Dr Oliver Mathematics
One or More Successive Impacts

In this note, we will look at one or more successive impacts.

1 One or More Successive Impacts: Right-Angles

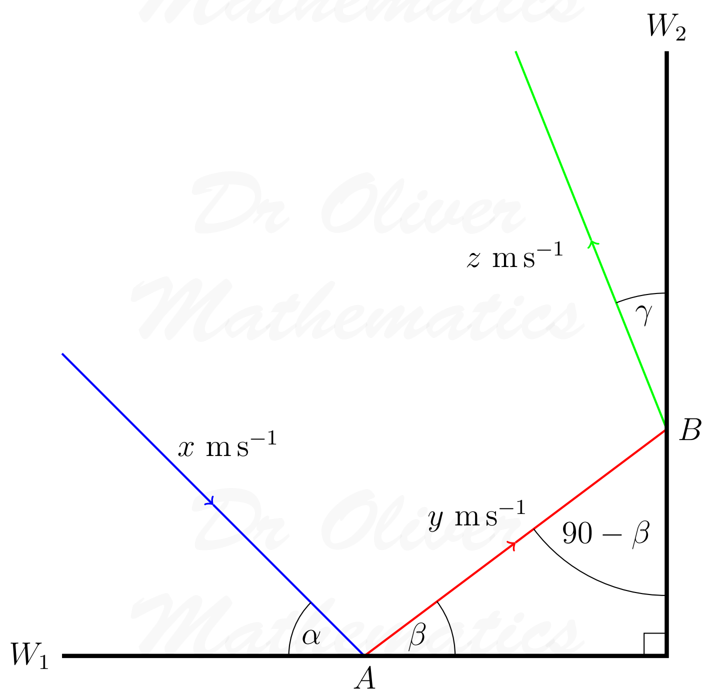
A small ball is projected along the floor towards with speed x ms^{-1} on a path that makes an angle α with W_1. The ball hits the wall and then hits W_2.

Immediately after hitting W_1, the ball is moving at y ms^{-1} and at an angle β to W_1.

Immediately after hitting W_2, the ball is moving at z ms^{-1} and at an angle γ to W_2.

The coefficient of restitution between the ball and W_1 is e_A.

The coefficient of restitution between the ball and W_2 is e_B.



What is the speed and direction after the second bounce?

We fill in the table, first column ...

A	Before	After	B	Before	After
Horizontally	$x\cos\alpha$		Horizontally		
Vertically	$x\sin\alpha$		Vertically		

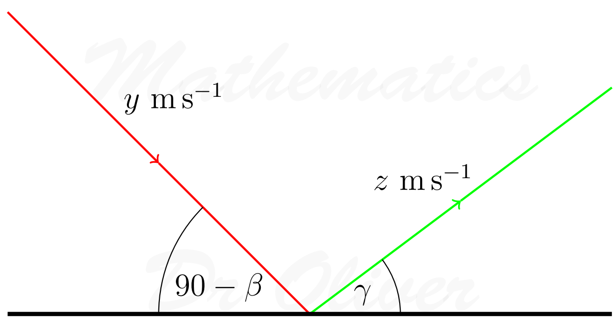
Table 1: completing the first column

…and we fill in the table, second column.

A	Before	After	B	Before	After
Horizontally	$x\cos\alpha$	$x\cos\alpha$	Horizontally		
Vertically	$x\sin\alpha$	$e_A x\sin\alpha$	Vertically		

Table 2: completing the second column

Turn the page a quarter-turn clockwise, so that BW_2 is horizontal, i.e.,



Then, we fill table, third column:

A	Before	After	B	Before	After
Horizontally	$x\cos\alpha$	$x\cos\alpha$	Horizontally	$x\cos(90-\beta)$	
Vertically	$x\sin\alpha$	$e_A x\sin\alpha$	Vertically	$x\sin(90-\beta)$	

But that's just

$$x\cos(90-\beta)=e_A x\sin\alpha$$

and

$$x\sin(90-\beta)=x\cos\alpha$$

because we interchanged the two rows:

A	Before	After	B	Before	After
Horizontally	$x\cos\alpha$	$x\cos\alpha$	Horizontally	$e_A x\sin\alpha$	
Vertically	$x\sin\alpha$	$e_A x\sin\alpha$	Vertically	$x\cos\alpha$	

Table 3: completing the third column

And finally, we fill in the fourth column:

A	Before	After	B	Before	After
Horizontally	$x\cos\alpha$	$x\cos\alpha$	Horizontally	$e_A x\sin\alpha$	$e_A x\sin\alpha$
Vertically	$x\sin\alpha$	$e_A x\sin\alpha$	Vertically	$x\cos\alpha$	$e_B x\cos\alpha$

Table 4: completing the fourth column

Hence,

$$\begin{aligned}\text{speed after 1st bounce} &= \sqrt{(e_A x\sin\alpha)^2 + (x\cos\alpha)^2}\\ &= \boxed{x\sqrt{(e_A\sin\alpha)^2 + (\cos\alpha)^2}}\end{aligned}$$

$$\begin{aligned}\text{direction after 1st bounce} &= \tan^{-1}\left(\frac{e_A\sin\alpha}{\cos\alpha}\right)\\ &= \boxed{\tan^{-1}(e_A\tan\alpha)}\end{aligned}$$

$$\text{kinetic energy after 1st bounce} = \boxed{\tfrac{1}{2}mx^2\left[(e_A\sin\alpha)^2 + (\cos\alpha)^2\right]}$$

and

$$\begin{aligned}\text{speed after 2nd bounce} &= \sqrt{(e_A x\sin\alpha)^2 + (e_B x\cos\alpha)^2}\\ &= \boxed{x\sqrt{(e_A\sin\alpha)^2 + (e_B\cos\alpha)^2}}\end{aligned}$$

$$\begin{aligned}\text{direction after 2nd bounce} &= \tan^{-1}\left(\frac{e_B\cos\alpha}{e_A\sin\alpha}\right)\\ &= \boxed{\tan^{-1}\left(\frac{e_B}{e_A}\cot\alpha\right)}\end{aligned}$$

$$\text{kinetic energy after 2nd bounce} = \boxed{\tfrac{1}{2}mx^2\left[(e_A\sin\alpha)^2 + (e_B\cos\alpha)^2\right].}$$

What happens if $e_A = e_B$, i.e., the coefficient of restitution are the same? Well,

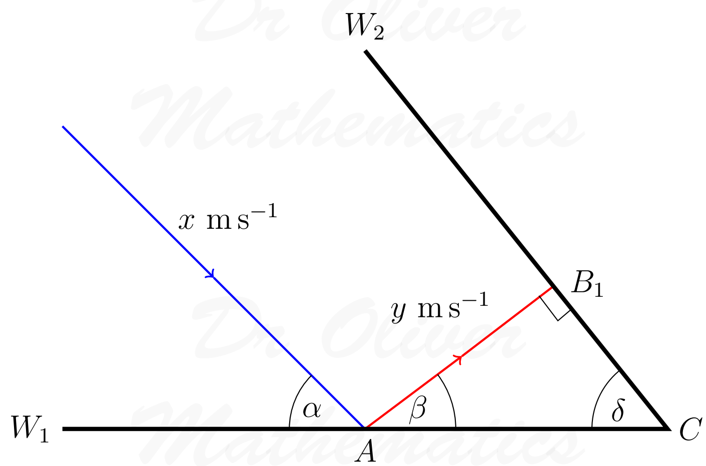
$$\begin{aligned}\tan^{-1}\left(\frac{e_B\cos\alpha}{e_A\sin\alpha}\right) &= \tan^{-1}(\cot\alpha) \\ &= \tan^{-1}(\tan(90-\alpha)) \\ &= 90-\alpha,\end{aligned}$$

and, hence, the path is parallel to the original path but it goes in the opposite direction.

2 One or More Successive Impacts: Non-Right Angles

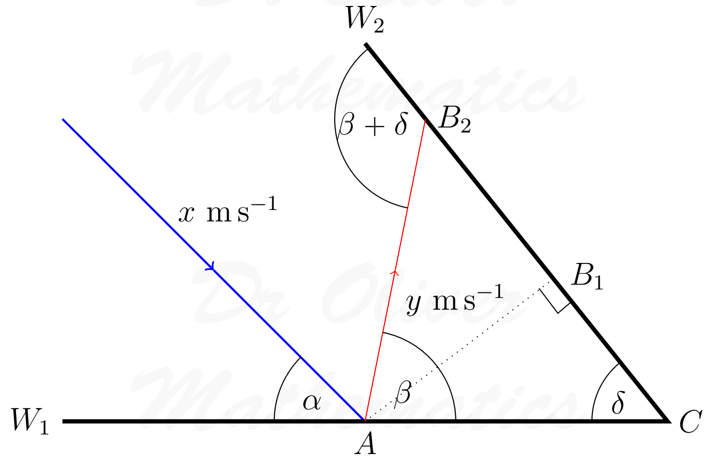
But what if the walls are not at right-angles? Well, it all hangs on the result of $\beta+\delta$.

Case: $\beta + \delta = 90°$:



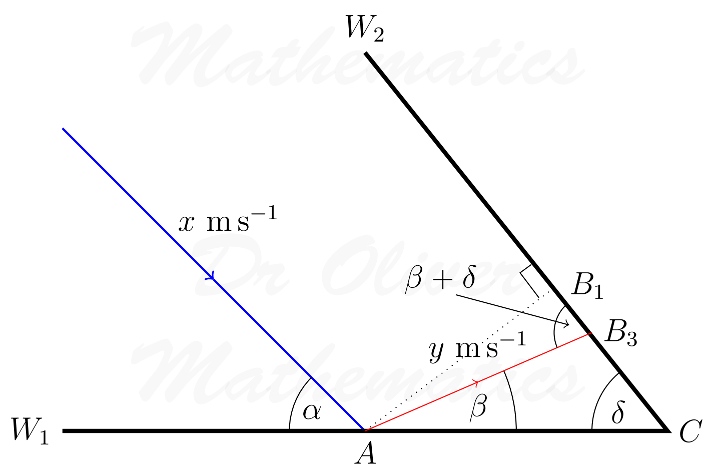
The ball into B_1 and then makes the opposite direction: the speed in reverse is less than than y m s^{-1} and we get a smaller angle (unless $e = 1$!).

Case: $\beta + \delta > 90°$:



We get a third bounce that takes it away from CW_2.

