Dr Oliver Mathematics
Mathematics
Differentiation Part 2
Past Examination Questions

This booklet consists of 25 questions across a variety of examination topics.
The total number of marks available is 229.

1. Figure 1 shows the plan of a stage in the shape of a rectangle joined to a semicircle.

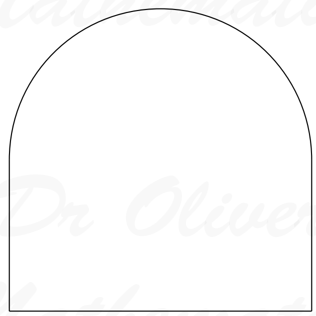

Figure 1: the stage

The length of the rectangular part is $2x$ metres and the width is y metres. The diameter of the semicircular part is $2x$ metres. The perimeter of the stage is 80 metres.

(a) Show the the area, $A\text{m}^2$, of the stage is given by (4)

$$A = 80x - \left(2 + \frac{\pi}{2}\right) x^2.$$

Solution

$$80 = 2x + 2y + \pi x \Rightarrow 2y = 80 - 2x - \pi x$$

and

$$\begin{aligned} A &= \tfrac{1}{2}\pi x^2 + 2xy \\ &= \tfrac{1}{2}\pi x^2 + 2x(40 - x - \tfrac{1}{2}\pi x) \\ &= \tfrac{1}{2}\pi x^2 + 80x - 2x^2 - \pi x^2 \\ &= 80x - 2x^2 - \tfrac{1}{2}\pi x^2 \\ &= \underline{\underline{80x - \left(2 + \frac{\pi}{2}\right) x^2}}. \end{aligned}$$

(b) Use calculus to find the value of x at which A has a stationary value. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \Rightarrow 80 - 2\left(2 + \frac{\pi}{2}\right)x = 0 \\ &\Rightarrow 2\left(2 + \frac{\pi}{2}\right)x = 80 \\ &\Rightarrow \left(2 + \frac{\pi}{2}\right)x = 40 \\ &\Rightarrow x = \frac{40}{2 + \frac{\pi}{2}} \\ &\Rightarrow \underline{\underline{x = \frac{80}{4 + \pi}}}.\end{aligned}$$

(c) Prove that the value of x you found in (b) gives the maximum value of A. (2)

Solution

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = -2\left(2 + \frac{\pi}{2}\right) < 0$$

and so the stationary point is a <u>maximum</u>.

(d) Calculate, to the nearest m^2, the maximum area of the stage. (2)

Solution

$$A = 448.079\,322\,8 \text{ (FCD)} = \underline{\underline{448 \text{ (nearest m}^2)}}.$$

2. Find the coordinates of the stationary point on the curve with equation $y = 2x^2 - 12x$.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 4x - 12 = 0 \\ &\Rightarrow 4x = 12 \\ &\Rightarrow x = 3 \\ &\Rightarrow y = -18,\end{aligned}$$

and $\underline{\underline{(3, -18)}}$ is a <u>minimum</u>.

3. The curve C has equation

$$y = 2x^3 - 5x^2 - 4x + 2.$$

(a) Find $\frac{dy}{dx}$. (2)

Solution

$$y = 2x^3 - 5x^2 - 4x + 2 \Rightarrow \underline{\underline{\frac{dy}{dx} = 6x^2 - 10x - 4}}.$$

(b) Using the result from part (a), find the coordinates of the turning points of C. (4)

Solution

$$\begin{aligned}\frac{dy}{dx} = 0 &\Rightarrow 6x^2 - 10x - 4 = 0 \\ &\Rightarrow 3x^2 - 5x - 2 = 0 \\ &\Rightarrow (3x+1)(x-2) = 0 \\ &\Rightarrow 3x + 1 = 0 \text{ or } x - 2 = 0 \\ &\Rightarrow x = -\tfrac{1}{3} \text{ or } x = 2,\end{aligned}$$

and we have $\underline{\underline{(-\frac{1}{3}, 2\frac{19}{27})}}$ and $\underline{\underline{(2, -10)}}$.

(c) Find $\frac{d^2y}{dx^2}$. (2)

Solution

$$\frac{dy}{dx} = 6x^2 - 10x - 4 \Rightarrow \underline{\underline{\frac{d^2y}{dx^2} = 12x - 10}}.$$

(d) Hence, or otherwise, determine the nature of the turning points of C. (2)

Solution

When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = -14 < 0$ and so we have a $\underline{\underline{\text{maximum}}}$.

When $x = 2$, $\frac{d^2y}{dx^2} = 14 > 0$ and so we have a $\underline{\underline{\text{minimum}}}$.

4. Figure 2 shows a sketch of part of the curve with equation $y = x^3 - 8x^2 + 20x$.

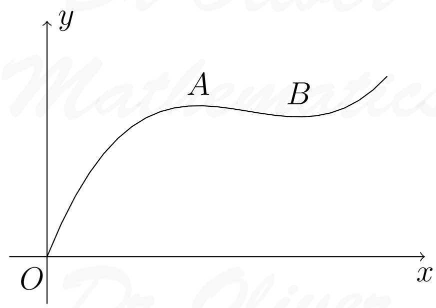

Figure 2: $y = x^3 - 8x^2 + 20x$

The curve has stationary points A and B.

(a) Use calculus to find the x-coordinates of A and B. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 3x^2 - 16x + 20 = 0\\ &\Rightarrow (3x-10)(x-2) = 0\\ &\Rightarrow \underline{\underline{x = 2 \text{ or } x = 3\tfrac{1}{3}}}.\end{aligned}$$

5. (3)

$$\mathrm{f}(x) = x^3 + 3x^2 + 5.$$

Find $\mathrm{f}''(x)$.

