Dr Oliver Mathematics
Further Mathematics
Complex Numbers: Loci
Past Examination Questions

This booklet consists of 17 questions across a variety of examination topics.
The total number of marks available is 188.

1. A transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, z \neq -\mathrm{i}.$$

   (a) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane. (4)

   (b) Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane. (6)

   (c) Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane. (2)

   (d) Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane. (2)

2. A complex number z is represented by a point P in the Argand diagram. Given that

$$|z - 3\mathrm{i}| = 3,$$

   (a) sketch the locus of P. (2)

   (b) Find the complex number z which satisfies both $|z - 3\mathrm{i}| = 3$ and $\arg(z - 3\mathrm{i}) = \frac{3}{4}\pi$. (4)

   The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2\mathrm{i}}{z}.$$

   (c) Show that T maps $|z - 3\mathrm{i}| = 3$ to a line in the w-plane, and give the cartesian equation of this line. (5)

3. In the Argand diagram the point P represents the complex number z. Given that $\arg\left(\frac{z-2\mathrm{i}}{z+2}\right) = \frac{\pi}{2}$,

   (a) sketch the locus of P, (2)

(b) deduce the value of $|z + 1 - \mathrm{i}|$. (2)

A transformation T from the z-plane to the w-plane is given by

$$w = \frac{2(1 + \mathrm{i})}{z + 2}, z \neq -2.$$

(c) Show that the locus of P in the z-plane is mapped to a straight line in the w-plane, and show this in the Argand diagram. (6)

4. The point P represents the complex number z on the Argand diagram, where

$$|z - 6 + 3\mathrm{i}| = 3|z + 2 - \mathrm{i}|.$$

(a) Show that the locus of P is a circle, giving the coordinates of the centre of the circle and the radius of this circle. (7)

The point Q represents the complex number z on the Argand diagram, where

$$\tan|\arg(z + 6)| = \tfrac{1}{2}.$$

(b) On the Argand diagram, sketch the locus of P and the locus of Q. (5)

(c) On your diagram, shade the region which satisfies both (2)

$$|z - 6 + 3\mathrm{i}| = 3|z + 2 - \mathrm{i}| \text{ and } \tan|\arg(z + 6)| = \tfrac{1}{2}.$$

5. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z + \mathrm{i}}{z}, z \neq 0.$$

(a) The transformation T maps the points on the line with equation $y = x$ in the z-plane, other than $(0, 0)$, to points on the line l in the w-plane. Find a cartesian equation of l. (5)

(b) Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has cartesian equation (7)

$$u^2 + v^2 - u + v = 0.$$

(c) On the same Argand diagram, sketch l and C. (3)

6. The point P represents the complex number z on the Argand diagram such that

$$|z - 3| = 2|z|.$$

(a) Show that, as z varies, the locus of P is a circle, and give the coordinates of the centre and the radius of the circle. (5)

The point Q represents the complex number z on the Argand diagram such that

$$|z + 3| = |z - \mathrm{i}\sqrt{3}|.$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies. (5)

(c) On your diagram shade the region which satisfies (2)

$$|z - 3| \geqslant 2|z| \text{ and } |z + 3| \geqslant |z - \mathrm{i}\sqrt{3}|.$$

7. The point P represents the complex number z on the Argand diagram. The locus of P is the curve C given by the equation

$$|z - 3| = 2|z - 4\mathrm{i}|.$$

(a) Show that C is a circle and give the coordinates of its centre and the value of its radius. (6)

The point Q represents the complex number w. The point Q is related to the point P by

$$w = \frac{12}{z}.$$

As P describes the curve C,

(b) show that the locus of Q is given by the equation (5)

$$|w - a| = k|w - \mathrm{i}b|,$$

where a, b, and $k \in \mathbb{R}$, stating the value of a, b, and k.

8. The point P represents the complex number z on the Argand diagram. Point P moves on the curve C given by the equation

$$|z - 4 + 4\mathrm{i}| = 2|z - 1 + \mathrm{i}|.$$

(a) Show that C is a circle whose equation may be written $|z| = k$, giving the exact value of k. (5)

(b) Draw an Argand diagram showing the circle C and the points representing the complex numbers $1 - \mathrm{i}$ and $4 - 4\mathrm{i}$. (3)

(c) For the points on the circle C, find the maximum and minimum values of $|z-4+4\mathrm{i}|$. (3)

The transformation T from the z-plane to the w-plane is given by

$$w = z + \frac{8}{z}.$$

(d) Show that T maps the curve C onto a line segment in the w-plane and define this line segment by giving its equation and the coordinates of its end points. (5)

9. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z}{z + \mathrm{i}}, z \neq -\mathrm{i}.$$

The circle with equation $|z| = 3$ is mapped by T onto the curve C.

