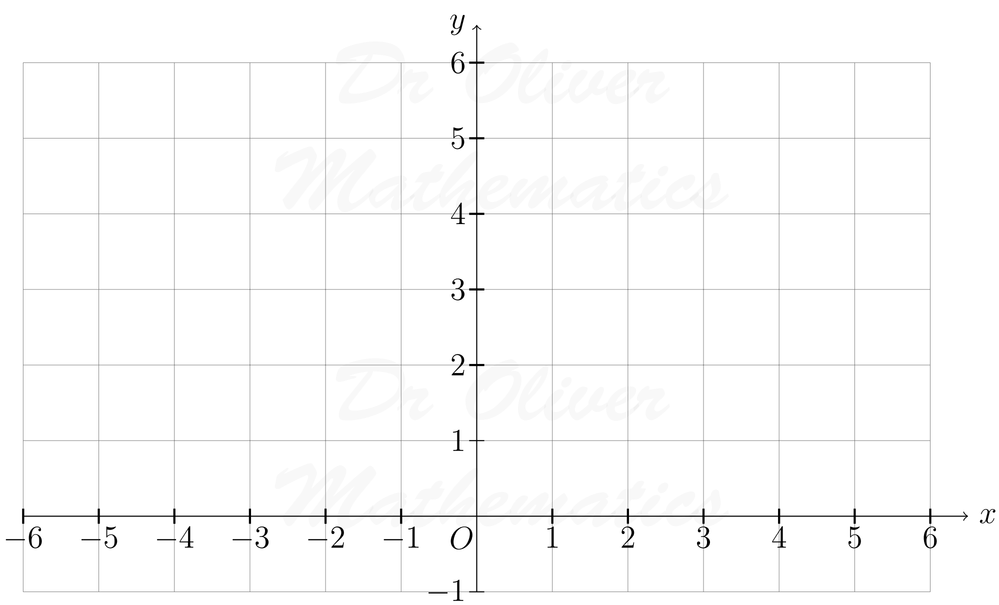
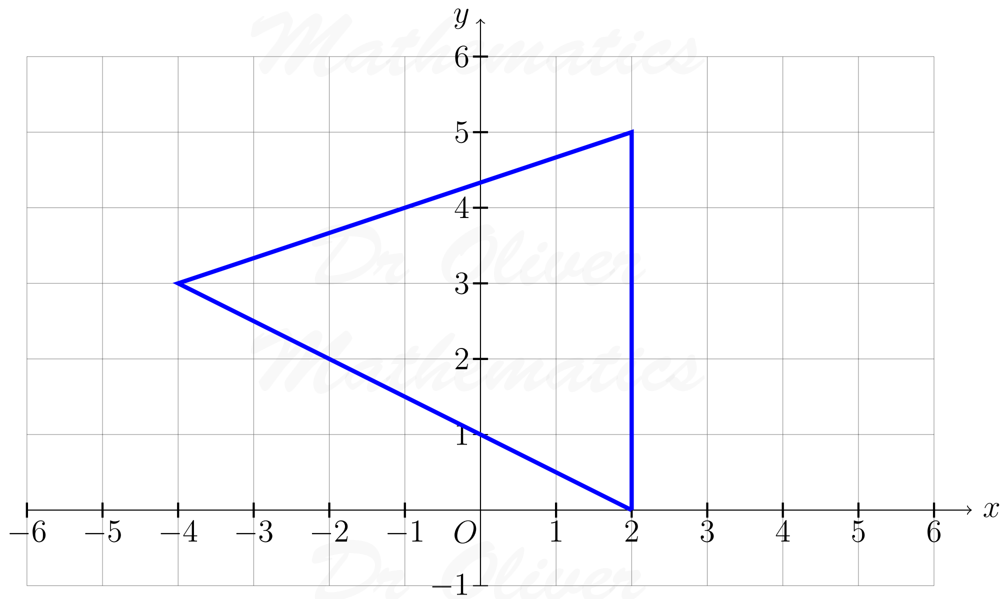
Dr Oliver Mathematics
AQA Further Maths Level 2
June 2016 Paper 2
2 hours

The total number of marks available is 105.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. A triangle has vertices $A(2, 5)$, $B(2, 0)$, and $C(-4, 3)$. (3)
   Work out the area of triangle ABC.
   You may use the grid to help you.