Solution

$$\begin{aligned}\mathrm{f}(x) = x^3 + 3x^2 + 5 &\Rightarrow \mathrm{f}'(x) = 3x^2 + 6x\\ &\Rightarrow \underline{\underline{\mathrm{f}''(x) = 6x + 6}}.\end{aligned}$$

6. A diesel lorry is driven from Birmingham to Bury at a steady speed of v kilometres per hour. The total cost of the journey, £C, is given by

$$C = \frac{1400}{v} + \frac{2v}{7}.$$

(a) Find the value of v for which C is a minimum. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}C}{\mathrm{d}v} = 0 &\Rightarrow -\frac{1400}{v^2} + \frac{2}{7} = 0\\ &\Rightarrow \frac{1400}{v^2} = \frac{2}{7}\\ &\Rightarrow v^2 = 4900\\ &\Rightarrow v = \pm 70.\end{aligned}$$

It is clearly **not** $v = -70$ so $\underline{\underline{v = 70}}$.

(b) Find $\frac{\mathrm{d}^2C}{\mathrm{d}v^2}$ and hence verify that C is a minimum for this value of v. (2)

Solution

$$\frac{\mathrm{d}C}{\mathrm{d}v} = -\frac{1400}{v^2} + \frac{2}{7} \Rightarrow \frac{\mathrm{d}^2C}{\mathrm{d}v^2} = \frac{2800}{v^3}.$$

Now,

$$v = 70 \Rightarrow \frac{\mathrm{d}^2C}{\mathrm{d}v^2} = \tfrac{2}{245} > 0$$

and this is a $\underline{\underline{\text{minimum}}}$.

(c) Calculate the minimum total cost of the journey. (2)

Solution

$$v = 70 \Rightarrow \underline{\underline{C = 40}}.$$

7. A solid brick in the shape of a cuboid measuring $2x$ cm by x cm by y cm.

The total surface area of the brick is 600 cm^2.

(a) Show that the volume, V cm^3, of the brick is given by (4)

$$V = 200x - \tfrac{4}{3}x^3.$$

Solution

$$600 = 4x^2 + 2xy + 4xy = 4x^2 + 6xy \Rightarrow y = \frac{600 - 4x^2}{6x}$$

and

$$\begin{aligned} V &= 2x^2 y \\ &= 2x^2 \left(\frac{600 - 4x^2}{6x} \right) \\ &= \frac{1200x^2 - 8x^4}{6x} \\ &= \underline{\underline{200x - \tfrac{4}{3}x^3}}. \end{aligned}$$

Given that x can vary,

(b) use calculus to find the maximum value of V, giving your answer to the nearest cm^3. (5)

Solution

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} = 0 &\Rightarrow 200 - 4x^2 = 0 \\ &\Rightarrow 4x^2 = 200 \\ &\Rightarrow x^2 = 50 \\ &\Rightarrow x = 5\sqrt{2}, \end{aligned}$$

as $x = -5\sqrt{2}$ makes no sense. Now,

$$\begin{aligned} x = 5\sqrt{2} \Rightarrow V &= \tfrac{2000}{3}\sqrt{2} \\ \Rightarrow V &= 942.809\,041\,6 \text{ (FCD)} \\ \Rightarrow \underline{\underline{V}} &\underline{\underline{= 943 \text{ (nearest cm}^3)}}. \end{aligned}$$

(c) Justify that the value of V you have found is a maximum. (2)

Solution

$$\frac{\mathrm{d}V}{\mathrm{d}x} = 200 - 4x^2 \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -8x$$

and

$$x = 5\sqrt{2} \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -40\sqrt{2} < 0$$

and this is a $\underline{\underline{\text{maximum}}}$.

8. Figure 3 shows an open-topped water tank, in the shape of a cuboid, which is made from sheet metal. The base of the tank is a rectangle x metres by y metres. The height of the tank is x metres.

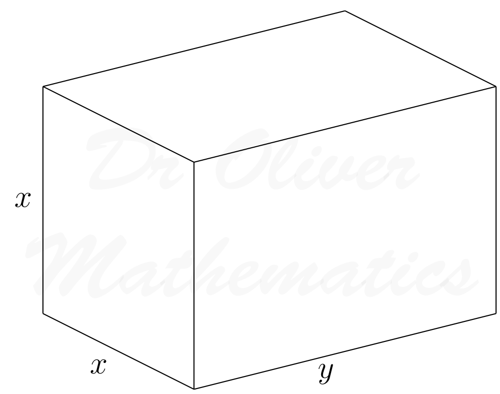

Figure 3: open-topped water tank

The capacity of the tank is 100 m^3.

(a) Show that the area A m^2 of the sheet metal used to make the tank is given by (4)

$$A = \frac{300}{x} + 2x^2.$$

Solution

$$100 = x^2 y \Rightarrow y = \frac{100}{x^2}$$

and

$$\begin{aligned} A &= 2x^2 + 3xy \\ &= 2x^2 + 3x\left(\frac{100}{x^2}\right) \\ &= \underline{\underline{\frac{300}{x} + 2x^2.}} \end{aligned}$$

(b) Use calculus to find the value of x for which A is stationary. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 &\Rightarrow -\frac{300}{x^2} + 4x = 0 \\ &\Rightarrow \frac{300}{x^2} = 4x \\ &\Rightarrow 4x^3 = 300 \\ &\Rightarrow x^3 = 75 \\ &\Rightarrow \underline{\underline{x = \sqrt[3]{75}}}.\end{aligned}$$

(c) Prove that this value of x gives a minimum value for A. (2)

Solution

$$\frac{\mathrm{d}A}{\mathrm{d}x} = -\frac{300}{x^2} + 4x \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{600}{x^3} + 4$$

and, when $x = \sqrt[3]{75}$, we have

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 12$$

and this is a minimum.