Case: $\beta + \delta < 90°$:

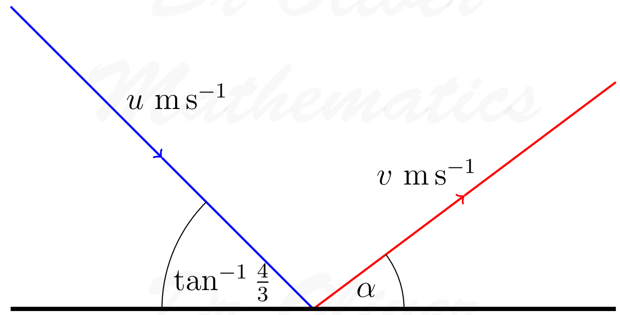


We get a third bounce that takes it to W_1C.

3 Problems

Here are some problems for you to try.

1. A smooth sphere, S, is moving on a smooth horizontal plane with speed u m s^{-1} when it collides with a smooth fixed horizontal plane. At the instant of collision, the direction of motion of S makes an angle of $\tan^{-1} \frac{4}{3}$ with the wall. The coefficient of restitution between S and the wall is $\frac{1}{3}$, as shown below.



Find the speed of S immediately after the collision.

Solution

	Before	After
Horizontally	$\frac{4}{5}u$	$v\cos\alpha$
Vertically	$\frac{3}{5}u$	$\frac{1}{3}v\sin\alpha$

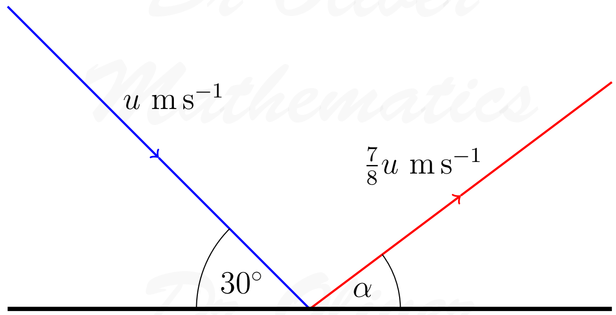
or

$$\leftrightarrow: \quad \tfrac{4}{5}u = v\cos\alpha$$
$$\updownarrow: \quad \tfrac{1}{3} = \frac{v\sin\alpha}{\frac{3}{5}u} \Rightarrow \tfrac{1}{5}u = v\sin\alpha.$$

We square and add them:

$$\begin{aligned} & (v\sin\alpha)^2 + (v\cos\alpha)^2 = (\tfrac{1}{5}u)^2 + (\tfrac{4}{5}u)^2 \\ \Rightarrow \quad & v^2(\sin^2\alpha + \cos^2\alpha) = \tfrac{1}{25}u^2 + \tfrac{16}{25}u^2 \\ \Rightarrow \quad & v^2 = \tfrac{17}{25}u^2 \\ \Rightarrow \quad & \underline{\underline{v = \tfrac{\sqrt{17}}{5}u}}. \end{aligned}$$

2. A smooth sphere, S, is moving on a smooth horizontal plane with speed u m s^{-1} when it collides with a smooth fixed horizontal plane. At the instant of collision, the direction of motion of S makes an angle of $30°$ with the wall. Immediately after the collision, the speed of of S is $\frac{7}{8}u$ m s^{-1}, as shown below.



Find the coefficient of restitution between S and the wall.

Solution

	Before	After
Horizontally	$u\cos 30°$	$\frac{7}{8}u\cos\alpha$
Vertically	$u\sin 30°$	$\frac{7}{8}u\sin\alpha$

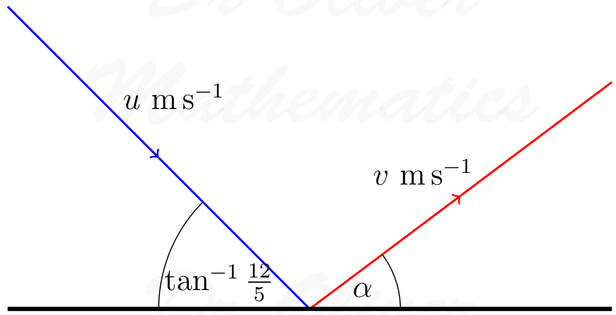
or

$$\leftrightarrow: \quad u\cos 30° = \tfrac{7}{8}u\cos\alpha \Rightarrow \tfrac{\sqrt{3}}{2} = \tfrac{7}{8}\cos\alpha$$
$$\updownarrow: \quad e = \frac{\frac{7}{8}u\sin\alpha}{u\sin 30°} \Rightarrow \tfrac{7}{8}\sin\alpha = \tfrac{1}{2}e.$$

We square and add them:

$$\begin{aligned}
& (\tfrac{7}{8}\sin\alpha)^2 + (\tfrac{7}{8}\cos\alpha)^2 = (\tfrac{\sqrt{3}}{2})^2 + (\tfrac{1}{2}e)^2 \\
\Rightarrow\ & \tfrac{49}{64}(\sin^2\alpha + \cos^2\alpha) = \tfrac{3}{4} + \tfrac{1}{4}e^2 \\
\Rightarrow\ & \tfrac{49}{64} = \tfrac{3}{4} + \tfrac{1}{4}e^2 \\
\Rightarrow\ & \tfrac{49}{64} = \tfrac{3}{4} + \tfrac{1}{4}e^2 \\
\Rightarrow\ & \tfrac{1}{4}e^2 = \tfrac{1}{64} \\
\Rightarrow\ & e^2 = \tfrac{1}{16} \\
\Rightarrow\ & \underline{\underline{e = \tfrac{1}{4}}}.
\end{aligned}$$

3. A smooth sphere, S, is moving on a smooth horizontal plane with speed u m s^{-1} when it collides with a smooth fixed horizontal plane. At the instant of collision, the direction of motion of S makes an angle of $\tan^{-1}\frac{12}{5}$ with the wall. The coefficient of restitution between S and the wall is $\frac{3}{5}$, as shown below.



Find the speed of S immediately after the collision.

Solution

	Before	After
Horizontally	$\frac{12}{13}u$	$v\cos\alpha$
Vertically	$\frac{5}{13}u$	$\frac{3}{5}v\sin\alpha$

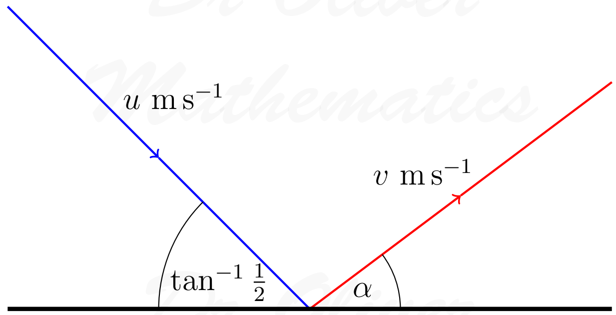
or

$$\leftrightarrow: \quad \tfrac{12}{13}u = v\cos\alpha$$
$$\updownarrow: \quad \tfrac{3}{5} = \frac{v\sin\alpha}{\frac{5}{13}u} \Rightarrow v\sin\alpha = \tfrac{3}{13}u.$$

We square and add them:

$$\begin{aligned} & (v\sin\alpha)^2 + (v\cos\alpha)^2 = (\tfrac{3}{13}u)^2 + (\tfrac{12}{13}u)^2 \\ \Rightarrow\ & v^2(\sin^2\alpha + \cos^2\alpha) = \tfrac{9}{169}u^2 + \tfrac{144}{169}u^2 \\ \Rightarrow\ & v^2 = \tfrac{153}{169}u^2 \\ \Rightarrow\ & \underline{\underline{v = \tfrac{3\sqrt{17}}{13}u}}. \end{aligned}$$

4. A smooth sphere, S, is moving on a smooth horizontal plane with speed u m s^{-1} when it collides with a smooth fixed horizontal plane. At the instant of collision, the direction of motion of S makes an angle of $\tan^{-1}\frac{1}{2}$ with the wall. Immediately after the collision, the speed of of S is $\frac{3}{4}u$ m s^{-1}, as shown below.



Find the coefficient of restitution between S and the wall.

Solution

	Before	After
Horizontally	$\frac{1}{\sqrt{5}}u$	$\frac{3}{4}u\cos\alpha$
Vertically	$\frac{2}{\sqrt{5}}u$	$\frac{3}{4}u\sin\alpha$

or

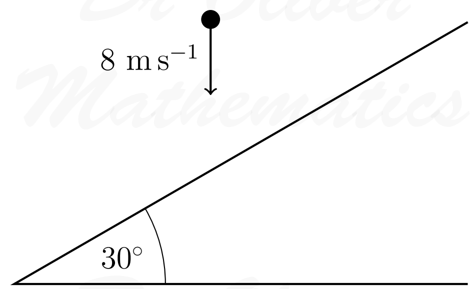
$$\leftrightarrow: \quad \tfrac{1}{\sqrt{5}}u = \tfrac{3}{4}u\cos\alpha$$

$$\updownarrow: \quad e = \frac{\frac{3}{4}u\sin\alpha}{\frac{2}{\sqrt{5}}u} \Rightarrow \tfrac{3}{4}u\sin\alpha = \tfrac{2}{\sqrt{5}}eu.$$

We square and add them:

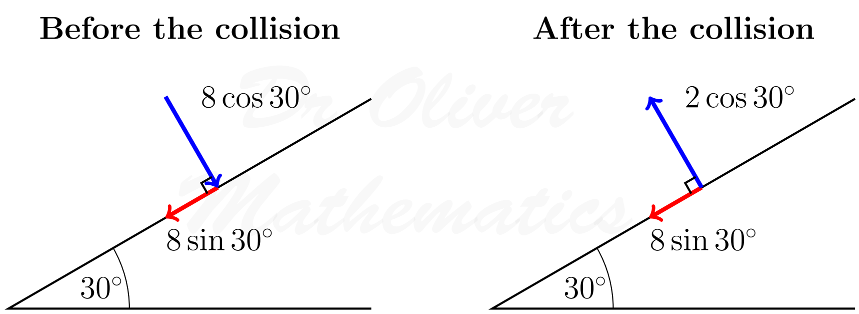
$$\begin{aligned} & (\tfrac{3}{4}u\sin\alpha)^2 + (\tfrac{3}{4}u\cos\alpha)^2 = (\tfrac{1}{\sqrt{5}}u)^2 + (\tfrac{2}{\sqrt{5}}eu)^2 \\ \Rightarrow\ & \tfrac{9}{16}(\sin^2\alpha + \cos^2\alpha) = \tfrac{1}{5} + \tfrac{4}{5}e^2 \\ \Rightarrow\ & \tfrac{9}{16} = \tfrac{1}{5} + \tfrac{4}{5}e^2 \\ \Rightarrow\ & \tfrac{29}{80} = \tfrac{4}{5}e^2 \\ \Rightarrow\ & e^2 = \tfrac{29}{64} \\ \Rightarrow\ & \underline{\underline{e = \tfrac{\sqrt{29}}{8}}}. \end{aligned}$$

5. A small smooth ball is falling vertically. The ball strikes a smooth plane, which is inclined at an angle 30° to the horizontal. Immediately before striking the plane, the ball has a speed of 8 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{4}$, as shown below.