(a) Show that C is a circle and find its centre and radius. (8)

The region $|z| < 3$ in the z-plane is mapped by T onto the region R in the w-plane.

(b) Shade the region R on an Argand diagram. (2)

10. A complex number z is represented by the point P in the Argand diagram.

(a) Given that $|z - 6| = |z|$, sketch the locus of P. (2)

(b) Find the complex numbers z which satisfies both $|z - 6| = |z|$ and $|z - 3 - 4\mathrm{i}| = 5$. (3)

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{30}{z}.$$

(c) Show that T maps $|z - 6| = |z|$ onto a circle in the w-plane and give the cartesian equation of this circle. (5)

11. The point P represents the complex number z on an Argand diagram, where

$$|z - \mathrm{i}| = 2.$$

The locus of P as z varies is the curve C.

(a) Find a cartesian equation of C. (2)

(b) Sketch the curve C. (2)

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + \mathrm{i}}{3 + \mathrm{i}z}, z \neq 3\mathrm{i}.$$

The point Q is mapped by T onto the point R. Given that R lies on the real axis,

(c) show that Q lies on C. (5)

12. The point P represents a complex number z on an Argand diagram such that

$$|z - 6\mathrm{i}| = 2|z - 3|.$$

(a) Show that, as z varies, the locus of P is a circle, stating the radius and the coordinates of the centre of this circle. (6)

The point Q represents the complex number $z$on an Argand diagram such that

$$\arg(z-6) = -\tfrac{3\pi}{4}.$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies. (4)

(c) Find the complex number for which both $|z-6\mathrm{i}| = 2|z-3|$ and $\arg(z-6) = -\frac{3\pi}{4}$. (4)

13. The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z+2\mathrm{i}}{\mathrm{i}z}, z \neq 0.$$

The transformation maps points on the real line in the z-plane onto a line in the w-plane. Find an equation of this line.

14. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{4(1-\mathrm{i})z - 8\mathrm{i}}{2(-1+\mathrm{i})z - \mathrm{i}}, z \neq \tfrac{1}{4} - \tfrac{1}{4}\mathrm{i}.$$

The transformation T maps the points on the line l with equation $y = x$ in the z-plane to a circle C in the w-plane.

(a) Show that (6)

$$w = \frac{ax^2 + bx\mathrm{i} + c}{16x^2 + 1},$$

where a, b, and c are real constants to be found.

(b) Hence show that the circle C has equation (4)

$$(u-3)^2 + v^2 = k^2,$$

where k is a constant to be found.

15. The transformation T maps from from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$. The transformation T is given by

$$w = \frac{z}{\mathrm{i}z+1}, z \neq \mathrm{i}.$$

The transformation T maps the line l in the z-plane onto the line with equation $v = -1$ in the w-plane.

(a) Find a cartesian equation of l in terms of x and y. (5)

The transformation T maps the line with equation $y = \frac{1}{2}$ in the z-plane onto the curve C in the w-plane.

(b) (i) Show that C is a circle with centre the origin. (4)

(ii) Write down a cartesian equation of C in terms of u and v. (2)

16. The transformation T maps from from the z-plane to the w-plane is given by

$$w = \frac{z}{z + 3\mathrm{i}}, z \neq -3\mathrm{i}.$$

The circle with equation $|z| = 2$ is mapped by T onto the curve C.

(a) (i) Show that C is a circle. (6)

(ii) Find the centre and radius of C. (2)

The region $|z| \leqslant 2$ in the z-plane is mapped by T onto the region R in to the w-plane.

(b) Shade the region R on an Argand diagram. (2)

17. The transformation T maps from from the z-plane to the w-plane is given by

$$w = \frac{z + 3\mathrm{i}}{1 + \mathrm{i}z}, z \neq \mathrm{i}.$$

The transformation T maps the circle $|z| = 1$ in the z-plane onto the line l in the w-plane.

(a) Find a cartesian equation of the line l. (5)

The circle $|z - a - b\mathrm{i}| = c$ in the z-plane is mapped by T onto the circle $|w| = 5$ in the w-plane.

(b) Find the exact values of the real constants a, b, and c. (6)