(d) Calculate the minimum area of sheet metal needed to make the tank. (2)

Solution

$$x = \sqrt[3]{75} \Rightarrow A = 106.706\,799\,1 \text{ (FCD)} = \underline{\underline{107 \text{ (3 sf)}}}.$$

9. Figure 4 shows a sketch of part of the curve with equation $y = 10 + 8x + x^2 - x^3$. (3)

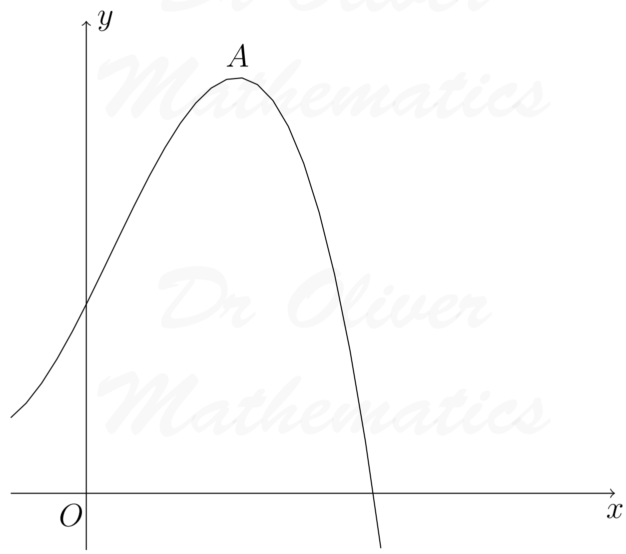

Figure 4: $y = 10 + 8x + x^2 - x^3$

The curve has a maximum turning point A.

Using calculus, show that the x-coordinate of A is 2.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 8 + 2x - 3x^2 = 0\\ &\Rightarrow (2 - x)(4 + 3x) = 0\\ &\Rightarrow \underline{\underline{x = 2}} \text{ or } x = -1\tfrac{1}{3}.\end{aligned}$$

10. A solid right circular cylinder has radius r cm and height h cm.

The total surface area of the cylinder is 800 cm^2.

(a) Show that the volume, V cm^3, of the cylinder is given by (4)

$$V = 400r - \pi r^3.$$

Solution

$$\begin{aligned} 800 = 2\pi rh + 2\pi r^2 &\Rightarrow 400 - \pi r^2 = \pi rh \\ &\Rightarrow 400 - \pi r^2 = \pi rh \\ &\Rightarrow h = \frac{400 - \pi r^2}{\pi r} \\ &\Rightarrow V = \pi r^2 \left(\frac{400 - \pi r^2}{\pi r} \right) \\ &\Rightarrow \underline{\underline{V = 400r - \pi r^3}}. \end{aligned}$$

Given that r varies,

(b) use calculus to find the maximum value of V, to the nearest cm^3. (6)

Solution

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}r} = 0 &\Rightarrow 400 - 3\pi r^2 = 0 \\ &\Rightarrow 3\pi r^2 = 400 \\ &\Rightarrow r^2 = \frac{400}{3\pi} \\ &\Rightarrow r = \sqrt{\frac{400}{3\pi}} \text{ (because } r = -\sqrt{\frac{400}{3\pi}} \text{ makes no sense)} \\ &\Rightarrow V = 1737.253\,376 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{V = 1737 \text{ (nearest cm}^3)}}. \end{aligned}$$

(c) Justify that the value of V you have found is a maximum. (2)

Solution

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 400 - 3\pi r^2 \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -6\pi r$$

and

$$r = \sqrt{\frac{400}{3\pi}} \Rightarrow \frac{\mathrm{d}^2V}{\mathrm{d}r^2} = -6\pi\sqrt{\frac{400}{3\pi}} < 0$$

which is a maximum.

11. Figure 5 shows a closed box used by a shop for packing pieces of cake.

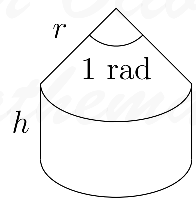

Figure 5: packing pieces of cake

The box is a right prism of height h cm. The cross-section is a sector of a circle. The sector has radius r cm and angle 1 radian.

The volume of the box is 300 cm^3.

(a) Show that the surface area of the box, S cm^2, is given by (5)

$$S = r^2 + \frac{1800}{r^2}.$$

Solution

$$\begin{aligned} \text{Arc length} &= r\theta = r \times 1 = r, \\ \text{sector area} &= \tfrac{1}{2}r^2\theta = \tfrac{1}{2}r^2 \times 1 = \tfrac{1}{2}r^2, \\ \text{surface area} &= 2 \text{ sectors} + 2 \text{ rectangles} + \text{curved face} \\ &= r^2 + 2rh + rh \\ &= r^2 + 3rh, \end{aligned}$$

and so we have

$$\begin{aligned} 300 = \tfrac{1}{2}r^2h \Rightarrow h &= \frac{600}{r^2} \\ \Rightarrow S &= r^2 + 3r\left(\frac{600}{r^2}\right) \\ \Rightarrow \underline{\underline{S = r^2 + \frac{1800}{r}}}. \end{aligned}$$

(b) Use calculus to find the value of r for which S is stationary. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}r} = 0 &\Rightarrow 2r - \frac{1800}{r^2} = 0\\ &\Rightarrow 2r = \frac{1800}{r^2}\\ &\Rightarrow r^3 = 900\\ &\Rightarrow \underline{\underline{r = \sqrt[3]{900}}}.\end{aligned}$$

(c) Prove that this value of r gives a minimum value for S. (2)

Solution

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 2r - \frac{1800}{r^2} \Rightarrow \frac{\mathrm{d}^2S}{\mathrm{d}r^2} = 2 + \frac{3600}{r^3}$$

and

$$r = \sqrt[3]{900} \Rightarrow \frac{\mathrm{d}^2S}{\mathrm{d}r^2} = 2 + \frac{3600}{900} = 6 > 0$$

which is a $\underline{\underline{\text{minimum}}}$.