Find the exact value of the speed of the ball immediately after the impact.

Solution



	Before	After
Parallel	$8\sin 30^\circ$	$8\sin 30^\circ$
Perpendicular	$8\cos 30^\circ$	$2\cos 30^\circ$

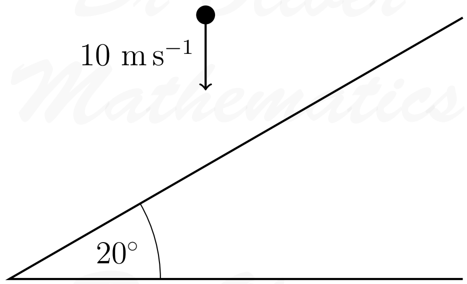
Now,

$$8\sin 30^\circ = 4 \text{ and } 2\cos 30^\circ = \sqrt{3}.$$

Hence, the speed of the ball immediately after the impact is

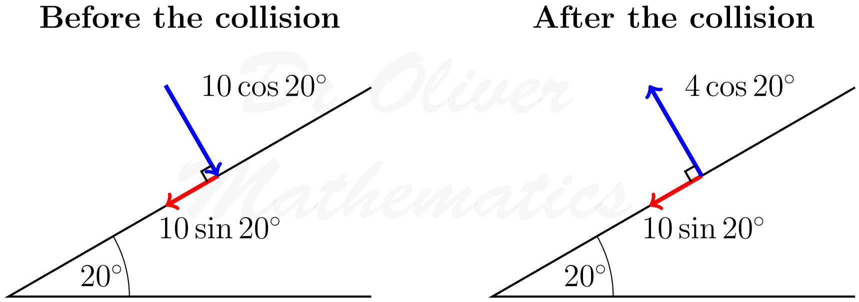
$$\begin{aligned}\sqrt{4^2 + (\sqrt{3})^2} &= \sqrt{16+3} \\ &= \underline{\underline{\sqrt{19} \text{ m s}^{-1}}}.\end{aligned}$$

6. A small smooth ball is falling vertically. The ball strikes a smooth plane, which is inclined at an angle 20° to the horizontal. Immediately before striking the plane, the ball has a speed of 10 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{2}{5}$, as shown below.



Find the speed, to 3 significant figures, of the ball immediately after the impact.

Solution

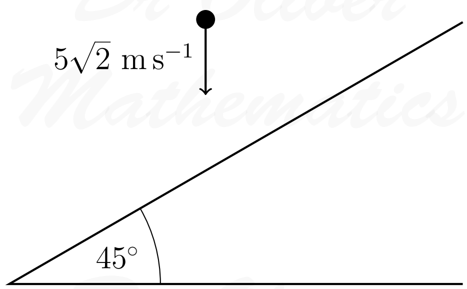


	Before	After
Parallel	$10\sin 20^\circ$	$10\sin 20^\circ$
Perpendicular	$10\cos 20^\circ$	$4\cos 20^\circ$

Hence, the speed of the ball immediately after the impact is

$$\begin{aligned}\sqrt{(10\sin 20^\circ)^2 + (4\cos 20^\circ)^2} &= 5.081\,941\,892 \text{ (FCD)}\\ &= \underline{\underline{5.08 \text{ m s}^{-1} \text{ (3 sf)}}}.\end{aligned}$$

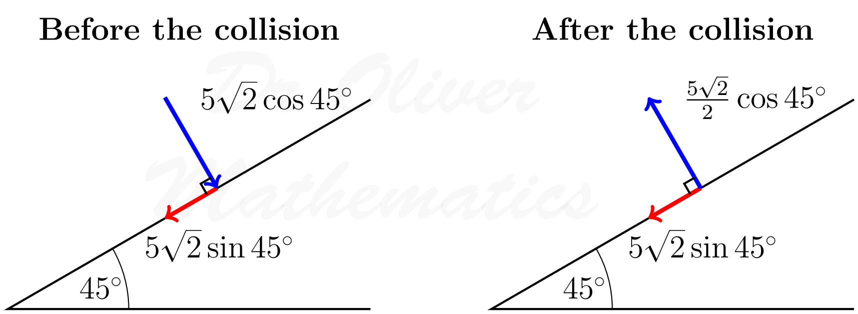
7. A small smooth ball of mass 750 g is falling vertically. The ball strikes a smooth plane, which is inclined at an angle 45° to the horizontal. Immediately before striking the plane, the ball has a speed of $5\sqrt{2}$ m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$, as shown below.



Find

(a) the speed, to 3 significant figures, of the ball immediately after the impact,

Solution



	Before	After
Parallel	$5\sqrt{2}\sin 45°$	$5\sqrt{2}\sin 45°$
Perpendicular	$5\sqrt{2}\cos 45°$	$\frac{5\sqrt{2}}{2}\cos 45°$

Now,

$$5\sqrt{2}\sin 45° = 5 \text{ and } \tfrac{5\sqrt{2}}{2}\cos 45° = 2\tfrac{1}{2}.$$

Hence, the speed of the ball immediately after the impact is

$$\begin{aligned}\sqrt{5^2 + (2\tfrac{1}{2})^2} &= 5.590\,169\,944 \text{ (FCD)} \\ &= \underline{\underline{5.59 \text{ m s}^{-1} \text{ (3 sf)}}}.\end{aligned}$$

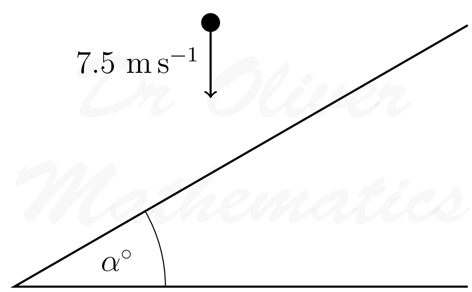
(b) the magnitude of the impulse received by the ball as it strikes the plane.

Solution

The impulse is perpendicular to the surface:

$$\begin{aligned} I &= 0.75\left[2\tfrac{1}{2} - (-5)\right] \\ &= \underline{\underline{5\tfrac{5}{8} \text{ N s}}}. \end{aligned}$$

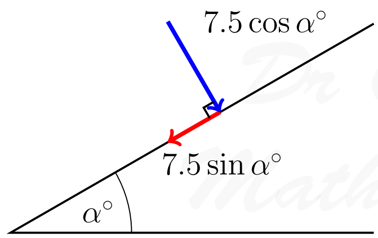
8. A small smooth ball is falling vertically. The ball strikes a smooth plane, which is inclined at an angle α° to the horizontal, where $\tan^{-1}\alpha = \frac{3}{4}$. Immediately before striking the plane, the ball has a speed of 7.5 m s^{-1}. Immediately after striking the plane, the ball has a speed of 5 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{2}{5}$, as shown below.



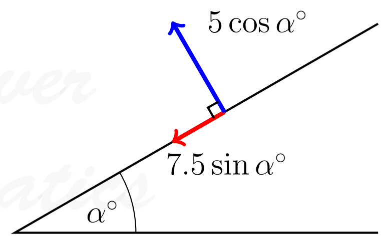
Find the coefficient of restitution, to 2 significant figures, between the ball and the plane.

Solution

Before the collision



After the collision



Now,

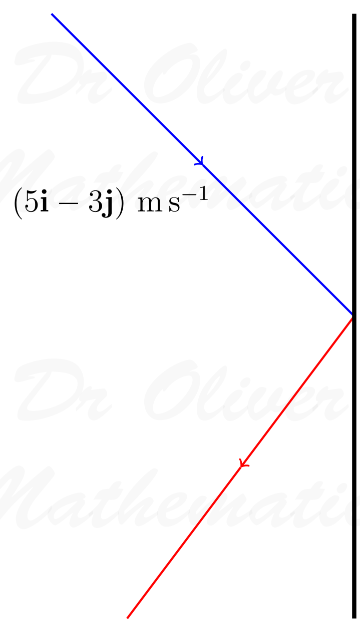
$$7.5 \sin\alpha^\circ = 4.5 \text{ and } 7.5\cos\alpha^\circ = 6.$$

	Before	After
Parallel	4.5	4.5
Perpendicular	6	$6e$

Combining the 'After' equations:

$$\begin{aligned} 4.5^2 + (6e)^2 = 5^2 &\Rightarrow 20.25 + 36e^2 = 25 \\ &\Rightarrow 36e^2 = 4.75 \\ &\Rightarrow e^2 = \tfrac{19}{144} \\ &\Rightarrow e = 0.363\,241\,578\,6 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{e = 0.36 \text{ (2 sf)}}}. \end{aligned}$$

9. A small smooth ball of mass 800 g is moving in the (x, y)-plane and collides with a smooth fixed vertical wall which contains the y-axis. The velocity of the ball just before impact is $(5\mathbf{i} - 3\mathbf{j})$ m s^{-1}. The coefficient of restitution between the sphere of the wall is $\frac{1}{2}$, as shown below.



Find

(a) the velocity of the ball immediately after the impact,

Solution

Suppose that the velocity of the ball immediately after the impact is

$$\mathbf{v} = (p\mathbf{i} + q\mathbf{j}).$$

	Before	After
Horizontally	5	$-\frac{1}{2} \times 5 = -2\frac{1}{2}$
Vertically	-3	-3

Hence,

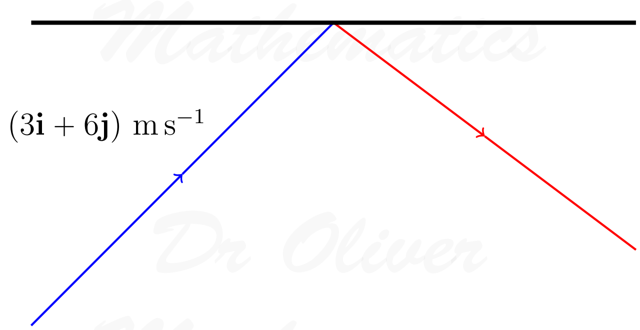
$$\mathbf{v} = \underline{\underline{(-2\tfrac{1}{2}\mathbf{i} - 3\mathbf{j})\ \mathrm{m\,s^{-1}}}}.$$

(b) the kinetic energy lost as a result of the impact.

