6}\pi$

D. Maximum value is 5 when $x = \frac{5}{6}\pi$

Solution

B

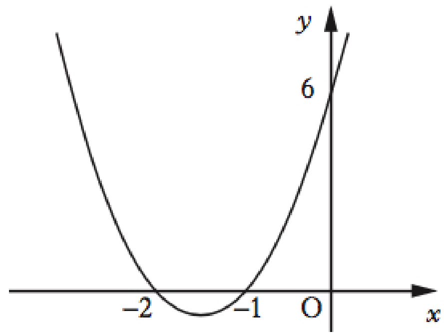
The maximum value is

$$2 - (-3) = 5$$

and it occurs

$$\begin{aligned}\sin(x - \tfrac{1}{3}\pi) = -1 &\Rightarrow x - \tfrac{1}{3}\pi = \tfrac{3}{2}\pi\\ &\Rightarrow x = \tfrac{11}{6}\pi.\end{aligned}$$

13. A parabola intersects the axes at $x = -2$, $x = -1$, and $y = 6$, as shown in the diagram. (2)



What is the equation of the parabola?

A. $y = 6(x - 1)(x - 2)$

B. $y = 6(x + 1)(x + 2)$

C. $y = 3(x - 1)(x - 2)$

D. $y = 3(x + 1)(x + 2)$

Solution

D

$x = -2$ and $x = -1$ are the roots which make the brackets $(x + 1)(x + 2)$. Now,

$$x = 0 \Rightarrow y = (0 + 1)(0 + 2) = 2$$

and so we need a factor of 3 in there.

14. Find (2)

$$\int (2x - 1)^{\frac{1}{2}} \, \mathrm{d}x$$

where $x > \frac{1}{2}$.

A. $\frac{1}{3}(2x - 1)^{\frac{3}{2}} + c$

B. $\frac{1}{2}(2x - 1)^{-\frac{1}{2}} + c$

C. $\frac{1}{2}(2x - 1)^{\frac{3}{2}} + c$

D. $\frac{1}{3}(2x - 1)^{-\frac{1}{2}} + c$

Solution

A

$$\int (2x-1)^{\frac{1}{2}}\,\mathrm{d}x = \tfrac{1}{3}(2x-1)^{\frac{3}{2}}.$$

15. If (2)

$$\mathbf{u} = k\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix},$$

where $k > 0$ and $\mathbf{u}$ is a unit vector, determine the value of k.

A. $\frac{1}{2}$

B. $\frac{1}{8}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{1}{\sqrt{10}}$

Solution

D

$$|\mathbf{u}| = \sqrt{(3^2 + (-1)^2 + 0} = \sqrt{10}$$

and value of k is the reciprocal of this.

16. If $y = 3\cos^4 x$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. (2)

A. $12\cos^3 x \sin x$

B. $12\cos^3 x$

C. $-12\cos^3 x \sin x$

D. $-12\sin^3 x$

Solution

C

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3\cos^3 x)\cdot 4 \cdot (-\sin x) = -12\cos^3 x \sin x.$$

17. Given that (2)

$$\mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \text{ and } \mathbf{a}.(\mathbf{a}+\mathbf{b}) = 7,$$

what is the value of $\mathbf{a}.\mathbf{b}$?

A. $\frac{7}{25}$

B. $-\frac{18}{5}$

C. -6

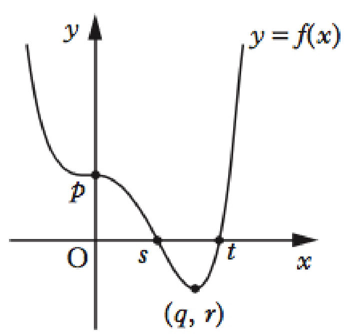
D. -18

Solution

D

$$\begin{aligned} \mathbf{a}.(\mathbf{a}+\mathbf{b}) = 7 &\Rightarrow \mathbf{a}.\mathbf{a} + \mathbf{a}.\mathbf{b} = 7 \\ &\Rightarrow |\mathbf{a}|^2 + \mathbf{a}.\mathbf{b} = 7 \\ &\Rightarrow (3^2 + 4^2 + 0) + \mathbf{a}.\mathbf{b} = 7 \\ &\Rightarrow 25 + \mathbf{a}.\mathbf{b} = 7 \\ &\Rightarrow \mathbf{a}.\mathbf{b} = -18. \end{aligned}$$

18. The graph of $y = \mathrm{f}(x)$ shown has stationary points at $(0, p)$ and (q, r). (2)



Here are two statements about $f(x)$:
(1) $\mathrm{f}(x) < 0$ for $s < x < t$;
(2) $\mathrm{f}'(x) < 0$ for $x < q$.
Which of the following is true?

A. Neither statement is correct.

B. Only statement (1) is correct.

C. Only statement (2) is correct.

D. Both statements are correct.

Solution

D

(1) and (2) are true.