Solution

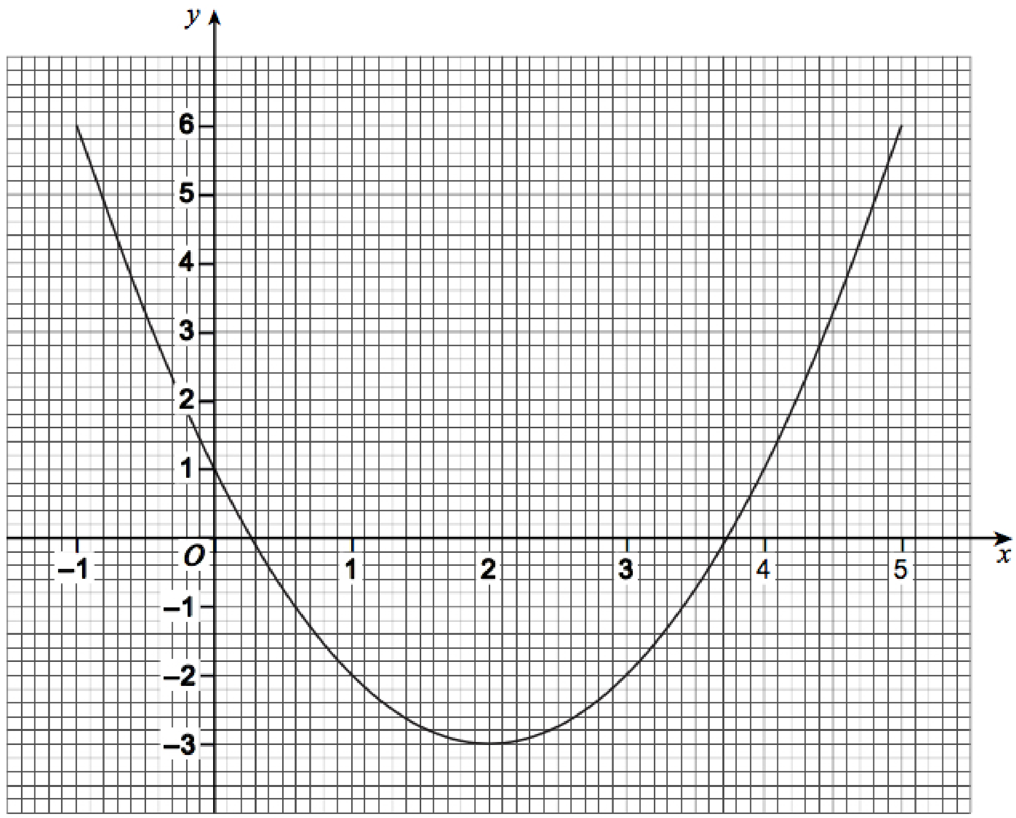


$$\text{Area} = \tfrac{1}{2} \times 5 \times 6$$
$$= \underline{\underline{15}}.$$

2. The function

$$\mathrm{f}(x) = x^2 - 4x + 1$$

has domain $-1 \leqslant x \leqslant 5$.
Here is the graph of $y = \mathrm{f}(x)$.



(a) Write down the equation of the line of symmetry of the graph. (1)

Solution

$x = 2$.

(b) Use the graph to work out the solutions of (2)

$$x^2 - 4x + 1 = 5.$$

Give your answers to 1 decimal place.

Solution

Compete the square:

$$\begin{aligned} x^2 - 4x + 1 = 5 &\Rightarrow x^2 - 4x + 4 = 8 \\ &\Rightarrow (x-2)^2 = 8 \\ &\Rightarrow x - 2 = \pm 2\sqrt{2} \\ &\Rightarrow x = 2 \pm 2\sqrt{2} \\ &\Rightarrow \underline{\underline{x = -0.8,\ 4.8\ (1\ \text{dp})}}. \end{aligned}$$

(c) Write down the range of $\mathrm{f}(x)$ for domain $-1 \leqslant x \leqslant 5$. (2)

Solution

$$\begin{aligned} \mathrm{f}(-1) &= (-1)^2 - 4(-1) + 1 = 6 \\ \mathrm{f}(2) &= 2^2 - 4(2) + 1 = -3 \\ \mathrm{f}(5) &= 5^2 - 4(5) + 1 = 6. \end{aligned}$$

Hence,

$$\underline{\underline{-3 \leqslant \mathrm{f}(x) \leqslant 6}}.$$

3. L is a straight line with equation

$$ax + by = c,$$

where a, b, and c are non-zero integers.

(a) At which point does L intersect the x-axis? (1)
Circle your answer.

$(\frac{a}{c}, 0)$ $(\frac{c}{a}, 0)$ $(\frac{b}{c}, 0)$ $(\frac{c}{b}, 0)$

Solution

$$y = 0 \Rightarrow ax = c \Rightarrow x = \tfrac{c}{a}.$$

$(\frac{a}{c}, 0)$ $\underline{\underline{(\frac{c}{a}, 0)}}$ $(\frac{b}{c}, 0)$ $(\frac{c}{b}, 0)$

(b) What is the gradient of a line parallel to L? (1)
Circle your answer.

$-\frac{b}{a}$ $\frac{b}{a}$ $-\frac{a}{b}$ $-\frac{a}{b}$

Solution

$$\begin{aligned} ax + by = c &\Rightarrow by = -ax + c \\ &\Rightarrow y = -\tfrac{a}{b}x + \tfrac{c}{b}. \end{aligned}$$

$$-\tfrac{b}{a} \qquad \tfrac{b}{a} \qquad \underline{\underline{-\tfrac{a}{b}}} \qquad -\ \tfrac{a}{b}$$

4. Work out the point of intersection of the lines (4)

$$2x + 3y = 11 \text{ and } 2y = 13 - 3x.$$

Solution

$$\begin{aligned} 2x + 3y &= 11 \quad (1) \\ 3x + 2y &= 13 \quad (2) \end{aligned}$$

E.g., do $3 \times (1)$ and $2 \times (2)$:

$$\begin{aligned} 6x + 9y &= 33 \quad (3) \\ 6x + 4y &= 26 \quad (4) \end{aligned}$$

and do $(3) - (4)$:

$$\begin{aligned} 5y = 7 &\Rightarrow y = 1\tfrac{2}{5} \\ &\Rightarrow 2x + 3(1\tfrac{2}{5}) = 11 \\ &\Rightarrow 2x + 4\tfrac{1}{5} = 11 \\ &\Rightarrow 2x = 6\tfrac{4}{5} \\ &\Rightarrow x = 3\tfrac{2}{5}. \end{aligned}$$

They intersect at $\underline{\underline{(3\tfrac{2}{5}, 1\tfrac{2}{5})}}$.

5. a, b, and c are numbers such that (4)

$$a < 0,\ b > 1, \text{ and } -1 < c < 1.$$

Tick the correct box for each statement.

