Dr Oliver Mathematics
Mathematics: Advanced Higher
2013 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Write down the binomial expansion of (4)

$$\left(3x - \frac{2}{x^2}\right)^4$$

and simplify your answer.

Solution

$$\begin{aligned} &\left(3x - \frac{2}{x^2}\right)^4 \\ = \;& (3x)^4 + 4(3x)^3\left(-\frac{2}{x^2}\right) + 6(3x)^2\left(-\frac{2}{x^2}\right)^2 + 4(3x)\left(-\frac{2}{x^2}\right)^3 + \left(-\frac{2}{x^2}\right)^4 \\ = \;& \underline{\underline{81x^4 - 216x + 216x^{-2} - 96x^{-5} + 16x^{-8}}}. \end{aligned}$$

2. Differentiate (3)

$$f(x) = e^{\cos x} \sin^2 x.$$

Solution

$$\begin{aligned} f'(x) &= (-\sin x\, e^{\cos x}) \sin^2 x + e^{\cos x}(2 \sin x \cos x) \\ &= \underline{\underline{\sin x\, e^{\cos x}(2\cos x - \sin^2 x)}}. \end{aligned}$$

3. Matrices **A** and **B** are defined by

$$\mathbf{A} = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}.$$

(a) Find $\mathbf{A}^2$. (1)

Solution

$$\mathbf{A}^2 = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 16-2p & 5p \\ -10 & 1-2p \end{pmatrix}}}.$$

(b) Find the value of p for which $\mathbf{A}^2$ is singular. (2)

Solution

$$\det(\mathbf{A}^2) = 0 \Rightarrow (16-2p)(1-2p) - 5p(-10) = 0$$

$\times$	16	$-2p$
1	16	$-2p$
$-2p$	$-32p$	$+4p^2$

$$\begin{aligned} &\Rightarrow (4p^2 - 34p + 16) + 50p = 0 \\ &\Rightarrow 4p^2 + 16p + 16 = 0 \\ &\Rightarrow 4(p^2 + 4p + 4) = 0 \\ &\Rightarrow 4(p+2)^2 = 0 \\ &\Rightarrow \underline{\underline{p = -2 \text{ (repeated)}}} \end{aligned}$$

(c) Find the values of p and x if $\mathbf{B} = 3\mathbf{A}^T$. (2)

Solution

$$\begin{aligned} \mathbf{B} = 3\mathbf{A}^T &\Rightarrow \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} = 3\begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}; \end{aligned}$$

hence, $\underline{\underline{p = \frac{1}{3}}}$ and $\underline{\underline{x = 12}}$.

4. The velocity, v, of a particle P at time t is given by

$$v = \mathrm{e}^{3t} + 2\mathrm{e}^{t}.$$

(a) Find the acceleration of P at time t. (2)

Solution

$$\underline{\underline{a = 3\mathrm{e}^{3t} + 2\mathrm{e}^{t}}}.$$

(b) Find the distance covered by P between $t = 0$ and $t = \ln 3$. (3)

Solution

$$s = \tfrac{1}{3}\mathrm{e}^{3t} + 2\mathrm{e}^{t} + c$$

and

$$\begin{aligned}
\text{distance} &= \left[\tfrac{1}{3}\mathrm{e}^{3t} + 2\mathrm{e}^{t} + c\right]_{t=0}^{\ln 3} \\
&= \left(\tfrac{1}{3}\mathrm{e}^{3\ln 3} + 2\mathrm{e}^{\ln 3} + c\right) - \left(\tfrac{1}{3} + 2 + c\right) \\
&= \left(\tfrac{1}{3}\mathrm{e}^{\ln 3^3} + 2\mathrm{e}^{\ln 3} + c\right) - \left(2\tfrac{1}{3} + c\right) \\
&= (9 + 6 + c) - \left(2\tfrac{1}{3} + c\right) \\
&= \underline{\underline{12\tfrac{2}{3}}}.
\end{aligned}$$

5. Use the Euclidean algorithm to obtain the greatest common divisor of 1 204 and 833, expressing it in the form (4)

$$1\,204a + 833b,$$

where a and b are integers.

Solution

$$\begin{aligned}
1\,204 &= 833 + 371 \\
833 &= 2 \times 371 + 91 \\
371 &= 4 \times 91 + 7 \\
91 &= 13 \times 7.
\end{aligned}$$

Hence, the greatest common divisor is $\underline{\underline{7}}$ and

$$\begin{aligned} 7 &= 371 - 4 \times 91 \\ &= 371 - 4(833 - 2 \times 371) \\ &= 9 \times 371 - 4 \times 833 \\ &= 9(1\,204 - 833) - 4 \times 833 \\ &= \underline{\underline{9 \times 1\,204 - 13 \times 833}}; \end{aligned}$$

hence, $\underline{\underline{a = 9}}$ and $\underline{\underline{b = -13}}$.