(d) Find, to the nearest cm^2, this minimum value for S. (2)

Solution

$$r = \sqrt[3]{900} \Rightarrow S = 279.650\,925\,5 \text{ (FCD)} = \underline{\underline{280 \text{ (nearest cm}^2)}}.$$

12. The curve C has equation $y = 12\sqrt{x} - x^{\frac{3}{2}} - 10$, $x > 0$.

(a) Use calculus to find the coordinates of the turning point on C. (7)

Solution

$$\begin{aligned}y = 12\sqrt{x} - x^{\frac{3}{2}} - 10 &\Rightarrow y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}} - 10\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{-\frac{1}{2}} - \tfrac{3}{2}x^{\frac{1}{2}},\end{aligned}$$

and we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 6x^{-\frac{1}{2}} - \tfrac{3}{2}x^{\frac{1}{2}} = 0\\ &\Rightarrow 6x^{-\frac{1}{2}} = \tfrac{3}{2}x^{\frac{1}{2}}\\ &\Rightarrow x = 4\\ &\Rightarrow y = 6\end{aligned}$$

and we have $\underline{\underline{(4,6)}}$ as the turning point.

(b) Find $\dfrac{d^2y}{dx^2}$. (2)

Solution

$$\frac{d^2y}{dx^2} = \underline{\underline{-3x^{-\frac{3}{2}} - \tfrac{3}{4}x^{-\frac{1}{2}}}}.$$

(c) State the nature of the turning point. (1)

Solution

When $x = 4$ we have $\dfrac{d^2y}{dx^2} = -\frac{3}{4}$ and so we have a maximum turning point.

13.

$$y = x^2 - k\sqrt{x}, \text{ where } k \text{ is a constant.}$$

(a) Find $\dfrac{dy}{dx}$. (2)

Solution

$$\begin{aligned} y = x^2 - k\sqrt{x} &\Rightarrow y = x^2 - kx^{\frac{1}{2}} \\ &\Rightarrow \underline{\underline{\frac{dy}{dx} = 2x - \tfrac{1}{2}kx^{-\frac{1}{2}}}}. \end{aligned}$$

(b) Given that y is decreasing at $x = 4$, find the set of possible value of k. (2)

Solution

$$\begin{aligned} 2 \times 4 - \tfrac{1}{2}k \times 4^{-\frac{1}{2}} < 0 &\Rightarrow 8 < \tfrac{1}{4}k \\ &\Rightarrow \underline{\underline{k > 32}}. \end{aligned}$$

14. The volume V cm^3 of a box, of height h cm, is given by

$$V = 4x(5-x)^2,\ 0 < x < 5.$$

(a) Find $\dfrac{\mathrm{d}V}{\mathrm{d}x}$. (4)

Solution

$$\begin{aligned} V &= 4x(5-x)^2 \\ &= 4x(25-10x+x^2) \\ &= 100x - 40x^2 + 4x^3 \end{aligned}$$

and so we have

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \underline{\underline{100 - 80x + 12x^2}}.$$

(b) Hence find the maximum volume of the box. (4)

Solution

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} = 0 &\Rightarrow 100 - 80x + 12x^2 = 0 \\ &\Rightarrow 3x^2 - 40x + 25 = 0 \\ &\Rightarrow (3x-5)(x-5) = 0 \\ &\Rightarrow x = \tfrac{5}{3} \\ &\Rightarrow \underline{\underline{V = 74\tfrac{2}{27}}}. \end{aligned}$$

(c) Use calculus to justify that the volume that you found in part (b) is a maximum. (2)

Solution

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = -80 + 24x,$$

and, when $x = \frac{5}{3}$, we have $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = -40 < 0$, which is a $\underline{\underline{\text{maximum}}}$.

15. A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm.

The volume of the cuboid 81 cm^3.

(a) Show that the total length, L cm, of the twelve edges of the cuboid is given by (3)

$$L = 12x + \frac{162}{x^2}.$$

Solution

Let y cm be the depth of the rectangle. Then,

$$81 = 2x \times x \times y \Rightarrow y = \frac{81}{2x^2}$$

and

$$\begin{aligned} L &= 8x + 4x + 4y \\ &= 12x + 4\left(\frac{81}{2x^2}\right) \\ &= 12x + \frac{324}{2x^2} \\ &= \underline{\underline{12x + \frac{162}{x^2}}}. \end{aligned}$$

(b) Use calculus to find the minimum value of L. (6)

Solution

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}x} = 0 &\Rightarrow 12 - \frac{324}{x^3} = 0 \\ &\Rightarrow 12 = \frac{324}{x^3} \\ &\Rightarrow x^3 = 27 \\ &\Rightarrow x = 3 \\ &\Rightarrow L = \underline{\underline{54}}. \end{aligned}$$

(c) Justify, by further differentiation, that the value of L that you have found is a minimum. (2)

Solution

$$\frac{\mathrm{d}L}{\mathrm{d}x} = 12 - \frac{324}{x^3} \Rightarrow \frac{\mathrm{d}^2L}{\mathrm{d}x^2} = \frac{972}{x^4},$$

and, for $x = 3$, $\frac{\mathrm{d}^2L}{\mathrm{d}x^2} = 12 > 0$; hence, this is the $\underline{\underline{\text{minimum}}}$.