	Always true	Sometimes true	Never true
$a^3 < 0$	☐	☐	☐
$b < 10a^2$	☐	☐	☐
$ab > 0$	☐	☐	☐
$b - c > 1$	☐	☐	☐

Solution

$a^3 < 0$: Always true
$b < 10a^2$: Sometimes true
$ab > 0$: Never true
$b - c > 0$: Sometimes true

6. For the curve $y = \mathrm{f}(x)$, (3)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}x - kx^4 + k,$$

where k is a constant.

When $x = -2$ the gradient of the curve is 12.

Work out the value of k.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 12 &\Rightarrow \tfrac{3}{2}(-2) - k[(-2)^4] + k = 12 \\ &\Rightarrow -3 - 16k + k = 12 \\ &\Rightarrow 15k = -15 \\ &\Rightarrow \underline{\underline{k = -1}}.\end{aligned}$$

7. Simplify fully (2)

$$(\tfrac{2}{3}x^3y)^3.$$

Solution

$$(\tfrac{2}{3}x^3y)^3 = \underline{\underline{\tfrac{8}{27}x^9y^3}}.$$

8. $D(-6, 4)$ and $E(-2, 9)$ are joined by a straight line. (3)

- P is a point on DE.
- $DP : PE = 3 : 5$.

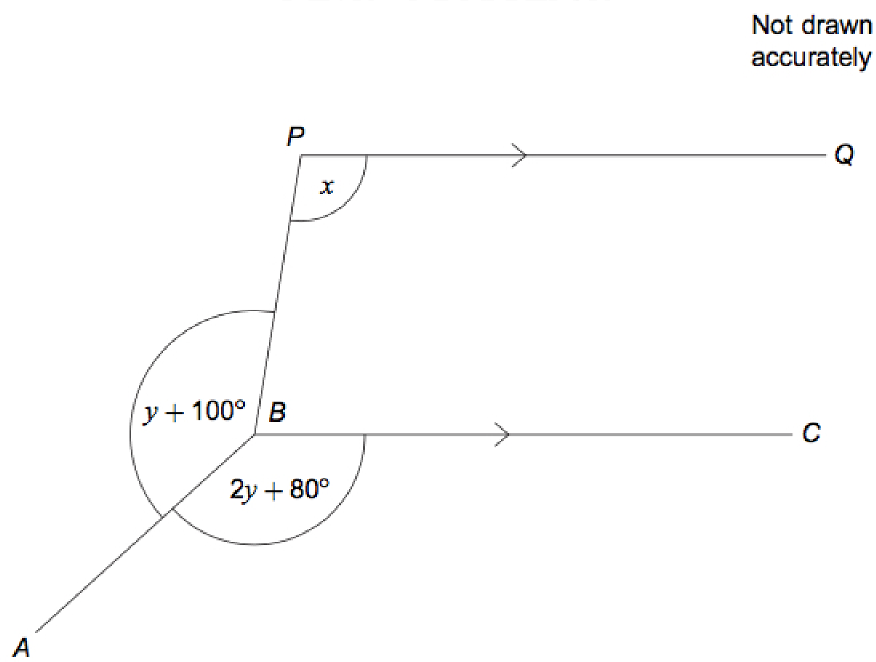
Work out the coordinates of P.

Solution

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OD} + \overrightarrow{DP} \\ &= \overrightarrow{OD} + \tfrac{3}{8}\overrightarrow{DE} \\ &= \begin{pmatrix} -6 \\ 4 \end{pmatrix} + \tfrac{3}{8}\begin{pmatrix} -2-(-6) \\ 9-4 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 4 \end{pmatrix} + \tfrac{3}{8}\begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} -6 \\ 4 \end{pmatrix} + \begin{pmatrix} 1\frac{1}{2} \\ 1\frac{7}{8} \end{pmatrix} \\ &= \begin{pmatrix} -4\frac{1}{2} \\ 5\frac{7}{8} \end{pmatrix};\end{aligned}$$

hence, $\underline{\underline{P(-4\tfrac{1}{2}, 5\tfrac{7}{8})}}$.

9. PQ is parallel to BC. (4)



Prove that

$$x = 3y.$$

Solution

$\angle PBC = 360 - (y + 100) - (2y + 80) = 180 - 3y$ (completing the angle at B).
Hence, $\underline{\underline{x = 3y}}$ (interior angles)

10. (a) Simplify (2)

$$\frac{x^2 - 7x + 10}{x^2 - 2x - 15}.$$

Solution

$$\left.\begin{array}{ll} \text{add to:} & -7 \\ \text{multiply to:} & +10 \end{array}\right\} -5,\ -2$$

$$x^2 - 7x + 10 = (x - 5)(x - 2).$$

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & -15 \end{array}\right\} -5,\ +3$$

$$x^2 - 2x - 15 = (x-5)(x+3).$$

Finally,

$$\frac{x^2 - 7x + 10}{x^2 - 2x - 15} = \frac{(x-5)(x-2)}{(x-5)(x+3)} = \underline{\underline{\frac{x-2}{x+3}}}.$$

(b) Factorise fully (2)

$$w^5x^3y^2 + w^2x^6y^3.$$

Solution

$$w^5x^3y^2 + w^2x^6y^3 = \underline{\underline{w^2x^3y^2(w^3 + x^3y)}}.$$

11. The x^2 term in the expansion of (3)

$$(3x+4)(x^2+px+5)$$

is $-23x^2$.

Work out the value of p.