6. Integrate (4)

$$\int \frac{\sec^2 3x}{1 + \tan 3x}\,\mathrm{d}x$$

with respect to x.

Solution

$$\begin{aligned} \int \frac{\sec^2 3x}{1 + \tan 3x}\,\mathrm{d}x &= \tfrac{1}{3} \int \frac{3\sec^2 3x}{1 + \tan 3x}\,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3} \ln|1 + \tan 3x| + c}}. \end{aligned}$$

7. Given that (4)

$$z = 1 - \sqrt{3}\mathrm{i},$$

write down $\bar{z}$ and express $\bar{z}^2$ in polar form.

Solution

$$z = 1 - \sqrt{3}\mathrm{i} = 2(\cos \tfrac{1}{3}\pi - \mathrm{i} \sin \tfrac{1}{3}\pi).$$

Now,

$$\bar{z} = \underline{\underline{2(\cos \tfrac{1}{3}\pi + \mathrm{i} \sin \tfrac{1}{3}\pi)}}$$

and

$$\begin{aligned} \bar{z}^2 &= \left[2(\cos \tfrac{1}{3}\pi + \mathrm{i} \sin \tfrac{1}{3}\pi)\right]^2 \\ &= \underline{\underline{4(\cos \tfrac{2}{3}\pi + \mathrm{i} \sin \tfrac{2}{3}\pi)}}. \end{aligned}$$

8. Use integration by parts to obtain (5)

$$\int x^2 \cos 3x \, \mathrm{d}x.$$

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos 3x \Rightarrow v = \tfrac{1}{3}\sin 3x$$

$$\int x^2 \cos 3x \, \mathrm{d}x = \tfrac{1}{3}x^2 \sin 3x - \tfrac{2}{3}\int x \sin 3x \, \mathrm{d}x$$

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sin 3x \Rightarrow v = -\tfrac{1}{3}\cos 3x$$

$$\begin{aligned}
&= \tfrac{1}{3}x^2 \sin 3x - \tfrac{2}{3}\left(-\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int \cos 3x \, \mathrm{d}x\right) \\
&= \tfrac{1}{3}x^2 \sin 3x + \tfrac{2}{9}x\cos 3x - \tfrac{2}{9}\int \cos 3x \, \mathrm{d}x \\
&= \underline{\underline{\tfrac{1}{3}x^2 \sin 3x + \tfrac{2}{9}x\cos 3x - \tfrac{2}{27}\sin 3x + c.}}
\end{aligned}$$

9. Prove by induction that, for all positive integers n, (6)

$$\sum_{r=1}^{n}(4r^3 + 3r^2 + r) = n(n+1)^3.$$

Solution

$\underline{n = 1}$: LHS $= 4 + 3 + 1 = 8$, RHS $= 1 \cdot 2^3 = 8$, and it is true for $n = 1$.

Suppose it is true for $n = k$, i.e.,

$$\sum_{r=1}^{k}(4r^3 + 3r^2 + r) = k(k+1)^3.$$

Well,

$$\begin{aligned}\sum_{r=1}^{k+1}(4r^3 + 3r^2 + r) &= \sum_{r=1}^{k}(4r^3 + 3r^2 + r) + [4(k+1)^3 + 3(k+1)^2 + (k+1)] \\ &= k(k+1)^3 + [4(k+1)^3 + 3(k+1)^2 + (k+1)] \\ &= (k+1)[k(k+1)^2 + 4(k+1)^2 + 3(k+1) + 1] \\ &= (k+1)[(k^3 + 2k^2 + k) + 4(k^2 + 2k + 1) + (3k+3) + 1] \\ &= (k+1)(k^3 + 6k^2 + 12k + 8) \\ &= (k+1)(k+2)^3 \\ &= (k+1)[(k+1) + 1]^3,\end{aligned}$$

and so it is true for $n = k + 1$.

Hence, by mathematical induction, it is true for all positive integers.

10. Describe the loci in the complex plane given by:

(a) $|z + \mathrm{i}| = 1$, (2)

Solution
It is a circle, radius 1, centre $(0, -1)$.

(b) $|z - 1| = |z + 5|$. (3)

Solution

$$\frac{1 + (-5)}{2} = -2;$$

hence, it is straight line $x = -2$.