16. Figure 6 shows a flowerbed.

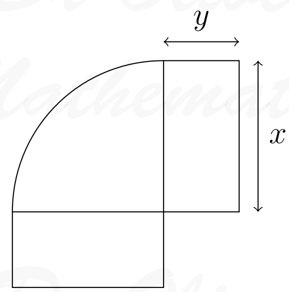

Figure 6: the flowerbed

Its shape is a quarter of a circle of radius x metres with two equal rectangles attached to it along its radii. Each rectangle has length equal to x metres and width equal to$y$ metres.

Given that the area of the flowerbed is 4 m^2,

(a) show that (3)

$$y = \frac{16 - \pi x^2}{8x}.$$

Solution

$$4 = \tfrac{1}{4}\pi x^2 + 2xy \Rightarrow 2xy = \frac{16 - \pi x^2}{4} \Rightarrow \underline{\underline{y = \frac{16 - \pi x^2}{8x}}}.$$

(b) Hence show that the perimeter P metres of the flowerbed is given by the equation (3)

$$P = \frac{8}{x} + 2x.$$

Solution

$$\begin{aligned} P &= \tfrac{1}{2}\pi x + 2x + 4y \\ &= \tfrac{1}{2}\pi x + 2x + 4\left(\frac{16 - \pi x^2}{8x}\right) \\ &= \tfrac{1}{2}\pi x + 2x + \frac{8}{x} - \tfrac{1}{2}\pi x^2 \\ &= \underline{\underline{\frac{8}{x} + 2x}}. \end{aligned}$$

(c) Use calculus to find the minimum value for P. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}x} = 0 &\Rightarrow -\frac{8}{x^2} + 2 = 0 \\ &\Rightarrow \frac{8}{x^2} = 2 \\ &\Rightarrow x^2 = 4 \\ &\Rightarrow x = 2 \text{ (as } x = -2 \text{ makes no sense)} \\ &\Rightarrow \underline{\underline{P = 8}}.\end{aligned}$$

(d) Find the width of each rectangle when the perimeter is a minimum. Give your answer to the nearest centimetre. (2)

Solution

$$y = \frac{16 - 4\pi}{16} = 0.214\,601\,836\,6 \text{ m (FCD)} = \underline{\underline{21 \text{ cm (2 sf)}}}.$$

17. A manufacturer produces pain relieving tablets. Each tablet is in the shape of a solid circular cylinder with base radius x mm and h mm.

Given that the volume of each tablet has to be 60 mm^3,

(a) express h in terms of x, (1)

Solution

$$60 = \pi x^2 h \Rightarrow \underline{\underline{h = \frac{60}{\pi x^2}}}.$$

(b) show that the surface area, A mm^2, of a tablet is given by (3)

$$A = 2\pi x^2 + \frac{120}{x}.$$

Solution

$$
\begin{aligned}
A &= 2\pi xh + 2\pi x^2 \\
&= 2\pi x\left(\frac{60}{\pi x^2}\right) + 2\pi x^2 \\
&= \underline{\underline{2\pi x^2 + \frac{120}{x}}}.
\end{aligned}
$$

The manufacturer needs to minimise the surface area, A mm^2, of a tablet.

(c) Use calculus to find the value of x for which A is a minimum. (5)

Solution

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}x} = 0 &\Rightarrow 4\pi x - \frac{120}{x^2} = 0 \\
&\Rightarrow 4\pi x = \frac{120}{x^2} \\
&\Rightarrow x^3 = \frac{30}{\pi} \\
&\Rightarrow x = \underline{\underline{\sqrt[3]{\frac{30}{\pi}}}}.
\end{aligned}
$$

(d) Calculate the minimum value of A, giving your answer to the nearest integer. (2)

Solution

When $x = \sqrt[3]{\frac{30}{\pi}}$,

$$A = 84.842\,875\,21 \text{ (FCD)} = \underline{\underline{85 \text{ (nearest integer)}}}.$$

(e) Show that this value of A is a minimum. (2)

Solution

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 4\pi x - \frac{120}{x^2} \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 4\pi + \frac{240}{x^3},$$

when, $x = \sqrt[3]{\frac{30}{\pi}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 4\pi + 2 > 0$ and we have a $\underline{\underline{\text{minimum}}}$.

18. The curve C has equation $y = 6 - 3x - \frac{4}{x^3}$, $x \neq 0$.

(a) Use calculus to show that the curve has a turning point P when $x = \sqrt{2}$. (4)

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow -3 + \frac{12}{x^4} = 0\\ &\Rightarrow 3 = \frac{12}{x^4}\\ &\Rightarrow x^4 = 4\\ &\Rightarrow x^2 = 2\\ &\Rightarrow \underline{\underline{x = \sqrt{2}}}\ldots\end{aligned}$$

(b) Find the x-coordinate of the other turning point Q on the curve. (1)

Solution

...and the other is $\underline{\underline{x = -\sqrt{2}}}$.

(c) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (1)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3 + \frac{12}{x^4} \Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{48}{x^5}}}.$$

(d) Hence, or otherwise, state with justification, the nature of each of these turning points P and Q. (3)

Solution

$x = \sqrt{2} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -6\sqrt{2} < 0$ and we have a $\underline{\underline{\text{maximum}}}$.

$x = -\sqrt{2} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6\sqrt{2} > 0$ and we have a $\underline{\underline{\text{minimum}}}$.

19. The curve with equation

$$y = x^2 - 32\sqrt{x} + 20,\ x > 0,$$

has a stationary point P.

Use calculus

(a) to find the coordinates of P, (6)

Solution

$$y = x^2 - 32\sqrt{x} + 20 \Rightarrow y = x^2 - 32x^{\frac{1}{2}} + 20$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 16x^{-\frac{1}{2}}.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 2x - 16x^{-\frac{1}{2}} = 0$$
$$\Rightarrow 2x = 16x^{-\frac{1}{2}}$$
$$\Rightarrow x^{\frac{3}{2}} = 8$$
$$\Rightarrow x = 4$$
$$\Rightarrow y = -28,$$

and so we have the point $\underline{\underline{(4, -28)}}$.