Solution

$\times$	x^2	$+px$	$+5$
$3x$	$3x^3$	$+3px^2$	$+15x$
$+4$	$+4x^2$	$+4px$	$+20$

Hence,

$$3p + 4 = -23 \Rightarrow 3p = -27$$
$$\Rightarrow \underline{\underline{p = -9}}.$$

12. Here are the first four terms of linear sequences X and Y and quadratic sequence Z.

$$\begin{aligned}\text{Sequence } X: &\quad 7 \quad 9 \quad 11 \quad 13 \quad \ldots\\ \text{Sequence } Y: &\quad 2 \quad 5 \quad 8 \quad 11 \quad \ldots\\ \text{Sequence } Z: &\quad 14 \quad 45 \quad 88 \quad 143 \quad \ldots\end{aligned}$$

(a) Work out the nth term of sequence X. (2)

Solution

Let the

$$n\text{th term} = an + b.$$

7		9		11		13
	2		2		2	

$a+b$		$2a+b$		$3a+b$		$4a+b$
	a		a		a	

We compare terms:

$$a = 2$$

and

$$\begin{aligned}a + b = 7 &\Rightarrow 2 + b = 7\\ &\Rightarrow b = 5.\end{aligned}$$

Hence,

$$n\text{th term} = \underline{\underline{2n + 5}}.$$

The nth term of sequence Y is $3n - 1$.

(b) Using your answer to part (a), or otherwise, work out the nth term of sequence Z. Give your answer in the form $an^2 + bn + c$, where a, b, and c are integers. (3)

Solution

Let nth term be

$$an^2 + bn + c.$$

Write down the sequence:	14		45		88		143
First line of differences:		31		43		55	
Second line of differences:			12		12		

Sequence:	$a+b+c$		$4a+2b+c$	$9a+3b+c$
First line:		$3a+b$	$5a+b$	
Second line:			$2a$	

We compare terms:

$$2a = 12 \Rightarrow a = 6,$$

$$\begin{aligned} 3a+b=31 &\Rightarrow 3\times 6 + b = 31 \\ &\Rightarrow b = 13, \end{aligned}$$

and

$$\begin{aligned} a+b+c=14 &\Rightarrow 6+13+c = 14 \\ &\Rightarrow c = -5; \end{aligned}$$

hence,

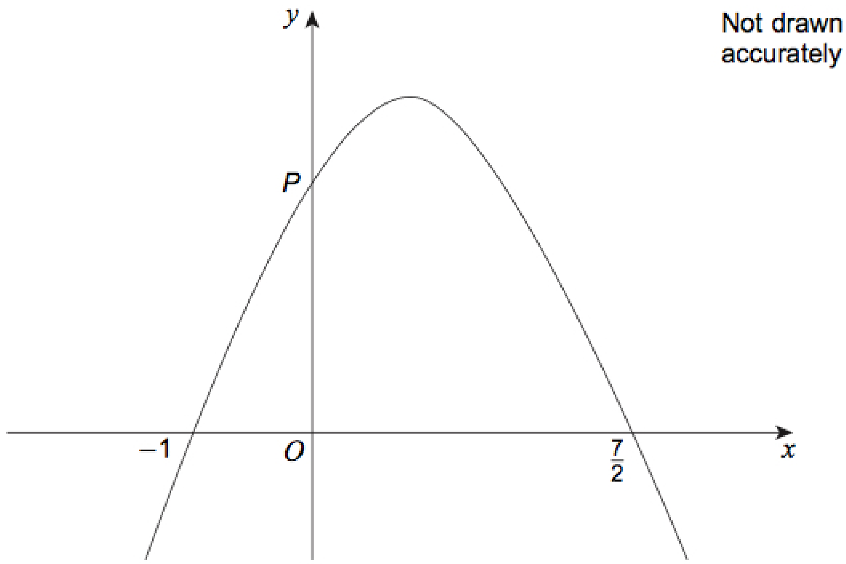
$$n\text{th term} = \underline{\underline{6n^2 + 13n - 5}}.$$

13. Here is a sketch of (4)

$$y = a + bx - 2x^2,$$

where a and b are constants.

The graph intersects the x-axis at $(-1, 0)$ and $(\frac{7}{2}, 0)$ and the y-axis at point P.



Work out the coordinates of point P.
You **must** show your working.

Solution

$$x = -1,\ y = 0 \Rightarrow 0 = a + b(-1) - 2[(-1)^2]$$
$$\Rightarrow 0 = a - b - 2$$
$$\Rightarrow a - b = 2 \quad (1)$$

and

$$x = \tfrac{7}{2},\ y = 0 \Rightarrow 0 = a + b(\tfrac{7}{2}) - 2[(\tfrac{7}{2})^2]$$
$$\Rightarrow 0 = a + \tfrac{7}{2}b - 24\tfrac{1}{2}$$
$$\Rightarrow a + \tfrac{7}{2}b = 24\tfrac{1}{2} \quad (2).$$

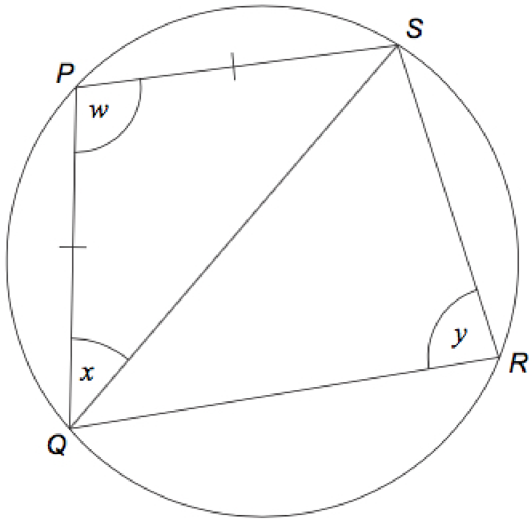
Do (2) − (1):

$$\tfrac{9}{2}b = 22\tfrac{1}{2} \Rightarrow b = 5$$
$$\Rightarrow a = 7.$$

Hence, $\underline{\underline{P(0, 7)}}$.