Solution

$$\begin{aligned}
\text{Initial KE} &= \tfrac{1}{2} \times 0.8 \times [5^2 + (-3)^2] \\
&= 13.6 \text{ J} \\
\text{Final KE} &= \tfrac{1}{2} \times 0.8 \times [(-2\tfrac{1}{2})^2 + (-3)^2] \\
&= 6.1 \text{ J} \\
\text{KE lost} &= 13.6 - 6.1 \\
&= \underline{\underline{7.5 \text{ J}}}.
\end{aligned}$$

10. A small smooth ball of mass 1 kg is moving in the (x, y)-plane and collides with a smooth fixed vertical wall which contains the x-axis. The velocity of the ball just before impact is $(3\mathbf{i} + 6\mathbf{j})\ \mathrm{m\,s^{-1}}$. The coefficient of restitution between the sphere of the wall is $\frac{1}{3}$, as shown below.



Find

(a) the speed of the ball immediately after the impact,

Solution

Suppose that the velocity of the ball immediately after the impact is

$$\mathbf{v} = (p\mathbf{i} + q\mathbf{j}).$$

	Before	After
Horizontally	3	3
Vertically	6	$\frac{1}{3} \times (-6) = -2$

Hence,

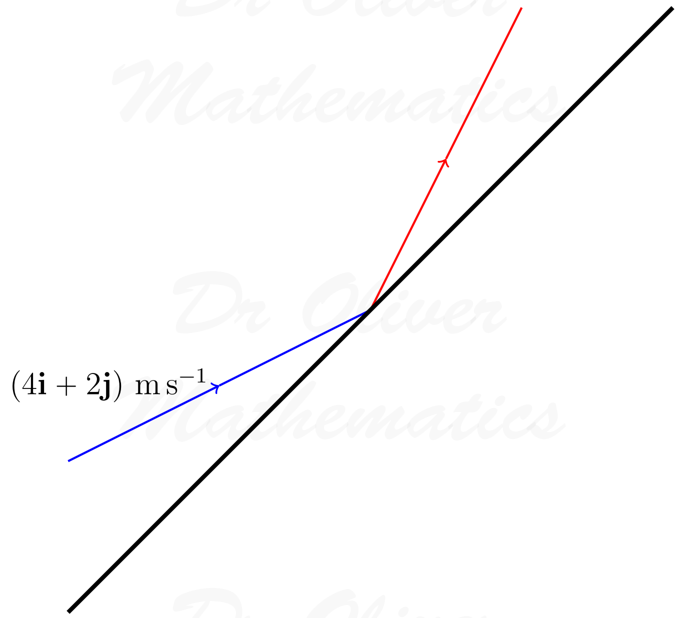
$$\begin{aligned} \text{speed} &= \sqrt{3^2 + (-2)^2} \\ &= \underline{\underline{\sqrt{13} \text{ m s}^{-1}}}. \end{aligned}$$

(b) the kinetic energy lost as a result of the impact.

Solution

$$\begin{aligned} \text{Initial KE} &= \tfrac{1}{2} \times 1 \times [3^2 + 6^2] \\ &= 22.5 \text{ J} \\ \text{Final KE} &= \tfrac{1}{2} \times 1 \times [3^2 + (-2)^2] \\ &= 6.5 \text{ J} \\ \text{KE lost} &= 22.5 - 6.5 \\ &= \underline{\underline{16 \text{ J}}}. \end{aligned}$$

11. A small smooth ball of mass 2 kg is moving in the (x, y)-plane and collides with a smooth fixed vertical wall which contains the line $y = x$. The velocity of the ball just before impact is $(4\mathbf{i} + 2\mathbf{j})$ m s^{-1}. The coefficient of restitution between the sphere of the wall is $\frac{1}{3}$, as shown below.



Find

(a) the velocity of the ball immediately after the impact,

Solution

Suppose that the velocity of the ball immediately after the impact is

$$\mathbf{v} = \mathbf{a} + \mathbf{b},$$

where $\mathbf{a}$ is parallel to the wall and $\mathbf{b}$ is perpendicular to the wall. In particular,

$$\tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \text{ and } \tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$$

are unit vectors parallel and perpendicular to the wall respectively.

Parallel:

$$\begin{aligned} \left[(4\mathbf{i} + 2\mathbf{j}).\tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})\right] \tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) &= \left[\tfrac{1}{\sqrt{2}}(4 + 2)\right] \tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \\ &= \tfrac{1}{2}(6)(\mathbf{i} + \mathbf{j}) \\ &= 3\mathbf{i} + 3\mathbf{j}. \end{aligned}$$

Perpendicular:

$$\begin{aligned} \tfrac{1}{3}\left[(4\mathbf{i} + 2\mathbf{j}).\tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})\right] \tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) &= \tfrac{1}{3}\left[\tfrac{1}{\sqrt{2}}(-4 + 2)\right] \tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \\ &= \tfrac{1}{6}(-2)(-\mathbf{i} + \mathbf{j}) \\ &= \tfrac{1}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j}. \end{aligned}$$

Hence,

$$\begin{aligned}\text{velocity} &= (3\mathbf{i} + 3\mathbf{j}) - (\tfrac{1}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j}) \\ &= \underline{\underline{\tfrac{8}{3}\mathbf{i} + \tfrac{10}{3}\mathbf{j}\ \mathrm{m\,s^{-1}}}}.\end{aligned}$$

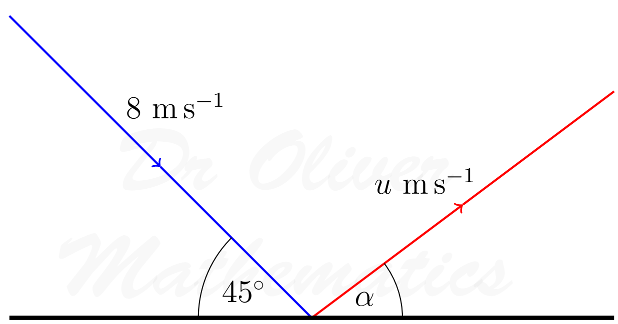
(b) the kinetic energy lost as a result of the impact.

Solution

$$\begin{aligned}\text{Initial KE} &= \tfrac{1}{2} \times 2 \times [4^2 + 2^2] \\ &= 20 \text{ J} \\ \text{Final KE} &= \tfrac{1}{2} \times 2 \times [(\tfrac{8}{3})^2 + (-\tfrac{10}{3})^2] \\ &= 18\tfrac{2}{9} \text{ J} \\ \text{KE lost} &= 20 - 18\tfrac{2}{9} \\ &= \underline{\underline{1\tfrac{7}{9}\text{ J}}}.\end{aligned}$$

12. A smooth snooker ball strikes a smooth cushion with a speed of 8 $\mathrm{m\,s^{-1}}$ at an angle of $45°$ to the cushion. Given that the coefficient of restitution between the sphere of the wall is $\frac{2}{5}$, find the direction and magnitude of the velocity of the ball after the impact.

Solution



	Before	After
Horizontally	$8\cos 45^\circ$	$8\cos 45^\circ$
Vertically	$8\sin 45^\circ$	$\frac{2}{5}\times 8\sin 45^\circ = \frac{16}{5}\sin 45^\circ$

$$\begin{aligned}\text{Magnitude} &= \sqrt{(8\cos 45^\circ)^2 + (\tfrac{16}{5}\sin 45^\circ)^2}\\ &= \underline{\underline{\tfrac{4}{5}\sqrt{58} \text{ or } 6.09 \text{ m s}^{-1} \text{ (3 sf)}}}\end{aligned}$$

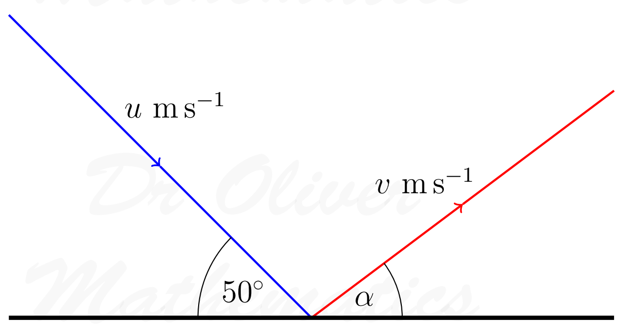
and

$$\begin{aligned}\text{direction} &= \tan^{-1}\left(\frac{\frac{16}{5}\sin 45^\circ}{8\cos 45^\circ}\right)\\ &= \tan^{-1}\tfrac{2}{5}\\ &= 21.801\,409\,49 \text{ (FCD)}\\ &= \underline{\underline{21.8^\circ \text{ (3 sf)}}}.\end{aligned}$$

13. A smooth snooker ball strikes a smooth cushion with a speed of u m s^{-1} at an angle of 50° to the cushion. The coefficient of restitution between the sphere of the wall is e.

(a) Show that the angle between the cushion and the rebound direction is independent of u.

Solution



	Before	After
Horizontally	$u\cos 50^\circ$	$v\cos\alpha^\circ$
Vertically	$u\sin 50^\circ$	$ev\sin\alpha^\circ$

$$\begin{aligned}\tan\alpha^\circ &= \frac{v\sin\alpha^\circ}{v\cos\alpha^\circ}\\ &= \frac{eu\sin 50^\circ}{u\cos 50^\circ}\\ &= \underline{\underline{e\tan 50^\circ}},\end{aligned}$$

which is independent of u.

(b) Find the value of e given that the ball rebounds at right angles to its original direction.