19. Solve (2)

$$6 - x - x^2 < 0.$$

A. $-3 < x < 2$

B. $x < -3$ or $x > 2$

C. $-2 < x < 3$

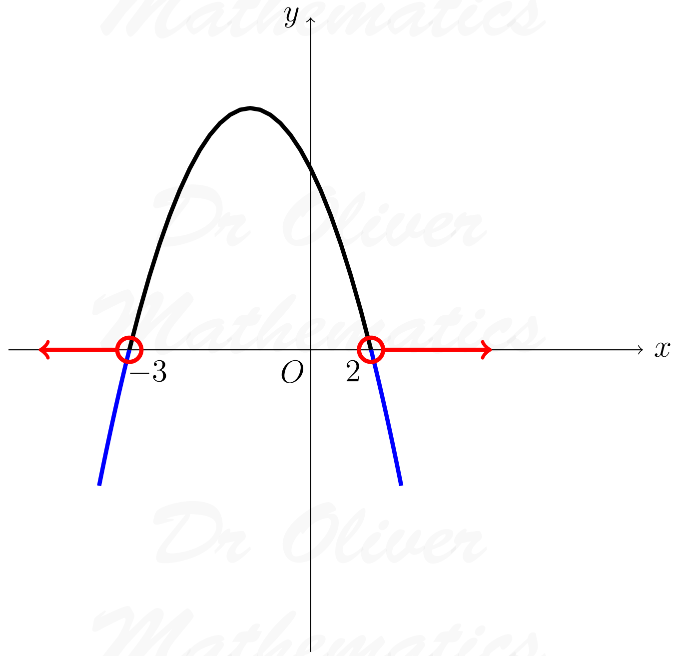
D. $x < -2$ or $x > 3$

Solution

B

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & (+6) \times (-1) = -6\end{array}\right\} -3,\ +2$$

$$\begin{aligned} 6 - x - x^2 < 0 &\Rightarrow 6 - 3x + 2x - x^2 < 0 \\ &\Rightarrow 3(2 - x) + x(2 - x) < 0 \\ &\Rightarrow (3 + x)(2 - x) < 0 \end{aligned}$$



$$\Rightarrow x < -3 \text{ or } x > 2.$$

20. Simplify (2)

$$\frac{\log_b 9a^2}{\log_b 3a},$$

where $a > 0$ and $b > 0$.

A. 2

B. $3a$

C. $\log_b 3a$

D. $\log_b(9a^2 - 3a)$.

Solution

A

$$\begin{aligned}\frac{\log_b 9a^2}{\log_b 3a} &= \frac{\log_b (3a)^2}{\log_b 3a} \\ &= \frac{2\log_b 3a}{\log_b 3a} \\ &= 2.\end{aligned}$$

Section B

21. (a) (i) Show that $(x-4)$ is a factor of $x^3 - 5x^2 + 2x + 8$. (6)

Solution
We use synthetic division:

4	1	−5	2	8
	↓	1	−4	−8
	1	−1	−2	0

As there is a remainder of zero, $(x-4)$ is a $\underline{\underline{\text{factor}}}$ of $x^3 - 5x^2 + 2x + 8$.

(ii) Factorise $x^3 - 5x^2 + 2x + 8$ fully.

Solution

$$\begin{aligned}x^3 - 5x^2 + 2x + 8 &= (x-4)(x^2 - x - 2) \\ &= \underline{\underline{(x-4)(x-2)(x+1)}}.\end{aligned}$$

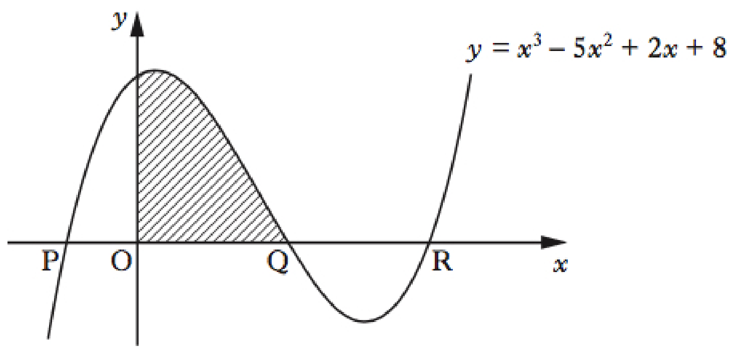
(iii) Solve $x^3 - 5x^2 + 2x + 8 = 0$.

Solution

$$\begin{aligned}x^3 - 5x^2 + 2x + 8 = 0 &\Rightarrow (x-4)(x-2)(x+1) = 0 \\ &\Rightarrow \underline{\underline{x = -1, x = 2, \text{ or } x = 4}}.\end{aligned}$$

The diagram shows the curve with equation

$$y = x^3 - 5x^2 + 2x + 8.$$



The curve crosses the x-axis at P, Q, and R.