14. P, Q, R, and S are points on the circumference of a circle. (4)

- $w : y = 7 : 5$.
- $PQ = PS$.



Not drawn accurately

Work out the size of angle x.

Solution

Because opposite angles in a cyclic quadrilateral add to 180°,

$$\begin{aligned} w + y = 180 &\Rightarrow w + \tfrac{5}{7}w = 180 \\ &\Rightarrow \tfrac{12}{7}w = 180 \\ &\Rightarrow w = 105 \end{aligned}$$

and, because base angles are equal,

$$x = \tfrac{1}{2}(180 - 105) = \underline{\underline{37\tfrac{1}{2}^\circ}}.$$

15. (a) Solve (2)

$$\tfrac{2}{5}\sqrt{x} = 1.$$

Solution

$$\begin{aligned}\tfrac{2}{5}\sqrt{x} = 1 &\Rightarrow \sqrt{x} = \tfrac{5}{2}\\ &\Rightarrow x = \left(2\tfrac{1}{2}\right)^2\\ &\Rightarrow \underline{\underline{x = 6\tfrac{1}{4}}}.\end{aligned}$$

(b) Solve (2)

$$x^3 = 5x^2.$$

Solution

$$\begin{aligned}x^3 = 5x^2 &\Rightarrow x^3 - 5x^2 = 0\\ &\Rightarrow x^2(x-5) = 0\\ &\Rightarrow \underline{\underline{x = 0 \text{ or } x = 5}}.\end{aligned}$$

16. Rearrange (4)

$$y = \frac{8(w-x)}{x}$$

to make x the subject.

Solution

$$\begin{aligned}y = \frac{8(w-x)}{x} &\Rightarrow xy = 8(w-x)\\ &\Rightarrow xy = 8w - 8x\\ &\Rightarrow xy + 8x = 8w\\ &\Rightarrow x(y+8) = 8w\\ &\Rightarrow \underline{\underline{x = \frac{8w}{y+8}}}.\end{aligned}$$

17. A cylinder has base radius x cm and height y cm. (3)
A hemisphere has radius $6y$ cm.

The cylinder and hemisphere have equal volumes.

Work out the value of

$$\frac{x}{y}.$$

You **must** show your working.

Solution

Equal volumes:

$$\begin{aligned}\pi \times x^2 \times y = \tfrac{1}{2} \times \tfrac{4}{3} \times \pi \times (6y)^3 &\Rightarrow x^2 y = 144y^3 \\ &\Rightarrow x^2 = 144y^2 \\ &\Rightarrow \frac{x^2}{y^2} = 144 \\ &\Rightarrow \underline{\underline{\frac{x}{y} = 12}},\end{aligned}$$

because $x > 0$ and $y > 0$.

18. Angle y is acute.

$$\tan y = \frac{p+1}{p-1},$$

where p is a constant greater than 1.

(a) Which of the statements below is correct? (1)
Circle your answer.

$y = 45^\circ$ $\quad$ $y < 45^\circ$ $\quad$ $y > 45^\circ$ $\quad$ y could be any acute angle

Solution

As $\tan y > 1$ and p is a constant greater than 1,

$y = 45^\circ$ $\quad$ $y < 45^\circ$ $\quad$ $\underline{\underline{y > 45^\circ}}$ $\quad$ y could be any acute angle

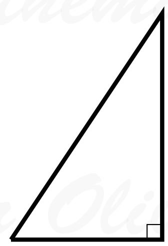
(b) Work out the expression for $\sin y$. (4)
Give your answer in the form

$$\frac{ap+b}{\sqrt{cp^2+d}},$$

where a, b, c, and d are integers.
You may use a diagram to help you.

Solution

$\times$	p	± 1
p	p^2	$\pm p$
± 1	$\pm p$	$+1$



The hypotuenuse is

$$\begin{aligned} \text{hyp} &= \sqrt{(p-1)^2 + (p+1)^2} \\ &= \sqrt{(p^2 - 2p + 1) + (p^2 + 2p + 1)} \\ &= \sqrt{2p^2 + 2} \end{aligned}$$