Solution

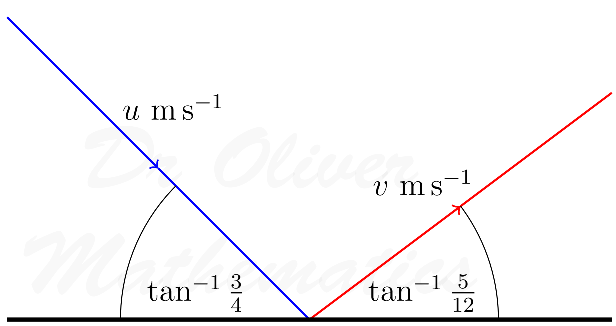
If $\alpha^\circ = 40^\circ$, then

$$\begin{aligned}\tan 40^\circ = e\tan 50^\circ &\Rightarrow e = \frac{\tan 40^\circ}{\tan 50^\circ}\\ &\Rightarrow e = 0.704\,088\,191 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{e = 0.70 \text{ (2 sf)}}}.\end{aligned}$$

14. A smooth snooker ball strikes a smooth cushion at an angle of $\tan^{-1}\frac{3}{4}$to the cushion. The ball rebounds at an angle of $\tan^{-1}\frac{5}{12}$to the cushion. Find

(a) the fraction of the kinetic energy of the ball lost in the collision,

Solution



	Before	After
Horizontally	$\frac{4}{5}u$	$\frac{12}{13}v$
Vertically	$\frac{3}{5}u$	$\frac{5}{13}ev$

Horizontally,

$$\begin{aligned} \tfrac{4}{5}u = \tfrac{12}{13}v &\Rightarrow v = \tfrac{4}{5}u \times \tfrac{13}{12} \\ &\Rightarrow v = \tfrac{13}{15}u \end{aligned}$$

and the kinetic energy lost is

$$\begin{aligned} \tfrac{1}{2}mu^2 - \tfrac{1}{2}m(\tfrac{13}{15}u)^2 &= \tfrac{1}{2}mu^2 - \tfrac{1}{2}m(\tfrac{169}{225}u^2) \\ &= \tfrac{1}{2}m(\tfrac{56}{225}u^2). \end{aligned}$$

Hence, the fraction of the kinetic energy of the ball lost in the collision is

$$\frac{\tfrac{1}{2}mu(\tfrac{56}{225}u^2)}{\tfrac{1}{2}mu^2} = \underline{\underline{\tfrac{56}{225}}}.$$

(b) the coefficient of restitution between the ball and the wall.

Solution

$$\begin{aligned} e &= \frac{\tfrac{5}{13}v}{\tfrac{3}{5}u} \\ &= \frac{\tfrac{1}{3}u}{\tfrac{3}{5}u} \\ &= \underline{\underline{\tfrac{5}{9}}}u. \end{aligned}$$

15. A small smooth sphere of mass m kg is moving velocity $(5\mathbf{i} - 2\mathbf{j})$ m s^{-1} when it hits a smooth wall. It rebounds from the wall with $(2\mathbf{i} + 2\mathbf{j})$ m s^{-1}. Find

(a) the magnitude and direction of the impulse received by the sphere,

Solution

$$\begin{aligned} \mathbf{I} &= m(\mathbf{v} - \mathbf{u}) \\ &= m\left[(2\mathbf{i} + 2\mathbf{j}) - (5\mathbf{i} - 2\mathbf{j})\right] \\ &= m(-3\mathbf{i} + 4\mathbf{j}); \end{aligned}$$

hence, the magnitude is

$$\begin{aligned}|\mathbf{I}| &= m\sqrt{(-3)^2 + 4^2} \\ &= \underline{\underline{5m \text{ N s}}}\end{aligned}$$

and the direction is $\underline{\underline{\text{parallel}}}$ to the unit vector $\underline{\underline{\frac{1}{5}(-3\mathbf{i} + 4\mathbf{j})}}$.

(b) the coefficient of restitution between the sphere and the wall.

Solution

Component of $(5\mathbf{i} - 2\mathbf{j})$:

$$\begin{aligned}\left[(5\mathbf{i} - 2\mathbf{j}).\tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j})\right]\tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) &= \left[\tfrac{1}{5}(-15 - 8)\right]\tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) \\ &= -\tfrac{23}{5} \cdot \tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j}).\end{aligned}$$

Component of $(2\mathbf{i} + 2\mathbf{j})$:

$$\begin{aligned}\left[(2\mathbf{i} + 2\mathbf{j}).\tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j})\right]\tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) &= \left[\tfrac{1}{5}(-6 + 8)\right]\tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j}) \\ &= \tfrac{2}{5} \cdot \tfrac{1}{5}(-3\mathbf{i} + 4\mathbf{j}).\end{aligned}$$

Hence,

$$\begin{aligned}\tfrac{2}{5} = \tfrac{23}{5}e \Rightarrow e &= \frac{\frac{2}{5}}{\frac{23}{5}} \\ \Rightarrow \underline{\underline{e = \tfrac{2}{23}}}.\end{aligned}$$

16. A small smooth sphere of mass 2 kg is moving velocity $(2\mathbf{i} + 3\mathbf{j})$ m s^{-1} when it hits a smooth wall. It rebounds from the wall with $(3\mathbf{i} - \mathbf{j})$ m s^{-1}. Find

(a) the magnitude and direction of the impulse received by the sphere,

Solution

$$\begin{aligned}\mathbf{I} &= m(\mathbf{v} - \mathbf{u}) \\ &= 2\left[(3\mathbf{i} - \mathbf{j}) - (2\mathbf{i} + 3\mathbf{j})\right] \\ &= 2(\mathbf{i} - 4\mathbf{j});\end{aligned}$$

hence, the magnitude is

$$\begin{aligned}|\mathbf{I}| &= 2\sqrt{(1^2 + (-4)^2} \\ &= \underline{\underline{2\sqrt{17}\text{ N s}}}\end{aligned}$$

and the direction is <u>parallel</u> to the unit vector $\underline{\underline{\frac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j})}}$.

(b) the coefficient of restitution between the sphere and the wall,

Solution

<u>Component of $(2\mathbf{i} + 3\mathbf{j})$</u>:

$$\begin{aligned}(2\mathbf{i} + 3\mathbf{j}).\tfrac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) &= \tfrac{1}{\sqrt{17}}(2 - 12) \\ &= -\tfrac{10}{\sqrt{17}}.\end{aligned}$$

<u>Component of $(3\mathbf{i} - \mathbf{j})$</u>:

$$\begin{aligned}(3\mathbf{i} - \mathbf{j}).\tfrac{1}{\sqrt{17}}(\mathbf{i} - 4\mathbf{j}) &= \tfrac{1}{\sqrt{17}}(3 + 4) \\ &= \tfrac{7}{\sqrt{17}}.\end{aligned}$$

Hence,

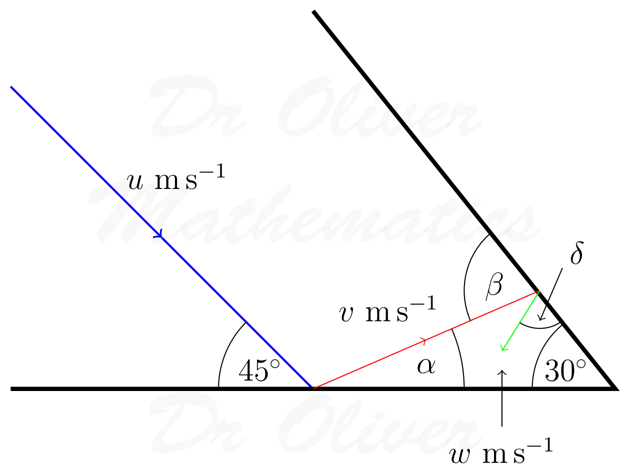
$$\begin{aligned}\tfrac{7}{\sqrt{17}} = \tfrac{10}{\sqrt{17}}e \Rightarrow e &= \frac{\frac{7}{\sqrt{17}}}{\frac{10}{\sqrt{17}}} \\ \Rightarrow \underline{\underline{e = \tfrac{7}{10}}}.\end{aligned}$$

(c) the kinetic energy lost by the sphere in the collision.

Solution

$$\begin{aligned}\text{Initial KE} &= \tfrac{1}{2} \times 2 \times [2^2 + 3^2] \\ &= 13\text{ J} \\ \text{Final KE} &= \tfrac{1}{2} \times 2 \times [3^2 + (-1)^2] \\ &= 10\text{ J} \\ \text{KE lost} &= 13 - 10\tfrac{2}{9} \\ &= \underline{\underline{3\text{ J}}}.\end{aligned}$$

17. Two smooth vertical wall stand on a smooth horizontal floor and intersect an at angle of 30°. A particle is projected along the floor with a speed of $u\ \mathrm{m\,s^{-1}}$ at 45° to one the walls and towards the intersections of the walls. The coefficient of restitution between the particle and the each wall is $\frac{1}{\sqrt{3}}$, as shown below.



Find the speed of the particle after one impact with each wall.

Solution

1st bounce	Before	After
Horizontally	$u\cos 45^\circ = \frac{1}{\sqrt{2}}u$	$v\cos\alpha$
Vertically	$u\sin 45^\circ = \frac{1}{\sqrt{2}}u$	$\frac{1}{\sqrt{3}}v\sin\alpha$

or

$$\leftrightarrow: \quad \tfrac{1}{\sqrt{2}}u = v\cos\alpha$$
$$\updownarrow: \quad \tfrac{1}{\sqrt{3}} = \frac{v\sin\alpha}{\frac{1}{\sqrt{2}}u} \Rightarrow \tfrac{1}{\sqrt{2}\cdot\sqrt{3}}u = v\sin\alpha.$$

Now,

$$\tan\alpha = \frac{v\sin\alpha}{v\cos\alpha} \Rightarrow \tan\alpha = \frac{\frac{1}{\sqrt{2}\cdot\sqrt{3}}u}{\frac{1}{\sqrt{2}}u}$$
$$\Rightarrow \tan\alpha = \tfrac{1}{\sqrt{3}}u$$
$$\Rightarrow \alpha = 30^\circ$$

and

$$\begin{aligned} v^2 &= (v\sin\alpha)^2 + (v\cos\alpha)^2 \\ &= (\tfrac{1}{\sqrt{6}}u)^2 + (\tfrac{1}{\sqrt{2}}u)^2 \\ &= \tfrac{1}{6}u^2 + \tfrac{1}{2}u^2 \\ &= \tfrac{2}{3}u^2. \end{aligned}$$

Next, the exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles:

$$\begin{aligned} \beta &= \alpha + 30 \\ &= 30 + 30 \\ &= 60^\circ, \end{aligned}$$

and we can get the 2nd bounce.