11. A curve has equation (6)

$$x^2 + 4xy + y^2 + 11 = 0.$$

Find the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $(-2, 3)$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}: \quad 2x + \left(4y + 4x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Now, at $(-2, 3)$:

$$-4 + \left(12 - 8\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 6\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow -2\frac{\mathrm{d}y}{\mathrm{d}x} = -8$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 4.}}$$

Now,

$$2x + \left(4y + 4x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow \quad (4x + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} = -2x - 4y$$
$$\Rightarrow \quad \left(4 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + (4x + 2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + = -2 - 4\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Finally, at $(-2, 3)$:

$$(4 + 8)\,(4) + (-8 + 6)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + = -2 - 4(4)$$
$$\Rightarrow \quad 48 - 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -18$$
$$\Rightarrow \quad 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 66$$
$$\Rightarrow \quad \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 33.}}$$

12. Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

(a) If n is a multiple of 9, then so is n^2. (3)

Solution

Let $n = 9p$ for some integer p. Then

$$\begin{aligned} n^2 &= (9p)^2 \\ &= 81p^2 \\ &= 9(9p^2) \end{aligned}$$

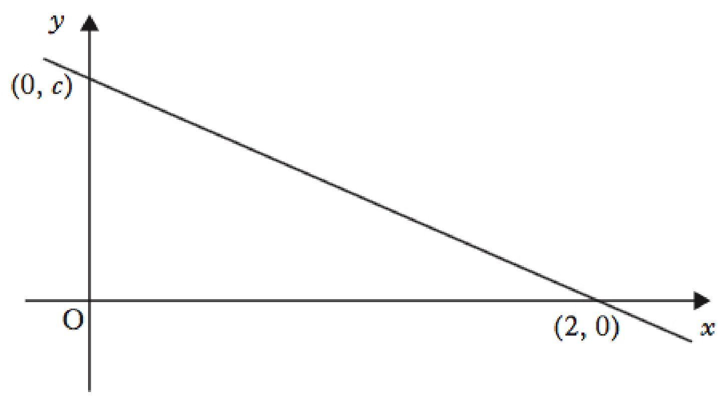
and so n^2 is a multiple of 9.

(b) If n^2 is a multiple of 9, then so is n. (1)

Solution

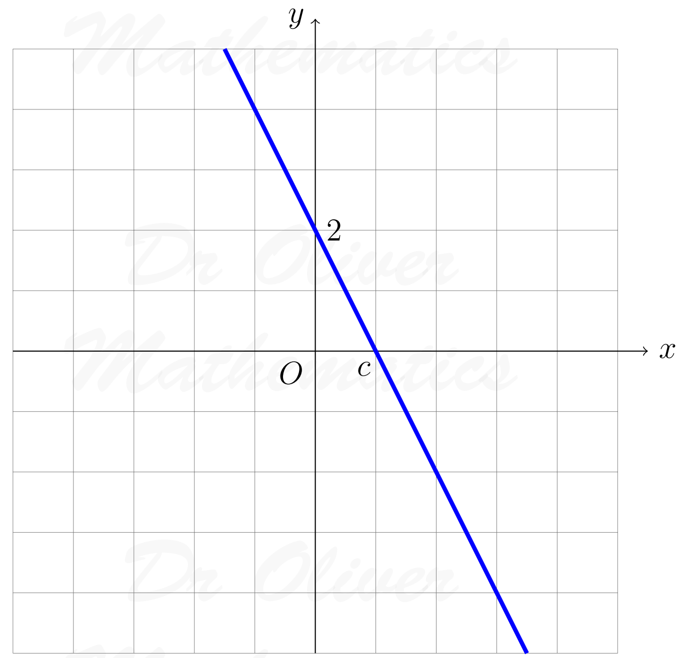
Counter-example: $n = 3$.

13. Part of the straight line graph of a function $f(x)$ is shown.



(a) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes. (2)

Solution



(b) State the value of k for which $\mathrm{f}(x) + k$ is an odd function. (1)

Solution

$k = -c$.

(c) Find the value of h for which $|\,\mathrm{f}(x+h)|$ is an even function. (2)

Solution

$h = 2$.

14. Solve the differential equation (11)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 4\mathrm{e}^{3x},$$

given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 0$.