(b) Determine the shaded area. (6)

Solution

$$\begin{aligned}\int_0^2 (x^3 - 5x^2 + 2x + 8)\,\mathrm{d}x &= \left[\tfrac{1}{4}x^4 - \tfrac{5}{3}x^3 + x^2 + 8x\right]_{x=0}^{2} \\ &= \left(4 - \tfrac{40}{3} + 4 + 16\right) - (0 - 0 + 0 + 0) \\ &= 24 - 13\tfrac{1}{3} \\ &= \underline{\underline{10\tfrac{2}{3}}}.\end{aligned}$$

22. (a) The expression (4)

$$\cos x - \sqrt{3}\sin x$$

can be written in the form

$$k\cos(x + a),$$

where $k > 0$ and $0 \leqslant a < 2\pi$.

Calculate the values of k and a.

Solution

Well,

$$k\cos(x + a) \equiv k\cos x\cos a - k\sin x\sin a$$

with

$$k\cos a = 1 \text{ and } k\sin a = \sqrt{3}.$$

Now,

$$\begin{aligned} k &= \sqrt{k^2} \\ &= \sqrt{(k\cos a)^2 + (k\sin a)^2} \\ &= \sqrt{1^2 + (\sqrt{3})^2} \\ &= \underline{\underline{2}} \end{aligned}$$

and

$$\tan a = \frac{k\sin a}{k\cos a} \Rightarrow \tan a = \sqrt{3}$$
$$= \underline{\underline{a = \tfrac{1}{3}\pi}}.$$

(b) Find the points of intersection of the graph of $y = \cos x - \sqrt{3}\sin x$ with the x- and y-axes, in the interval $0 \leqslant x < 2\pi$. (3)

Solution

$\underline{y - \text{intercept}}$: $x = 0 \Rightarrow y = 1$ and the y-intercept is $\underline{\underline{(0, 1)}}$.

$\underline{x - \text{intercepts}}$:

$$\begin{aligned} \cos x - \sqrt{3}\sin x = 0 &\Rightarrow 2\cos(x + \tfrac{1}{3}\pi) = 0 \\ &\Rightarrow x + \tfrac{1}{3}\pi = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi \\ &\Rightarrow \tfrac{1}{6}\pi, \tfrac{7}{6}\pi; \end{aligned}$$

hence, the x-intercepts are $\underline{\underline{(\tfrac{1}{6}\pi, 0)}}$ and $\underline{\underline{(\tfrac{7}{6}\pi, 0)}}$

23. (a) Find the equation of l_1, the perpendicular bisector of the line joining $P(3, -3)$ to $Q(-1, 9)$. (4)

Solution

$$\text{Midpoint} = \left(\frac{3 + (-1)}{2}, \frac{-3 + 9}{2}\right) = (1, 3).$$

Now,

$$\begin{aligned}\text{gradient of } PQ &= \frac{9-(-3)}{-1-3} \\ &= \frac{12}{-4} \\ &= -3\end{aligned}$$

and the gradient of the perpendicular bisector is $\frac{1}{3}$. Finally, the equation of l_1 is

$$\begin{aligned}y - 3 = \tfrac{1}{3}(x-1) &\Rightarrow y - 3 = \tfrac{1}{3}x - \tfrac{1}{3} \\ &\Rightarrow \underline{\underline{y = \tfrac{1}{3}x + \tfrac{8}{3}}}.\end{aligned}$$

(b) Find the equation of l_2 which is parallel to PQ and passes through $R(1, -2)$. (2)

Solution

$$\begin{aligned}y + 2 = -3(x-1) &\Rightarrow y + 2 = -3x + 3 \\ &\Rightarrow \underline{\underline{y = -3x + 1}}.\end{aligned}$$

(c) Find the point of intersection of l_1 and l_2. (3)

Solution

Eliminate y:

$$\begin{aligned}\tfrac{1}{3}x + \tfrac{8}{3} = -3x + 1 &\Rightarrow \tfrac{10}{3}x = -\tfrac{5}{3} \\ &\Rightarrow x = -\tfrac{1}{2} \\ &\Rightarrow y = 1\tfrac{1}{2} + 1 \\ &\Rightarrow y = 2\tfrac{1}{2};\end{aligned}$$

hence, the point is $\underline{\underline{(-\tfrac{1}{2}, 2\tfrac{1}{2})}}$.

(d) Hence find the shortest distance between PQ and l_2. (2)

Solution

$$
\begin{aligned}
\text{Shortest distance} &= \sqrt{(1-(-\tfrac{1}{2}))^2+(3-2\tfrac{1}{2})^2} \\
&= \sqrt{(\tfrac{3}{2})^2+(\tfrac{1}{2})^2} \\
&= \sqrt{\tfrac{9}{4}+\tfrac{1}{4}} \\
&= \sqrt{\tfrac{10}{4}} \\
&= \underline{\underline{\tfrac{1}{2}\sqrt{10}}}.
\end{aligned}
$$