2nd bounce	Before	After
Horizontally	$v\cos 60^\circ = \frac{\sqrt{6}}{6}u$	$w\cos\delta$
Vertically	$v\sin 60^\circ = \frac{\sqrt{2}}{2}u$	$\frac{1}{\sqrt{3}}w\sin\delta$

or

$$\begin{aligned} &\leftrightarrow: \quad \tfrac{\sqrt{6}}{6}u = w\cos\delta \\ &\updownarrow: \quad \tfrac{1}{\sqrt{3}} = \frac{w\sin\delta}{\frac{\sqrt{2}}{2}u} \Rightarrow \tfrac{\sqrt{6}}{6}u = w\sin\delta. \end{aligned}$$

Finally,

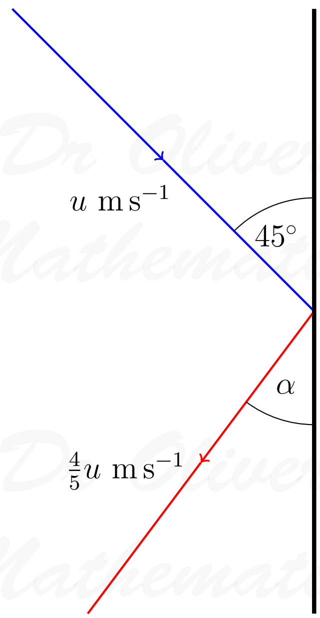
$$\begin{aligned} w^2 &= (w\sin\delta)^2 + (w\cos\delta)^2 \\ &= (\tfrac{\sqrt{6}}{6}u)^2 + (\tfrac{\sqrt{6}}{6}u)^2 \\ &= \tfrac{1}{6}u^2 + \tfrac{1}{6}u^2 \\ &= \tfrac{1}{3}u^2 \end{aligned}$$

and, hence,

$$\underline{\underline{w = \tfrac{\sqrt{3}}{3}u.}}$$

18. A smooth sphere, S, is moving on a smooth horizontal plane with speed u m s^{-1} when

it collides with a smooth fixed vertical wall. At the instant of collision, the direction of motion of S makes an angle of 45° with the wall. Immediately after the collision, the speed of S $\frac{4}{5}u$ m s^{-1} as shown below.



Find the coefficient of restitution between S and the wall.

Solution

	Before	After
Horizontally	$u \sin 45^\circ = \frac{1}{\sqrt{2}}u$	$\frac{4}{5}u \sin \alpha$
Vertically	$u \cos 45^\circ = \frac{1}{\sqrt{2}}u$	$\frac{4}{5}eu \cos \alpha$

or

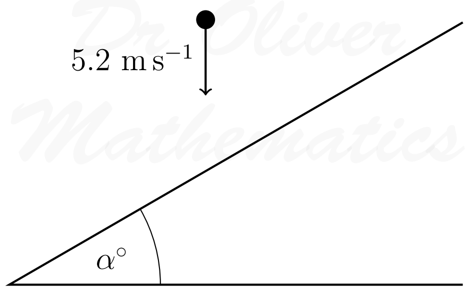
$$\leftrightarrow: \quad \frac{1}{\sqrt{2}}u = \frac{4}{5}u \sin \alpha$$

$$\updownarrow: \quad e = \frac{\frac{4}{5}u \cos \alpha}{\frac{1}{\sqrt{2}}u} \Rightarrow \frac{1}{\sqrt{2}}eu = \frac{4}{5}u \cos \alpha.$$

We square and add them:

$$\left(\tfrac{4}{5}u\right)^2 = \left(\tfrac{1}{\sqrt{2}}u\right)^2 + \left(\tfrac{1}{\sqrt{2}}eu\right)^2 \Rightarrow \tfrac{16}{25} = \tfrac{1}{2} + \tfrac{1}{2}e^2$$
$$\Rightarrow \tfrac{1}{2}e^2 = \tfrac{7}{50}$$
$$\Rightarrow e^2 = \tfrac{7}{25}$$
$$\Rightarrow \underline{\underline{e = \tfrac{\sqrt{7}}{5}}}.$$

19. A small smooth ball of mass $\frac{1}{2}$ kg is falling vertically. The ball strikes a smooth plane, which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. Immediately before striking the plane, the ball has a speed of 5.2 m s^{-1}. The coefficient of restitution between the ball and the plane is $\frac{1}{4}$, as shown below.

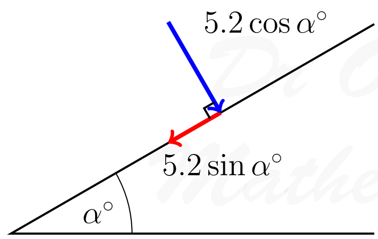


Find

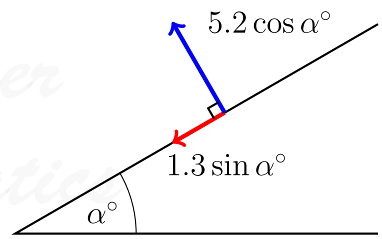
(a) the speed, to 3 significant figures, of the ball immediately after the impact,

Solution

Before the collision



After the collision



	Before	After
Parallel	$5.2\sin\alpha^\circ$	$5.2\sin\alpha^\circ$
Perpendicular	$5.2\cos\alpha^\circ$	$1.3\cos\alpha^\circ$

Hence, the speed of the ball immediately after the impact is

$$\begin{aligned}\sqrt{(5.2\sin\alpha^\circ)^2 + (1.3\cos\alpha^\circ)^2} &= \sqrt{2^2 + (1.2)^2} \\ &= 2.332\,380\,758 \text{ (FCD)} \\ &= \underline{\underline{2.33 \text{ m s}^{-1} \text{ (3 sf)}}}.\end{aligned}$$

(b) the magnitude of the impact received by the ball as it strikes the plane.

Solution

$$\begin{aligned}|\mathbf{I}| &= \tfrac{1}{2}[1.2 - (-4.8)] \\ &= \underline{\underline{3 \text{ N s}}}.\end{aligned}$$

20. A small smooth ball of mass 500 g is moving in the (x, y)-plane and collides with a smooth fixed vertical wall which contains the line $x + y = 3$. The velocity of the ball just before impact is $(-4\mathbf{i} - 2\mathbf{j})$ m s^{-1}. The coefficient of restitution between the sphere of the wall is $\frac{1}{2}$.Find

(a) the velocity of the ball immediately after the impact,

Solution

Suppose that the velocity of the ball immediately after the impact is

$$\mathbf{v} = \mathbf{a} + \mathbf{b},$$

where **a** is parallel to the wall and **b** is perpendicular to the wall. In particular,

$$\tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \text{ and } \tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$$

are unit vectors parallel and perpendicular to the wall respectively.

Parallel:

$$\begin{aligned}\left[(-4\mathbf{i} - 2\mathbf{j}).\tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})\right]\tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) &= \left[\tfrac{1}{\sqrt{2}}(4 - 2)\right]\tfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) \\ &= \tfrac{1}{2}(2)(-\mathbf{i} + \mathbf{j}) \\ &= -\mathbf{i} + \mathbf{j}.\end{aligned}$$

Perpendicular:

$$\tfrac{1}{2}\left[(-4\mathbf{i}-2\mathbf{j}).\tfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})\right]\tfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \tfrac{1}{2}\left[\tfrac{1}{\sqrt{2}}(-4-2)\right]\tfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$$
$$= \tfrac{1}{4}(-6)(\mathbf{i}+\mathbf{j})$$
$$= -\tfrac{3}{2}\mathbf{i}-\tfrac{3}{2}\mathbf{j}.$$

Hence,

$$\text{velocity} = (-\mathbf{i}+\mathbf{j}) - (-\tfrac{3}{2}\mathbf{i}-\tfrac{3}{2}\mathbf{j})$$
$$= \underline{\underline{\tfrac{1}{2}\mathbf{i}+\tfrac{5}{2}\mathbf{j}\ \mathrm{m\,s^{-1}}}}.$$

(b) the kinetic energy lost as a result of the impact.

Solution

$$\text{Initial KE} = \tfrac{1}{2}\times 0.5\times[(-4)^2+(-2)^2]$$
$$= 5\ \mathrm{J}$$
$$\text{Final KE} = \tfrac{1}{2}\times 0.5\times[(\tfrac{1}{2})^2+(\tfrac{5}{2})^2]$$
$$= 1.625\ \mathrm{J}$$
$$\text{KE lost} = 5-1.625$$
$$= \underline{\underline{3.375\ \mathrm{J}}}.$$

21. A small smooth sphere of mass m kg is moving velocity $(6\mathbf{i}+3\mathbf{j})\ \mathrm{m\,s^{-1}}$ when it hits a smooth wall. It rebounds from the wall with $(2\mathbf{i}-2\mathbf{j})\ \mathrm{m\,s^{-1}}$. Find

(a) the magnitude and direction of the impulse received by the sphere,

Solution

$$\mathbf{I} = m(\mathbf{v}-\mathbf{u})$$
$$= m\left[(2\mathbf{i}-2\mathbf{j})-(6\mathbf{i}+3\mathbf{j})\right]$$
$$= m(-4\mathbf{i}-5\mathbf{j});$$

hence, the magnitude is

$$|\mathbf{I}| = m\sqrt{(-4)^2+(-5)^2}$$
$$= \underline{\underline{m\sqrt{41}\ \mathrm{N\,s}}}$$

and the direction is parallel to the unit vector $\underline{\underline{\frac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})}}$.

(b) the coefficient of restitution between the sphere and the wall.

Solution

Component of $(6\mathbf{i}+3\mathbf{j})$:

$$\left[(6\mathbf{i}+3\mathbf{j}).\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})\right]\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j}) = \left[\tfrac{1}{\sqrt{41}}(-24-15)\right]\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$$
$$= -\tfrac{39}{\sqrt{41}}\cdot\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j}).$$

Component of $(2\mathbf{i}+2\mathbf{j})$:

$$\left[(2\mathbf{i}-2\mathbf{j}).\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})\right]\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j}) = \left[\tfrac{1}{\sqrt{41}}(-8+10)\right]\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j})$$
$$= \tfrac{2}{\sqrt{41}}\cdot\tfrac{1}{\sqrt{41}}(-4\mathbf{i}-5\mathbf{j}).$$

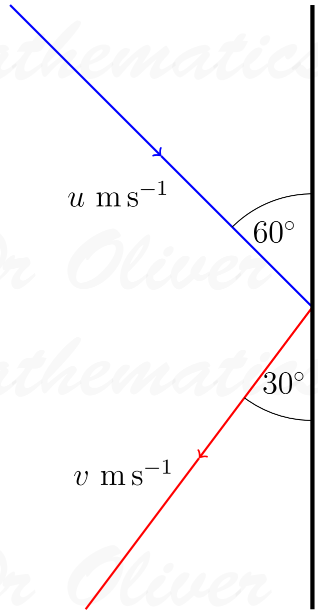
Hence,

$$\tfrac{2}{\sqrt{41}} = \tfrac{39}{\sqrt{41}}e \Rightarrow e = \frac{\frac{2}{\sqrt{41}}}{\frac{39}{\sqrt{41}}}$$
$$\Rightarrow \underline{\underline{e = \tfrac{2}{39}}}.$$

22. A smooth ball is moving on a smooth horizontal plane when it collides with a smooth fixed vertical wall. The coefficient of restitution between the ball and the wall is e. Immediately before the collision, the direction of motion of the ball makes an angle of 60° with the wall. Immediately after the collision, the direction of motion of the ball makes an angle of 30° with the wall.