(b) to determine the nature of the stationary point P. (3)

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 16x^{-\frac{1}{2}} \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 + 8x^{-\frac{3}{2}};$$

now, $x = 4 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3 > 0$ and we have a $\underline{\underline{\text{minimum}}}$.

20. Using calculus, find the coordinates of the stationary point on the curve with equation (6)

$$y = 2x + 3 + \frac{8}{x^2},\ x > 0.$$

Solution

$$y = 2x + 3 + \frac{8}{x^2} \Rightarrow y = 2x + 3 + 8x^{-2}$$
$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 16x^{-3}.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 2 - 16x^{-3} = 0$$
$$\Rightarrow 2 = 16x^{-3}$$
$$\Rightarrow x^3 = 8$$
$$\Rightarrow x = 2$$
$$\Rightarrow y = 9,$$

and so we have the point $\underline{\underline{(2, 9)}}$.

21. Figure 7 shows a closed letter box $ABFEHGCD$, which is made to be attached to a wall of a house.

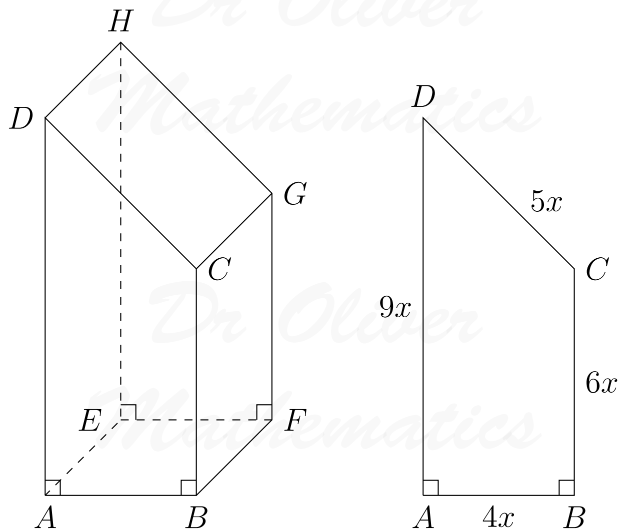

Figure 7: a letter box

The letter box is a right prism of length y cm. The base $ABFE$ of the prism is a rectangle. The total surface area of the six faces is S cm^2.

The cross-sectional $ABCD$ of the letter box is a trapezium with edges of lengths $DA = 9x$ cm, $AB = 4x$ cm, $BC = 6x$ cm, and $CD = 5x$ cm

The angle $DAB = 90°$ and the angle $ABC = 90°$.

The volume of the letter box is 9000 cm^3.

(a) Show that (2)

$$y = \frac{320}{x^2}.$$

Solution

$$\begin{aligned} & 9600 = \left[(9x \times 4x) - \tfrac{1}{2} \times 3x \times 4x\right] y \\ \Rightarrow\ & 9600 = (36x^2 - 6x^2)y \\ \Rightarrow\ & 9600 = 30x^2 y \\ \Rightarrow\ & \underline{\underline{y = \frac{320}{x^2}}}. \end{aligned}$$

(b) Hence show that the surface area of the letter box, S cm^2, is given by (4)

$$S = 60x^2 + \frac{7680}{x}.$$

Solution

$$\begin{aligned} S &= (9x + 4x + 6x + 5x)y + 2 \times 30x^2 \\ &= 24xy + 60x^2 \\ &= 24x\left(\frac{320}{x^2}\right) + 60x^2 \\ &= \underline{\underline{60x^2 + \frac{7680}{x}}}. \end{aligned}$$

(c) Use calculus to find the minimum value of S. (6)

Solution

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}x} = 0 &\Rightarrow 120x - \frac{7680}{x^2} = 0 \\ &\Rightarrow 120x = \frac{7680}{x^2} \\ &\Rightarrow x^3 = 64 \\ &\Rightarrow x = 4 \\ &\Rightarrow \underline{\underline{S = 2880}}. \end{aligned}$$

(d) Justify, by further differentiation, that the value of S you have found is a minimum. (2)

Solution

$$\frac{\mathrm{d}S}{\mathrm{d}x} = 120x - \frac{7680}{x^2} \Rightarrow \frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 120 + \frac{15\,360}{x^3},$$

and, when $x = 4$, $\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 360 > 0$, which is a minimum.

22. Figure 8 shows the plan of a pool.

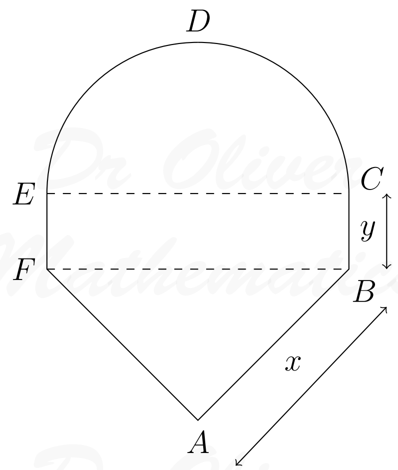

Figure 8: the pool

The shape of the pool $ABCDEFA$ consists of a rectangle $BCEF$ joined to an equilateral triangle BFA and a semi-circle CDE, as shown in the figure.