and, hence,

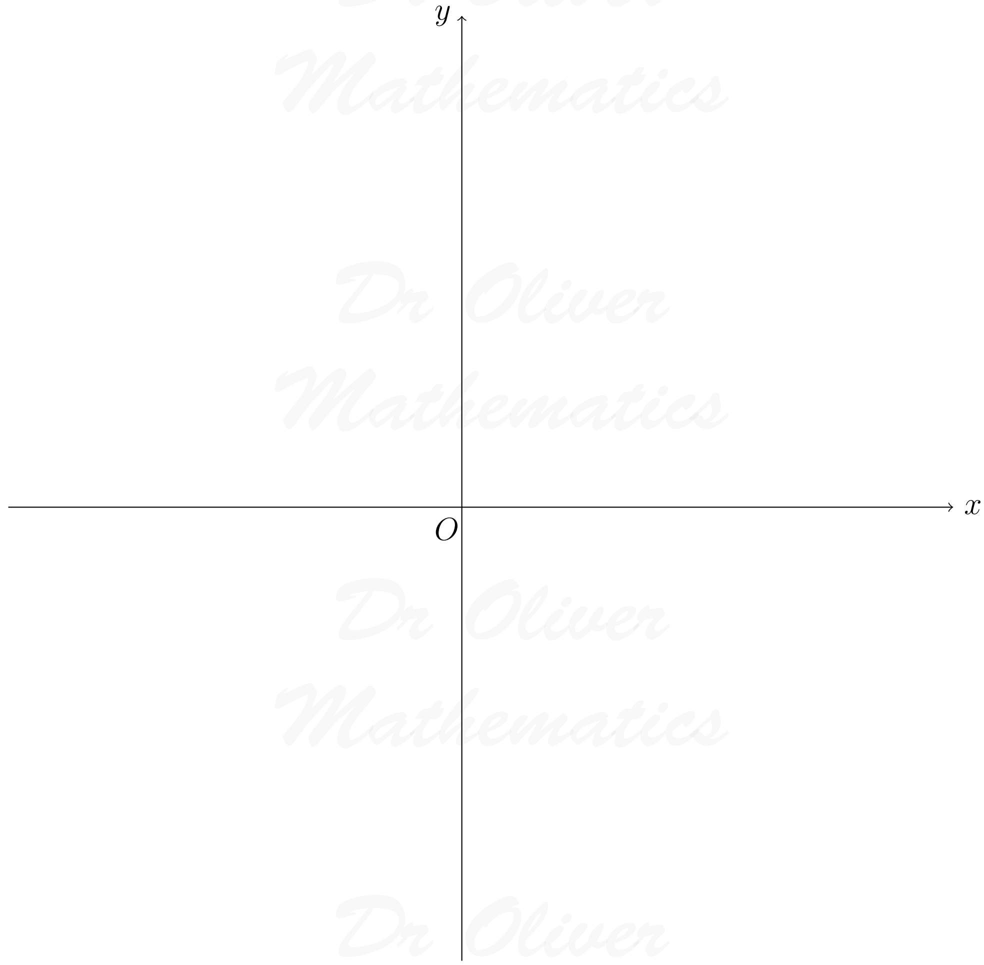
$$\underline{\underline{\sin y = \frac{p+1}{\sqrt{2p^2+2}}}};$$

$\underline{\underline{a = 1}}$, $\underline{\underline{b = 1}}$, $\underline{\underline{c = 2}}$, and $\underline{\underline{d = 2}}$.

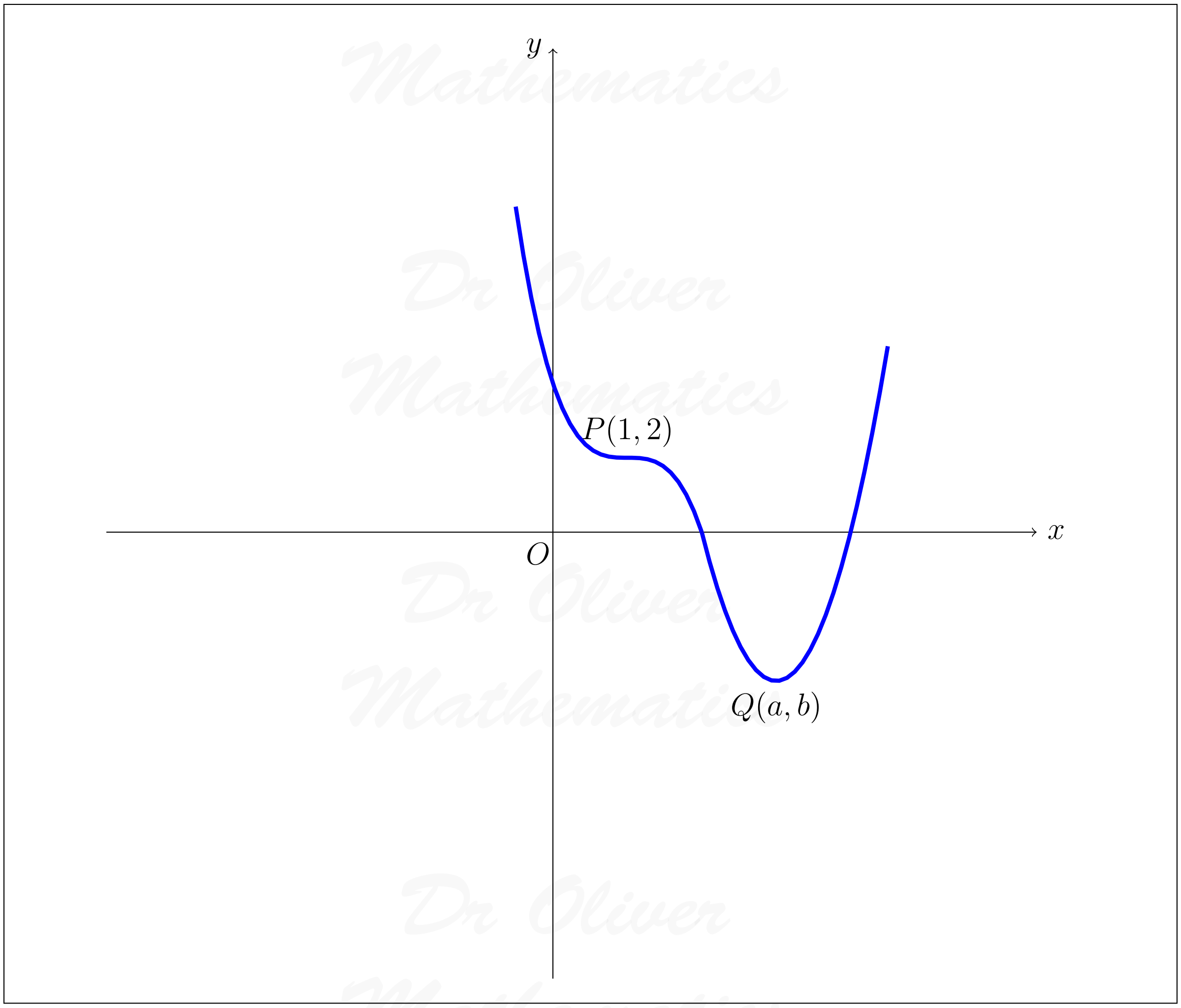
19. The continuous curve $y = \mathrm{g}(x)$ has exactly two stationary points. (3)
The stationary points are