(a) Find the fraction of the kinetic energy of the ball which is lost in the impact.

Solution



	Before	After
Horizontally	$u\cos 60^\circ = \frac{1}{2}u$	$v\cos 30^\circ = \frac{\sqrt{3}}{2}v$
Vertically	$u\sin 60^\circ = \frac{\sqrt{3}}{2}u$	$ev\sin 30^\circ = \frac{1}{2}ev$

Horizontally,

$$\tfrac{1}{2}u = \tfrac{\sqrt{3}}{2}v \Rightarrow v = \tfrac{1}{\sqrt{3}}u$$

and, hence, the fraction of the kinetic energy of the ball which is lost in the impact is

$$\begin{aligned}\frac{\frac{1}{2}mu^2 - \frac{1}{2}mv^2}{\frac{1}{2}mu^2} &= \frac{u^2 - (\frac{1}{\sqrt{3}}u)^2}{u^2} \\ &= \frac{u^2 - \frac{1}{3}u^2}{u^2} \\ &= 1 - \tfrac{1}{3} \\ &= \underline{\underline{\tfrac{2}{3}}}.\end{aligned}$$

(b) Find the value of e.

Solution

Vertically,

$$\begin{aligned} e &= \frac{\frac{1}{2}v}{\frac{\sqrt{3}}{2}u} \\ &= \frac{\frac{1}{2}(\frac{1}{\sqrt{3}}u)}{\frac{\sqrt{3}}{2}u} \\ &= \underline{\underline{\tfrac{1}{3}}}. \end{aligned}$$

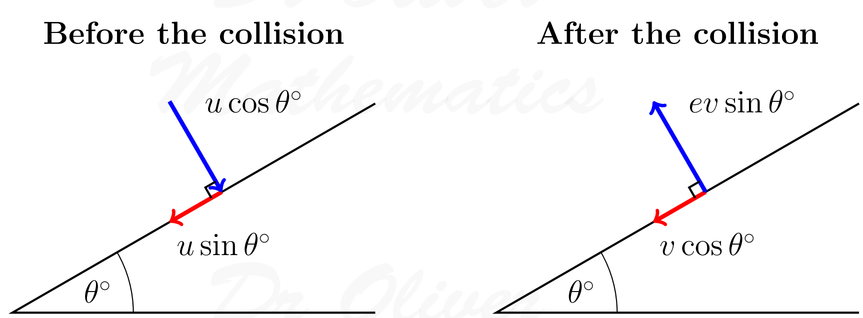
23. A smooth uniform sphere P of mass m kg is falling vertically and strikes a fixed smooth inclined plane with speed $5.2\ \mathrm{m\,s^{-1}}$. The plane is inclined at an angle of $\theta°$, $\theta < 45$, to the horizontal. The coefficient of restitution between the ball and the wall is e. Immediately after P strikes the plane, P moves horizontally.

(a) Show that

$$e = \tan^2\theta.$$

Solution

Let the speed of P immediately after the impact be v.



Parallel,

$$u\sin\theta = v\cos\theta \quad (1).$$

Perpendicular,

$$e = \frac{v\sin\theta}{u\cos\theta} \Rightarrow eu\cos\theta = v\sin\theta \quad (2).$$

Now, (2) ÷ (1):

$$\frac{eu\cos\theta}{u\sin\theta} = \frac{v\sin\theta}{v\cos\theta} \Rightarrow e\cot\theta = \tan\theta$$
$$\Rightarrow \underline{\underline{e = \tan^2\theta}},$$

as required.

(b) Show that the magnitude of the impulse exerted by P on the plane is

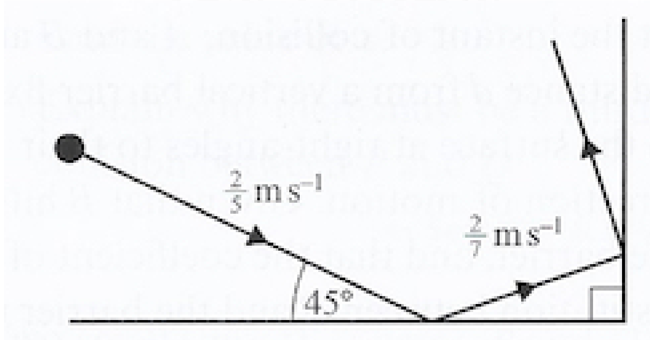
$$mu\sec\theta.$$

Solution

$$\begin{aligned} |\mathbf{I}| &= m[v\sin\theta - (-u\cos\theta)] \\ &= m(v\sin\theta + u\cos\theta) \\ &= m\left(\frac{u\sin^2\theta}{\cos\theta} + u\cos\theta\right) \\ &= mu\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta}\right) \\ &= mu\left(\frac{1}{\cos\theta}\right) \\ &= \underline{\underline{mu\sec\theta}}, \end{aligned}$$

as required.

24. Two smooth vertical walls stand on a smooth horizontal surface and intersect at right angles. A smooth sphere of mass 0.8 kg is moving across the surface such that it collides with the first wall at a speed of $\frac{2}{5}$ m s^{-1} at an angle of 45°. The coefficient of restitution between the ball and both walls is e. After the first collision, the sphere is moving with speed $\frac{2}{7}$ m s^{-1}, as shown in the figure below.



Find

(a) the direction in which the sphere is moving after the first impact, (2)

Solution

Let the direction be α°. Then

	Before	After
Horizontally	$\frac{2}{5}\cos 45^\circ$	$\frac{2}{7}\cos\alpha^\circ$
Vertically	$\frac{2}{5}\sin 45^\circ$	$\frac{2}{7}e\sin\alpha^\circ$

Now,

$$\begin{aligned}\tfrac{2}{5}\cos 45^\circ = \tfrac{2}{7}\cos\alpha^\circ &\Rightarrow \cos\alpha^\circ = \tfrac{7}{5}\cos 45^\circ\\ &\Rightarrow \cos\alpha^\circ = \tfrac{7\sqrt{2}}{10}\\ &\Rightarrow \alpha = 8.130\,102\,354 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{\alpha = 8.13 \text{ (3 sf)}}}.\end{aligned}$$

(b) the value of e. (2)

Solution

Now,

$$\begin{aligned}\cos\alpha^\circ = \tfrac{7\sqrt{2}}{10} &\Rightarrow \cos^2\alpha^\circ = \tfrac{49}{50}\\ &\Rightarrow \sin^2\alpha^\circ = \tfrac{1}{50}\\ &\Rightarrow \sin\alpha^\circ = \tfrac{\sqrt{2}}{10}\end{aligned}$$

and

$$
\begin{aligned}
e &= \frac{\frac{2}{7} \times \frac{\sqrt{2}}{10}}{\frac{2}{5}\sin 45^\circ} \\
&= \underline{\underline{\tfrac{1}{7}}}.
\end{aligned}
$$

The sphere then moves on to collide with the second wall.

(c) Calculate the kinetic energy of the sphere after the second collision. (6)

Solution

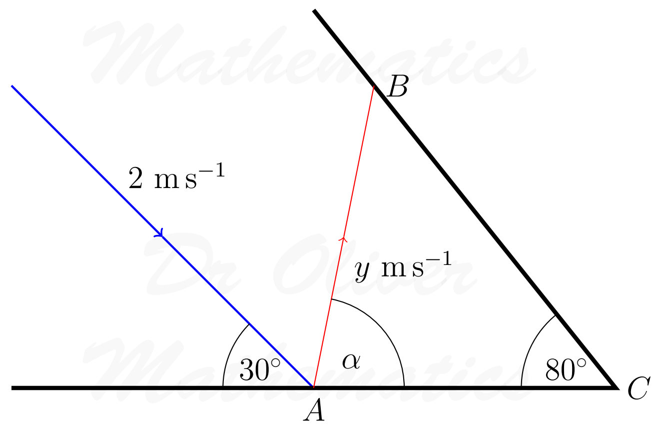
$$
\begin{aligned}
\text{Kinetic energy} &= \tfrac{1}{2} \times 0.8 \times (\tfrac{2}{5})^2 \times \left[(\tfrac{1}{7}\sin 45^\circ)^2 + (\tfrac{1}{7}\cos 45^\circ)^2\right] \\
&= \tfrac{8}{6\,125} \text{ or } 1.306\,122\,449 \times 10^{-3}\text{J (FCD)} \\
&= \underline{\underline{\tfrac{8}{6\,125} \text{ or } 1.31 \times 10^{-3}\text{J (3 sf)}}}.
\end{aligned}
$$

25. Two smooth vertical walls stand on a smooth horizontal surface and intersect at an angle of 80°. A smooth sphere of mass 0.3 kg is moving across the surface such that it collides with the first wall at a speed of 2 $\text{m}\,\text{s}^{-1}$ at an angle of 30° and towards the intersection of both walls. The sphere then collides with both walls. The coefficient of restitution between the ball and both walls is 0.6. (8)

Work out the total kinetic energy lost during the two collisions.

Solution

No picture? Well, we will sort that out!



$$\begin{aligned}\text{Initial KE} &= \tfrac{1}{2} \times 0.3 \times 2^2\\ &= 0.6 \text{ J.}\end{aligned}$$

1st bounce	Before	After
Horizontally	$2\cos 30^\circ$	$2\cos 30^\circ$
Vertically	$2\sin 30^\circ$	$0.6 \times 2\sin 30^\circ = 1.2\sin 30^\circ$

Hence,

$$\begin{aligned}\text{speed after 1st bounce} &= \sqrt{(2\cos 30^\circ)^2 + (1.2\sin 30^\circ)^2}\\ &= \sqrt{(\sqrt{3})^2 + (0.6)^2}\\ &= \tfrac{2}{5}\sqrt{21} \text{ m s}^{-1}\end{aligned}$$

and

$$\begin{aligned}\text{direction} &= \tan^{-1}\frac{1.2\sin 30^\circ}{2\cos 30^\circ}\\ &= \tan^{-1}(\tfrac{3}{5}\tan 30^\circ)\\ &= 19.106\,605\,35^\circ \text{ (FCD).}\end{aligned}$$

Now, we want $\angle ABC$:

$$\begin{aligned}\angle ABC &= 180 - 19.106\ldots - 80\\ &= 80.893\,394\,65^\circ \text{ (FCD).}\end{aligned}$$

2nd bounce	Before	After
Horizontally	$\frac{2}{5}\sqrt{21}\cos 80.893\ldots^\circ$	$\frac{2}{5}\sqrt{21}\cos 80.893\ldots^\circ$
Vertically	$\frac{2}{5}\sqrt{21}\sin 80.893\ldots^\circ$	$\frac{6}{25}\sqrt{21}\sin 80.893\ldots^\circ$