Given that $AB = x$ metres, $EF = y$ metres, and that the area of the pool is 50 m^2,

(a) show that (3)

$$y = \frac{50}{x} - \frac{x}{8}(\pi + 2\sqrt{3}).$$

Solution

$$
\begin{aligned}
50 &= \frac{\sqrt{3}}{2}x^2 + xy + \tfrac{1}{2}\pi(\tfrac{1}{2}x)^2 \\
\Rightarrow \quad xy &= 50 - \frac{\sqrt{3}}{2}x^2 - \tfrac{1}{8}\pi x^2 \\
\Rightarrow \quad y &= \frac{50}{x} - \frac{\sqrt{3}}{2}x - \tfrac{1}{8}\pi x \\
\Rightarrow \quad y &= \frac{50}{x} - \frac{x}{8}(\pi + 2\sqrt{3}).
\end{aligned}
$$

(b) Hence show that the perimeter, P metres, of the pool is given by (3)

$$P = \frac{100}{x} + \frac{x}{4}(\pi + 8 - 2\sqrt{3}).$$

Solution

$$
\begin{aligned}
P &= \pi(\tfrac{1}{2}x) + 2y + 2x \\
&= \tfrac{1}{2}\pi x + 2\left(\frac{50}{x} - \frac{x}{8}(\pi + 2\sqrt{3})\right) + 2x \\
&= \tfrac{1}{2}\pi x + \frac{100}{x} - \frac{x}{4}(\pi + 2\sqrt{3}) + 2x \\
&= \frac{100}{x} - \frac{x}{4}(\pi + 2\sqrt{3} - 8 - 2\pi) \\
&= \frac{100}{x} - \frac{x}{4}(-\pi + 2\sqrt{3} - 8) \\
&= \frac{100}{x} + \frac{x}{4}(\pi + 8 - 2\sqrt{3}).
\end{aligned}
$$

(c) Use calculus to find the minimum value of P, giving your answer to 3 significant figures. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}x} = 0 &\Rightarrow -\frac{100}{x^2} + \tfrac{1}{4}(\pi + 8 - 2\sqrt{3}) = 0 \\ &\Rightarrow \frac{100}{x^2} = \tfrac{1}{4}(\pi + 8 - 2\sqrt{3}) \\ &\Rightarrow x^2 = \frac{400}{\pi + 8 - 2\sqrt{3}} \\ &\Rightarrow x = \sqrt{\frac{400}{\pi + 8 - 2\sqrt{3}}} \\ &\Rightarrow P = 27.708\,285\,83 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{P = 27.7 \text{ (3 sf)}}},\end{aligned}$$

as $x = -\sqrt{\dfrac{400}{\pi + 8 - 2\sqrt{3}}}$ makes no sense.

(d) Justify, by further differentiation, that the value of P you have found is a minimum. (2)

Solution

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{100}{x^2} + \tfrac{1}{4}(\pi + 8 - 2\sqrt{3}) \Rightarrow \frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{200}{x^3},$$

and, $x = \sqrt{\dfrac{400}{\pi + 8 - 2\sqrt{3}}}$, we have $\dfrac{\mathrm{d}^2P}{\mathrm{d}x^2} = 0.531\ldots > 0$ which is a $\underline{\underline{\text{minimum}}}$.

23. A solid glass cylinder, which is used in an expensive laser amplifier, has a volume of 75π cm^3.

The cost of polishing the surface area of this grass cylinder is £2 per cm^2 for the curved surface area and £3 per cm^2 for the circular top and base areas.

Given that the radius of the cylinder is r cm,

(a) show that the cost of the polishing, £C, is given by (4)

$$C = 6\pi r^2 + \frac{300\pi}{r}.$$

Solution

$$75\pi = \pi r^2 h \Rightarrow h = \frac{75}{r^2}$$

and we have

$$\begin{aligned} C &= 2 \times 2\pi r h + 3 \times \pi r^2 + 3 \times \pi r^2 \\ &= 6\pi r^2 + 4\pi r \left(\frac{75}{r^2}\right) \\ &= \underline{\underline{6\pi r^2 + \frac{300\pi}{r}}}. \end{aligned}$$

(b) Use calculus to find the minimum cost of the polishing, giving your answer to the nearest pound. (5)

Solution

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}r} = 0 &\Rightarrow 12\pi r - \frac{300\pi}{r^2} = 0 \\ &\Rightarrow 12r = \frac{300}{r^2} \\ &\Rightarrow r^3 = 25 \\ &\Rightarrow r = \sqrt[3]{25} \\ &\Rightarrow C = 483.484\,308\,5 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{C = 483 \text{ (nearest pound)}}}. \end{aligned}$$

(c) Justify that the answer that you have obtained in part (b) is a minimum. (1)

Solution

$$\frac{\mathrm{d}C}{\mathrm{d}r} = 12\pi r - \frac{300\pi}{r^2} \Rightarrow \frac{\mathrm{d}^2C}{\mathrm{d}r^2} = 12\pi + \frac{600\pi}{r^3},$$

and, when $r = \sqrt[3]{25}$, $\frac{\mathrm{d}^2C}{\mathrm{d}r^2} = 36\pi > 0$ and this is a minimum.

24. Figure 9 shows a plan of a sheep enclosure.

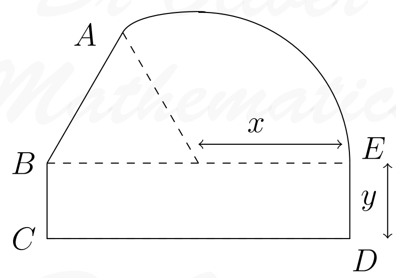

Figure 9: the sheep enclosure

The enclosure $ABCDEA$, as shown in the figure, consists of a rectangle $BCDE$ joined to an equilateral triangle BFA and a sector FEA of a circle with radius x metres and centre F.

The points B, F, and E lie on a straight line with $FE = x$ metres and $10 \leqslant x \leqslant 25$.

(a) Find, in m^2, the exact area of the sector FEA, giving your answer in terms of x, in its simplest form. (2)

Solution

It is one-third of a circle and so the area of $FEA = \underline{\underline{\tfrac{1}{3}\pi x^2}}$.