Solution

Complementary function:

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3 \text{ (repeated)}$$

and hence the complementary function is

$$y = \mathrm{e}^{3x}(A + Bx).$$

Particular integral: try

$$\begin{aligned} y = Cx^2\mathrm{e}^{3x} &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2xC\mathrm{e}^{3x} + 3Cx^2\mathrm{e}^{3x} = (2Cx + 3Cx^2)\mathrm{e}^{3x} \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2C\mathrm{e}^{3x} + 6Cx\mathrm{e}^{3x}) + (6Cx\mathrm{e}^{3x} + 9Cx^2\mathrm{e}^{3x}) \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (2C + 12Cx + 9Cx^2)\mathrm{e}^{3x}. \end{aligned}$$

Now,

$$\begin{aligned} &\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 4\mathrm{e}^{3x} \\ \Rightarrow\quad &(2C + 12Cx + 9Cx^2)\mathrm{e}^{3x} - 6(2Cx + 3Cx^2)\mathrm{e}^{3x} + 9(Cx^2\mathrm{e}^{3x}) = 4\mathrm{e}^{3x} \\ \Rightarrow\quad &2C + 12Cx + 9Cx^2 - 12Cx - 18Cx^2 + 9Cx^2 = 4 \\ \Rightarrow\quad &2C = 4 \\ \Rightarrow\quad &C = 2. \end{aligned}$$

Hence the particular integral is $y = 2x^2\mathrm{e}^{3x}$.

General solution: hence the general solution is

$$y = \mathrm{e}^{3x}(A + Bx + 2x^2).$$

Next,

$$\begin{aligned} x = 0, y = 1 &\Rightarrow 1 = 1(A + 0 + 0) \\ &\Rightarrow A = 1. \end{aligned}$$

Now,

$$y = \mathrm{e}^{3x}(1 + Bx + 2x^2) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{3x}(1 + Bx + 2x^2) + \mathrm{e}^{3x}(B + 4x)$$

and

$$\begin{aligned} x = 0, \frac{\mathrm{d}y}{\mathrm{d}x} = -1 &\Rightarrow -1 = 3(1 + 0 + 0) + (B + 0) \\ &\Rightarrow B = -4. \end{aligned}$$

Hence, the particular solution is

$$\underline{\underline{y = \mathrm{e}^{3x}(1 - 4x + 2x^2)}}.$$

15. (a) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$, and $C(0, 0, 5)$. (4)

Solution

Well,

$$\overrightarrow{AB} = \mathbf{i} - \mathbf{j} \text{ and } \overrightarrow{AC} = -\mathbf{j} + \mathbf{k}.$$

Now,

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} \\ &= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}. \end{aligned}$$

Finally,

$$2x - 2y + z = (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}).(5\mathbf{k}) \Rightarrow \underline{\underline{2x - 2y + z = 5}}.$$

π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.

(b) Find an equation of the plane π_2. (2)

Solution

$$-y + z = (-\mathbf{j} + \mathbf{k}).(-\mathbf{j} + 3\mathbf{k}) \Rightarrow \underline{\underline{-y + z = 4}}.$$

(c) Determine the acute angle between planes π_1 and π_2. (3)

Solution

Let θ be the acute angle between the planes. Then

$$\begin{aligned} & (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}).(-\mathbf{j} + \mathbf{k}) = |2\mathbf{i} - 2\mathbf{j} + \mathbf{k}||-\mathbf{j} + \mathbf{k}| \cos\theta \\ \Rightarrow \quad & 3 = 3 \cdot \sqrt{2} \cdot \cos\theta \\ \Rightarrow \quad & \cos\theta = \tfrac{1}{\sqrt{2}} \\ \Rightarrow \quad & \underline{\underline{\theta = 45^\circ}}. \end{aligned}$$

16. In an environment without enough resources to support a population greater than 1 000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(1\,000 - P).$$

(a) Show that (4)

$$\ln\left(\frac{P}{1\,000-P}\right) = 1\,000t + C,$$

for some constant C.

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} = P(1\,000-P) &\Rightarrow \frac{1}{P(1\,000-P)}\,\mathrm{d}P = \mathrm{d}t\\ &\Rightarrow \left(\frac{\frac{1}{1\,000}}{P} - \frac{\frac{1}{1\,000}}{1\,000-P}\right)\mathrm{d}P = \mathrm{d}t\\ &\Rightarrow \int\left(\frac{1}{P} - \frac{1}{1\,000-P}\right)\mathrm{d}P = \int 1\,000\,\mathrm{d}t\\ &\Rightarrow \ln P - \ln(1\,000-P) = 1\,000t + C\\ &\Rightarrow \underline{\underline{\ln\left(\frac{P}{1\,000-P}\right) = 1\,000t + C,}}\end{aligned}$$

as required.

(b) Hence show that (3)

$$P(t) = \frac{1\,000K}{K + \mathrm{e}^{-1\,000t}},$$

for some constant K.