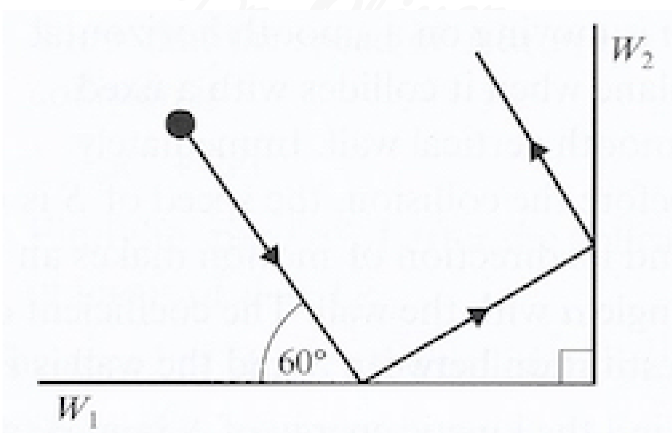
Next,

$$\begin{aligned}\text{final KE} &= \tfrac{1}{2} \times 0.3 \times \left[(\tfrac{2}{5}\sqrt{21}\cos 80.893\ldots^\circ)^2 + (\tfrac{6}{25}\sqrt{21}\sin 80.893\ldots^\circ)^2\right] \\ &= 0.189\,520\,126\,2 \text{ m s}^{-1}\end{aligned}$$

and, hence, the total kinetic energy lost during the two collisions is

$$\begin{aligned}0.6 - 0.189\ldots &= 0.410\,479\,873\,8 \text{ (FCD)} \\ &= \underline{\underline{0.410\text{J (3 sf)}}}.\end{aligned}$$

26. Two smooth vertical walls, W_1 and W_2, stand on a smooth horizontal surface and intersect at right angles. A small smooth sphere is moving with speed 4 m s^{-1} when it hits W_1 at an angle of 60°. It rebounds from the wall with speed 3 m s^{-1}and goes on to hit W_2.



(a) The coefficient of restitution between the sphere and W_1. (4)

Solution

	Before	After
Horizontally	$4\cos 60°$	$4\cos 60°$
Vertically	$4\sin 60°$	x

Now,

$$(4\cos 60°)^2 + x^2 = 3^2 \Rightarrow 4 + x^2 = 9$$
$$\Rightarrow x^2 = 5$$
$$\Rightarrow x = \sqrt{5}.$$

Hence,

$$e = \frac{\sqrt{5}}{4\sin 60°}$$
$$= \underline{\underline{\tfrac{\sqrt{15}}{6}}}.$$

Assuming that the coefficient of restitution between the sphere and W_2 is 0.35,

(b) work out the speed of the sphere and direction in which it is moving after it collides with W_2. (6)

Solution

	Before	After
Horizontally	$\sqrt{5}$	$\sqrt{5}$
Vertically	$4\cos 60°$	$0.35 \times 4\cos 60° = 0.7$

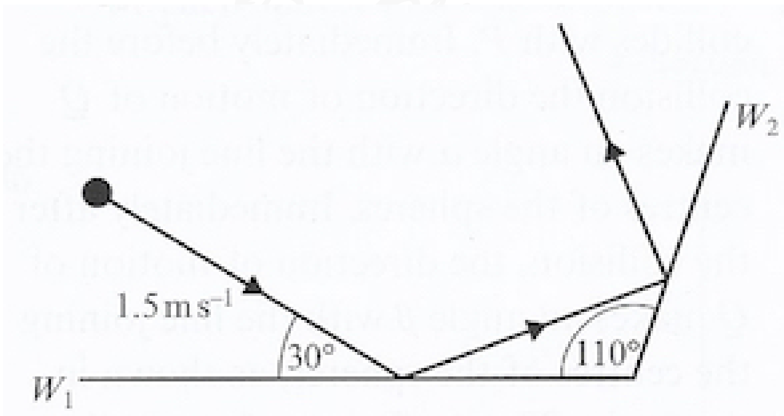
Hence,

$$\text{speed} = \sqrt{(\sqrt{5})^2 + (0.7)^2}$$
$$= \underline{\underline{\tfrac{3}{10}\sqrt{61} \text{ or } 2.34 \text{ m s}^{-1}}}$$

and

$$\begin{aligned}\text{direction} &= \tan^{-1}\frac{0.7}{\sqrt{5}} \\ &= \tan^{-1}(\tfrac{7}{50}\sqrt{5}) \\ &= 17.382\,703\,99 \text{ (FCD)} \\ &= \underline{\underline{17.4^\circ \text{ (3 sf)}}}.\end{aligned}$$

27. Two smooth vertical walls, W_1 and W_2, stand on a smooth horizontal surface and intersect at an angle of 110°. A small smooth sphere of mass 1.6 kg is projected across the surface with speed 1.5 m s^{-1} at an angle of 30° to wall W_1 and towards the intersection of the walls. The coefficient of restitution between the sphere and wall W_1 is 0.8.



(a) Work out the speed and direction of motion of the sphere after the first collision. (6)

Solution

1st bounce	Before	After
Horizontally	$1.5\cos 30^\circ$	$1.5\cos 30^\circ$
Vertically	$1.5\sin 30^\circ$	$0.8 \times 1.5\sin 30^\circ = 1.2\sin 30^\circ$

Hence,

$$\begin{aligned}\text{speed after 1st bounce} &= \sqrt{(1.5\cos 30^\circ)^2 + (1.2\sin 30^\circ)^2} \\ &= \sqrt{(\tfrac{3}{4}\sqrt{3})^2 + (0.6)^2} \\ &= \underline{\underline{\tfrac{3}{20}\sqrt{91} \text{ or } 1.43 \text{ m s}^{-1} \text{ (3 sf)}}}\end{aligned}$$

and

$$\begin{aligned}\text{direction} &= \tan^{-1}\left(\frac{1.2\sin 30^\circ}{1.5\cos 30^\circ}\right)\\ &= \tan^{-1}(\tfrac{4}{5}\tan 30^\circ)\\ &= 24.791\,280\,9 \text{ (FCD)}\\ &= \underline{\underline{24.8^\circ \text{ (3 sf)}}}.\end{aligned}$$

The sphere then moves on to collide with W_2. Given that after the second collision, the sphere has kinetic energy 1.35 J,

(b) work out the coefficient of restitution between the sphere and wall W_2. (8)

Solution

$$180 - 24.791\ldots - 110 = 45.208\,719\,1 \text{ (FCD)}.$$

Now,

2nd bounce	Before	After
Horizontally	$\frac{3}{20}\sqrt{91}\cos 45.208\ldots^\circ$	$\frac{3}{20}\sqrt{91}\cos 45.208\ldots^\circ$
Vertically	$\frac{3}{20}\sqrt{91}\sin 45.208\ldots^\circ$	$e \times \frac{3}{20}\sqrt{91}\sin 45.208\ldots^\circ$

Finally,

$$\begin{aligned}&\tfrac{1}{2} \times 1.6 \times \left[(\tfrac{3}{20}\sqrt{91}\cos 45.208\ldots^\circ)^2 + (\tfrac{3}{20}\sqrt{91}e\sin 45.208\ldots^\circ)^2\right] = 1.35\\ \Rightarrow\ &(\tfrac{3}{20}\sqrt{91}\cos 45.208\ldots^\circ)^2 + (\tfrac{3}{20}\sqrt{91}e\sin 45.208\ldots^\circ)^2 = 1\tfrac{11}{16}\\ \Rightarrow\ &(\tfrac{3}{20}\sqrt{91}e\sin 45.208\ldots^\circ)^2 = 0.671\,208\,639\,8 \text{ (FCD)}\\ \Rightarrow\ &\tfrac{3}{20}\sqrt{91}e\sin 45.208\ldots^\circ = 0.819\,273\,238\,8 \text{ (FCD)}\\ \Rightarrow\ &e = 0.806\,780\,694\,7 \text{ (FCD)}\\ \Rightarrow\ &\underline{\underline{e = 0.807 \text{ (3 sf)}}}.\end{aligned}$$

28. Two smooth vertical walls, W_1 and W_2, stand on a smooth horizontal surface and intersect at an angle of 100°. A small smooth sphere of mass 1.7 kg is projected across the surface with speed 8 $\mathrm{m\,s^{-1}}$ at an angle of 25° to wall W_1 and towards the intersection of the walls. The coefficient of restitution between the sphere and walls W_1 and W_2 are 0.6 and 0.7 respectively. (10)

Calculate the total kinetic energy lost by the sphere.

Solution

First,

$$\begin{aligned}\text{initial KE} &= \tfrac{1}{2} \times 1.7 \times 8^2 \\ &= 54.4 \text{ J}.\end{aligned}$$

1st bounce	Before	After
Horizontally	$8\cos 25°$	$8\cos 25°$
Vertically	$8\sin 25°$	$0.6 \times 8\sin 45° = 4.8\sin 25°$

Hence,

$$\begin{aligned}\text{speed after 1st bounce} &= \sqrt{(8\cos 25°)^2 + (4.8\sin 25°)^2} \\ &= 7.528\,897\,014 \text{ m s}^{-1} \text{ (3 sf)}\end{aligned}$$

and

$$\begin{aligned}\text{direction} &= \tan^{-1}\left(\frac{4.8\sin 25°}{8\cos 25°}\right) \\ &= 15.630\,801\,27° \text{ (FCD)}.\end{aligned}$$

Now, the new direction is

$$180 - 100 - 15.630\ldots = 64.369\,198\,73° \text{ (FCD)}.$$

2nd bounce	Before	After
Horizontally	$7.528\ldots\cos 64.369\ldots°$	$7.528\ldots\cos 64.369\ldots°$
Vertically	$7.528\ldots\sin 64.369\ldots°$	$0.7 \times 7.528\ldots\sin 64.369\ldots°$

Hence,

$$\begin{aligned}\text{final KE} &= \tfrac{1}{2} \times 1.7 \times \left[(7.528\ldots\cos 64.369\ldots°)^2 + (0.7 \times 7.528\ldots\sin 64.369\ldots°)^2\right] \\ &= 28.206\,971\,3 \text{ J (FCD)}.\end{aligned}$$

Finally,

$$\begin{aligned}\text{total kinetic energy lost} &= 54.4 - 28.206\ldots \\ &= 26.193\,028\,7 \text{ (FCD)} \\ &= \underline{\underline{26.2 \text{ J (3 sf)}}}.\end{aligned}$$