Dr Oliver Mathematics
Advance Level Further Mathematics
Further Pure Mathematics 1: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Use Simpson's rule with 4 intervals to estimate (5)

$$\int_{0.4}^{2} \mathrm{e}^{x^2}\,\mathrm{d}x.$$

2. Given that k is a real non-zero constant and that (4)

$$y = x^3 \sin kx,$$

use Leibnitz's theorem to show that

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = (k^2x^2 + A)k^3 x \cos kx + B(k^2x^2 + C)k^2 \sin kx,$$

where A, B, and C are integers to be determined.

3.
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2 \quad \text{(I)}.$$

(a) Show that (4)

$$\frac{\mathrm{d}^5 y}{\mathrm{d}x^5} = ay\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} + b\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + c\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)^2,$$

where a, b, and c are integers to be determined.

(b) Hence find a series solution, in ascending powers of x as far as the term in x^5, of the differential equation (I), given that $y = 1$ at $x = 0$. (5)

4. The parabola C has equation (8)

$$y^2 = 16x.$$

The distinct points $P(p^2, 4p)$ and $Q(q^2, 4q)$ lie on C, where $p \neq q$.

The tangent to C at P and the tangent to C at Q meet at the point $R(-28, 6)$.

Show that the area of triangle PQR is $1\,331$.

5. (8)

$$I = \int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x,\ 0 < x < \tfrac{1}{4}\pi.$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \tfrac{1}{5}\ln\left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right) + k,$$

where k is an arbitrary constant.

6. The concentration of a drug in the bloodstream of a patient, t hours after the drug has been administered, where $t \leqslant 6$, is modelled by the differential equation

$$t^2\frac{\mathrm{d}^2C}{\mathrm{d}t^2} - 5t\frac{\mathrm{d}C}{\mathrm{d}t} + 8C = t^3 \quad \text{(I)},$$

where C is measured in micrograms per litre.

(a) Show that the transformation $t = \mathrm{e}^x$ transforms equation (I) into the equation (5)

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} - 6\frac{\mathrm{d}C}{\mathrm{d}x} + 8C = \mathrm{e}^{3x} \quad \text{(II)}.$$

(b) Hence find the general solution for the concentration C at time t hours. (7)

Given that when $t = 6$, $C = 0$ and $\frac{\mathrm{d}C}{\mathrm{d}t} = -36$,

(c) find the maximum concentration of the drug in the bloodstream of the patient. (5)

7. With respect to a fixed origin O, the points A, B, and C have coordinates $(3, 4, 5)$, $(10, -1, 5)$, and $(4, 7, -9)$. respectively.

The plane Π has equation

$$4x - 8y + z = 2.$$

The line segment AB meets the plane Π at the point P and the line segment BC meets the plane Π at the point Q.

(a) Show that, to 3 significant figures, the area of quadrilateral $APQC$ is 38.5. (6)

The point D has coordinates $(k, 4, -1)$, where k is a constant.
Given that the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, and $\overrightarrow{AD}$ form three edges of a parallelepiped of volume 226,

(b) find the possible values of the constant k. (4)

8. The hyperbola H has equation

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

The line l_1 is the tangent to H at the point $P(4\cosh\theta, 3\sinh\theta)$.

The line l_1 meets the x-axis at the point A.

The line l_2 is the tangent to H at the point $(4, 0)$.

The lines l_1 and l_2 meet at the point B and the midpoint of AB is the point M.

(a) Show that, as θ varies, a Cartesian equation for the locus of M is (11)

$$y^2 = \frac{9(4-x)}{4x},\ p < x < q,$$

where p and q are values to be determined.

Let S be the focus of H that lies on the positive x-axis.

(b) Show that the distance from M to S is greater than 1. (3)