Solution

$$
\begin{aligned}
&\ln\left(\frac{P}{1\,000-P}\right) = 1\,000t + C \\
\Rightarrow\quad &\frac{P}{1\,000-P} = \mathrm{e}^{1\,000t+C} \\
\Rightarrow\quad &\frac{P}{1\,000-P} = K\mathrm{e}^{1\,000t} \text{ (for some constant } k) \\
\Rightarrow\quad &P = (1\,000-P)K\mathrm{e}^{1\,000t} \\
\Rightarrow\quad &P = 1\,000K\mathrm{e}^{1\,000t} - PK\mathrm{e}^{1\,000t} \\
\Rightarrow\quad &P + PK\mathrm{e}^{1\,000t} = 1\,000K\mathrm{e}^{1\,000t} \\
\Rightarrow\quad &P(1 + K\mathrm{e}^{1\,000t}) = 1\,000K\mathrm{e}^{1\,000t} \\
\Rightarrow\quad &P = \frac{1\,000K\mathrm{e}^{1\,000t}}{1+K\mathrm{e}^{1\,000t}} \\
\Rightarrow\quad &\underline{\underline{P = \frac{1\,000K}{K+\mathrm{e}^{-1\,000t}}}},
\end{aligned}
$$

as required.

(c) Given that (3)

$$P(0) = 200,$$

determine at what time t, $P(t) = 900$.

Solution

$$
\begin{aligned}
P(0) = 200 \Rightarrow 200 &= \frac{1\,000K}{K+1} \\
&\Rightarrow 200(K+1) = 1\,000K \\
&\Rightarrow 200K + 200 = 1\,000K \\
&\Rightarrow 800K = 200 \\
&\Rightarrow K = \tfrac{1}{4}
\end{aligned}
$$

and so

$$P = \frac{250}{\frac{1}{4} + \mathrm{e}^{-1\,000t}}.$$

Finally,

$$\begin{aligned} P(t) = 900 \Rightarrow 900 &= \frac{250}{\frac{1}{4} + \mathrm{e}^{-1\,000t}} \\ \Rightarrow \tfrac{1}{4} + \mathrm{e}^{-1\,000t} &= \tfrac{5}{18} \\ \Rightarrow \mathrm{e}^{-1\,000t} &= \tfrac{1}{36} \\ \Rightarrow \mathrm{e}^{1\,000t} &= 36 \\ \Rightarrow 1\,000t &= \ln 36 \\ \Rightarrow t &= \tfrac{1}{1\,000} \ln 36. \end{aligned}$$

17. (a) Write down the sums to infinity of the geometric series (2)

$$1 + x + x^2 + x^3 + \dots$$

and

$$1 - x + x^2 - x^3 + \dots$$

valid for $|x| < 1$.

Solution

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

and

$$\begin{aligned} 1 - x + x^2 - x^3 + \dots &= 1 + (-x) + (-x)^2 + (-x)^3 + \dots \\ &= \frac{1}{1-(-x)} \\ &= \frac{1}{1+x}. \end{aligned}$$

(b) Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$, (5)

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \dots\right).$$

Solution

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots \\ \Rightarrow \quad \int \frac{1}{1-x}\,\mathrm{d}x &= \int (1 + x + x^2 + x^3 + \ldots)\,\mathrm{d}x \\ \Rightarrow \quad -\ln(1-x) &= x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \ldots + c. \end{aligned}$$

Now,

$$x = 0 \Rightarrow 0 = 0 + c$$

which means

$$-\ln(1-x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \ldots$$

Similarly,

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \ldots$$

Finally,

$$\begin{aligned} \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\ &= (x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \ldots) \\ &\qquad + (x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \ldots) \\ &= \underline{\underline{2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots\right)}}, \end{aligned}$$

as required.

(c) Show how this series can be used to evaluate $\ln 2$. (2)

Solution

$$\begin{aligned} 2 = \frac{1+x}{1-x} &\Rightarrow 2(1-x) = 1 + x \\ &\Rightarrow 2 - 2x = 1 + x \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{3}}}. \end{aligned}$$

(d) Hence determine the value of $\ln 2$ correct to 3 decimal places. (1)

Solution

$$
\begin{aligned}
\ln 2 &= 2\left[\left(\tfrac{1}{3}\right) + \tfrac{1}{3}\left(\tfrac{1}{3}\right)^3 + \tfrac{1}{5}\left(\tfrac{1}{3}\right)^5 + \dots\right] \\
&= 2\left[\tfrac{1}{3} + \tfrac{1}{81} + \tfrac{1}{1\,215} + \dots\right] \\
&= \underline{\underline{0.693 \text{ (3 dp)}}}.
\end{aligned}
$$