- a point of inflection at $P(1, 2)$ and
- a minimum point at $Q(a, b)$ where $a > 1$ and $b < 0$.

On the axes below, sketch the curve.
Label points P and Q on your sketch.



Solution



20. Under the transformation represented by (4)

$$\begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix},$$

the image of point $P(a, 2)$ is point Q.

Can point Q be the same as point P?
You **must** show your working.

Solution

$$\begin{pmatrix} -1 & -3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ 2 \end{pmatrix} = \begin{pmatrix} -a-6 \\ 2a+8 \end{pmatrix}.$$

So,

$$a = -a - 6 \Rightarrow 2a = -6$$
$$\Rightarrow a = -3.$$

Do that work for the y-coordinate? Well,

$$2(-3) + 8 = -6 + 8 = 2 :$$

it does! So, point Q can the <u>same</u> as point P.

21. Solve (6)

$$\frac{3}{x-2} + \frac{2}{x-1} = 5.$$

Do **not** use trial and improvement.
Write your solutions to 3 significant figures.

Solution

$\times$	x	-1
x	x^2	$-x$
-2	$-2x$	$+2$

Multiply by $(x-1)(x-2)$:

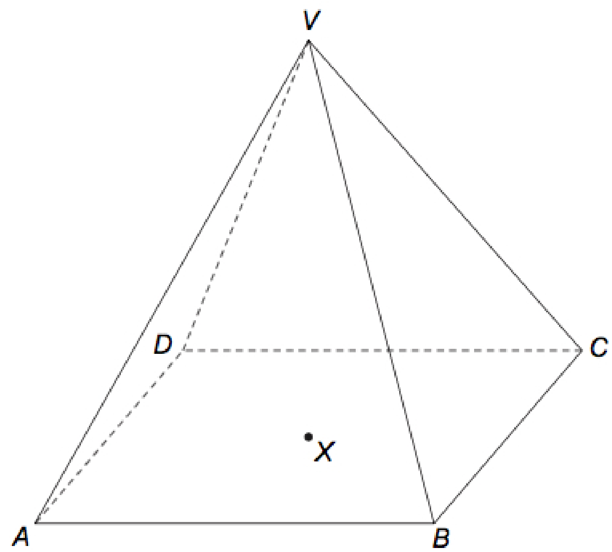
$$\frac{3}{x-2} + \frac{2}{x-1} = 5 \Rightarrow 3(x-1) + 2(x-2) = 5(x-1)(x-2)$$
$$\Rightarrow (3x-3) + (2x-4) = 5(x^2 - 3x + 2)$$
$$\Rightarrow 5x - 7 = 5x^2 - 15x + 10$$
$$\Rightarrow 5x^2 - 20x + 17 = 0$$

$a = 5$, $b = -20$, and $c = 17$:

$$\Rightarrow x = \frac{20 \pm \sqrt{20^2 - 4 \times 5 \times 17}}{2 \times 5}$$
$$\Rightarrow x = \frac{20 \pm \sqrt{60}}{10}$$
$$\Rightarrow x = 1.225\,403\,331,\ 2.774\,596\,669 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 1.23,\ 2.77 \text{ (3 sf)}}}.$$

22. Pyramid $VABCD$ has a horizontal rectangular base. (4)

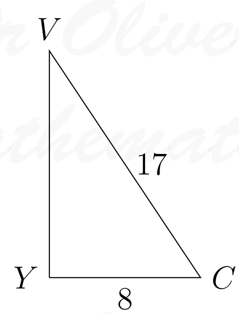
- X is the centre of the base.
- V is vertically above X.
- $VB = VC = 17$ cm.
- $AB = 22$ cm.
- $BC = 16$ cm



Work out the angle between the planes VBC and $ABCD$.

Solution

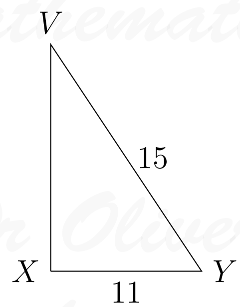
Let Y be the midpoint of BC. Then $BY = YC = 8$ cm.



Now,

$$\begin{aligned} VY^2 + YC^2 = VC^2 &\Rightarrow VY^2 + 8^2 = 17^2 \\ &\Rightarrow VY^2 + 64 = 289 \\ &\Rightarrow VY^2 = 225 \\ &\Rightarrow VY = 15 \text{ cm.} \end{aligned}$$

Next, $XY = \frac{1}{2}AB = 11$ cm.



Finally,

$$\begin{aligned} \cos = \frac{\text{adj}}{\text{hyp}} &\Rightarrow \cos \angle VYX = \frac{11}{15} \\ &\Rightarrow \angle VYX = 42.833\,428\,07 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\angle VYX = 42.8^\circ \text{ (3 sf)}}}. \end{aligned}$$

23. Shape A maps to shape B by an enlargement, scale factor 3, centre the origin. (5)
Shape B maps to shape C by a rotation through 180°, centre the origin.