Given that $BC = y$ metres, where $y > 0$, and the area of the enclosure is 1000 m^2,

(b) show that (3)

$$y = \frac{500}{x} - \frac{x}{24}(4\pi + 3\sqrt{3}).$$

Solution

$BF = AB = AF = FE = x$ and now

$$\begin{aligned} 1000 = 2xy + \frac{\sqrt{3}}{4}x^2 + \tfrac{1}{3}\pi x^2 &\Rightarrow 2xy = 1000 - \frac{\sqrt{3}}{4}x^2 - \tfrac{1}{3}\pi x^2 \\ &\Rightarrow y = \frac{500}{x} - \frac{\sqrt{3}}{8}x - \tfrac{1}{6}\pi x \\ &\Rightarrow \underline{\underline{y = \frac{500}{x} - \frac{x}{24}(4\pi + 3\sqrt{3})}}. \end{aligned}$$

(c) Hence show that the perimeter P metres of the enclosure is given by (3)

$$P = \frac{1000}{x} + \frac{x}{12}(4\pi + 36 - 3\sqrt{3}).$$

Solution

$$\begin{aligned} P &= 2y + x + \tfrac{2}{3}\pi x + 2x \\ &= \frac{1000}{x} - \frac{x}{12}(4\pi + 3\sqrt{3}) + x + \tfrac{2}{3}\pi x + 2x \\ &= \frac{1000}{x} - \frac{x}{12}(4\pi + 3\sqrt{3}) + \frac{x}{12}(12 + 8\pi + 24) \\ &= \frac{1000}{x} - \frac{x}{12}(4\pi + 3\sqrt{3}) + \frac{x}{12}(36 + 8\pi) \\ &= \underline{\underline{\frac{1000}{x} + \frac{x}{12}(4\pi + 36 - 3\sqrt{3})}}. \end{aligned}$$

(d) Use calculus to find the minimum value for P, giving your answer to the nearest metre. (5)

Solution

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}x} = 0 &\Rightarrow -\frac{1000}{x^2} + \frac{1}{12}(4\pi + 36 - 3\sqrt{3}) = 0 \\ &\Rightarrow \frac{1000}{x^2} = \frac{1}{12}(4\pi + 36 - 3\sqrt{3}) \\ &\Rightarrow x^2 = \frac{12000}{4\pi + 36 - 3\sqrt{3}} \\ &\Rightarrow x = \sqrt{\frac{12000}{4\pi + 36 - 3\sqrt{3}}} \\ &\Rightarrow P = 120.236\,181\,7 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{P = 120 \text{ (nearest metre)}}}. \end{aligned}$$

(e) Justify, by further differentiation, that the value of P you have found is a minimum. (2)

Solution

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{1000}{x^2} + \frac{1}{12}(4\pi + 36 - 3\sqrt{3}) \Rightarrow \frac{\mathrm{d}^2P}{\mathrm{d}x^2} = \frac{2000}{x^3},$$

and, as $x = \sqrt{\dfrac{12000}{4\pi + 36 - 3\sqrt{3}}}$, we have

$$\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = 0.434\ldots > 0$$

and so we have a minimum.

25. Figure 10 shows a sketch of part of the curve with equation (3)

$$y = 4x^3 + 9x^2 - 30x - 8, \; -0.5 \leqslant x \leqslant 2.2.$$

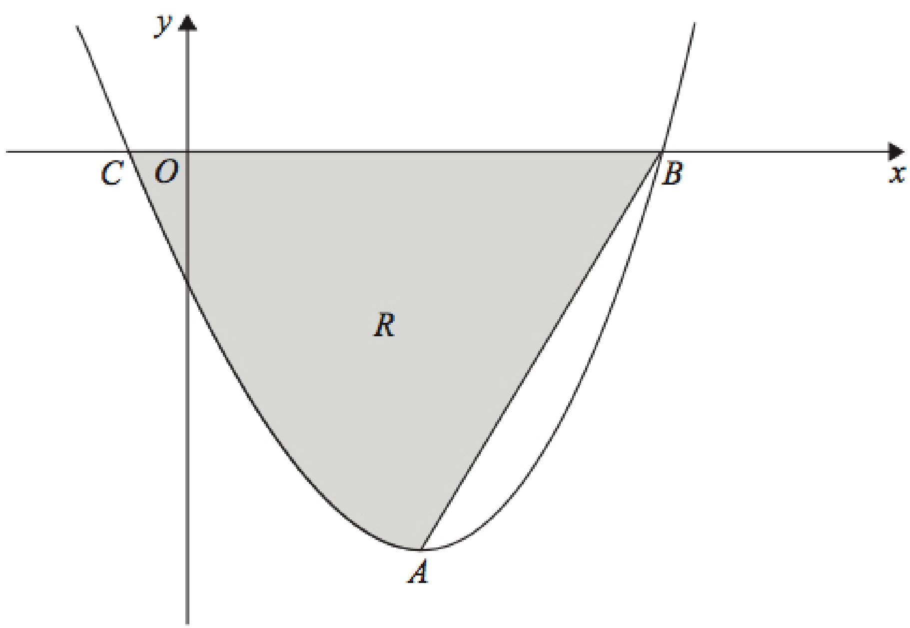

Figure 10: $y = 4x^3 + 9x^2 - 30x - 8$

The curve has a turning point at the point A. Using calculus, show that the x-coordinate of A is 1.

Solution

$$y = 4x^3 + 9x^2 - 30x - 8 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 18x - 30$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 12x^2 + 18x - 30 = 0 \\ &\Rightarrow 2x^2 + 3x - 5 = 0 \\ &\Rightarrow (2x+5)(x-1) = 0 \\ &\Rightarrow x = -\tfrac{5}{2} \text{ or } \underline{\underline{x = 1}}.\end{aligned}$$