Shape A can be mapped to shape C by a single transformation.

Use matrices to show that the single transformation is an enlargement, centre the origin. State the scale factor of the enlargement.

Solution

Well,

$$\mathbf{M}_{AB} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \text{ and } \mathbf{M}_{BC} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now,

$$\mathbf{M}_{BC}\mathbf{M}_{AB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}.$$

Hence, the single transformation is an enlargement, centre the origin, scale factor -3.

24. (5)

$$f(x) = \frac{x}{2x+1},$$

for positive values of x.

Work out

$$f(x+1) - f(x).$$

Give your answer as a fraction in its simplest form.
You **must** show your working.

Solution

$$f(x+1) = \frac{(x+1)}{2(x+1)+1} = \frac{x+1}{2x+3}$$

and

$$f(x+1) - f(x) = \frac{x+1}{2x+3} - \frac{x}{2x+1} = \frac{(x+1)(2x+1)}{(2x+1)(2x+3)} - \frac{x(2x+3)}{(2x+1)(2x+3)}$$

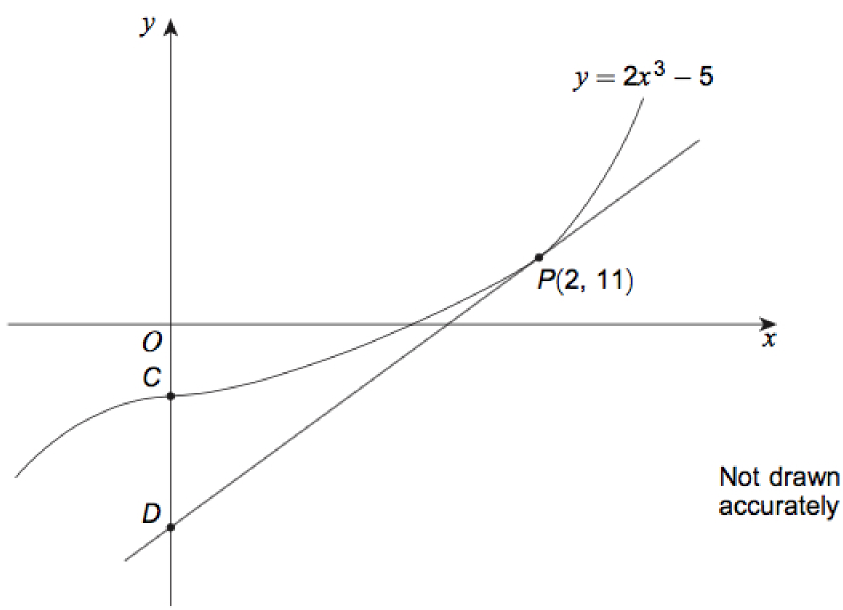
$\times$	x	$+1$
$2x$	$2x^2$	$+2x$
$+1$	$+x$	$+1$

$$= \frac{2x^2+3x+1}{(2x+1)(2x+3)} - \frac{(2x^2+3x)}{(2x+1)(2x+3)}$$
$$= \frac{(2x^2+3x+1)-(2x^2+3x)}{(2x+1)(2x+3)}$$
$$= \frac{1}{(2x+1)(2x+3)}.$$

25. The curve (6)

$$y = 2x^3 - 5$$

intersects the y-axis at C.
The tangent to the curve at $P(2, 11)$ intersects the y-axis at D.



Work out the length CD.

Solution

Well, for the curve,

$$x = 0 \Rightarrow y = -5$$

so $C(0, -5)$. Now,

$$y = 2x^3 - 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6(2^2) = 24.$$

Next, for the line,

$$y - 11 = 24(x - 2)$$

and

$$\begin{aligned} x = 0 &\Rightarrow y - 11 = 24(0 - 2) \\ &\Rightarrow y - 11 = -48 \\ &\Rightarrow y = -37 \end{aligned}$$

so $D(0, -37)$. Finally,

$$CD = 37 - 5 = \underline{\underline{32}}.$$

26. (a) Prove that (2)

$$\sin^2 x - 3\cos^2 x \equiv 4\sin^2 x - 3.$$

Solution

$$\begin{aligned} \sin^2 x - 3\cos^2 x &\equiv \sin^2 x - 3(1 - \sin^2 x) \\ &\equiv \underline{\underline{4\sin^2 x - 3}}, \end{aligned}$$

as required.

(b) Hence, or otherwise, work out the values of x between 0° and 360° for which (4)

$$\sin^2 x - 3\cos^2 x = 0.$$

Solution

$$\begin{aligned}\sin^2 x - 3\cos^2 x = 0 &\Rightarrow 4\sin^2 x - 3 = 0\\ &\Rightarrow 4\sin^2 x = 3\\ &\Rightarrow \sin^2 x = \tfrac{3}{4}\\ &\Rightarrow \sin x = \pm\tfrac{\sqrt{3}}{2}.\end{aligned}$$

$\underline{\sin x = \frac{\sqrt{3}}{2}}$:

$$\sin x = \tfrac{\sqrt{3}}{2} \Rightarrow \underline{\underline{x = 60,\ 120}}.$$

$\underline{\sin x = -\frac{\sqrt{3}}{2}}$:

$$\sin x = -\tfrac{\sqrt{3}}{2} \Rightarrow \underline{\underline{x = 240,\ 300}}.$$