Dr Oliver Mathematics
Further Mathematics: Core Pure Mathematics 1
June 2022: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1.
$$f(z) = z^3 + az + 52, \text{ where } a \text{ is a real constant.}$$

Given that $2 - 3i$ is a root of the equation $f(z) = 0$,

(a) write down the other complex root. (1)

(b) Hence, (4)

(i) solve completely $f(z) = 0$,

(ii) determine the value of a.

(c) Show all the roots of the equation $f(z) = 0$ on a single Argand diagram. (1)

2. **In this question you must show all stages of your working.** (4)
Solutions relying entirely on calculator technology are not acceptable.

Determine the values of x for which

$$64\cosh^4 x - 64\cosh^2 x - 9 = 0.$$

Give your answers in the form $q \ln 2$ where q is rational and in simplest form.

3. (a) Determine the general solution of the differential equation (3)

$$\cos x \frac{dy}{dx} + y \sin x = e^{2x} \cos^2 x,$$

giving your answer in the form $y = f(x)$.

Given that $y = 3$ when $x = 0$,

(b) determine the smallest positive value of x for which $y = 0$. (3)

4. (a) Use the method of differences to prove that for $n > 2$, (4)

$$\sum_{r=2}^{n} \ln\left(\frac{r+1}{r-1}\right) \equiv \ln\left(\frac{n(n+1)}{2}\right).$$

(b) Hence find the exact value of (3)

$$\sum_{r=51}^{100} \ln\left(\frac{r+1}{r-1}\right).$$

Give your answer in the form $a\ln\left(\frac{b}{c}\right)$, where a, b, and c are integers to be determined.

5.

$$\mathbf{M} = \begin{pmatrix} a & 2 & -3 \\ 2 & 3 & 0 \\ 4 & a & 2 \end{pmatrix},$$

where a is a constant.

(a) Show that $\mathbf{M}$ is non-singular for all values of a. (2)

(b) Determine, in terms of a, $\mathbf{M}^{-1}$. (4)

6. (a) Express as partial fractions (3)

$$\frac{2x^2+3x+6}{(x+1)(x^2+4)}.$$

(b) Hence, show that (4)

$$\int_0^2 \frac{2x^2+3x+6}{(x+1)(x^2+4)}\,\mathrm{d}x = \ln(a\sqrt{2}) + b\pi,$$

where a and b are constants to be determined.

7. Given that $z = a + b\mathrm{i}$ is a complex number where a and b are real constants,

(a) show that $zz^\star$ is a real number. (2)

Given that

- $zz^\star = 18$ and
- $\frac{z}{z^\star} = \frac{7}{9} + \frac{4\sqrt{2}}{9}\mathrm{i}$,

(b) determine the possible complex numbers z. (5)

8. (a) Given show that (5)

$$z^n + \frac{1}{z^n} = 2\cos n\theta,\ n \in \mathbb{N},$$

show that

$$32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$$

Figure 1 shows a solid paperweight with a flat base.

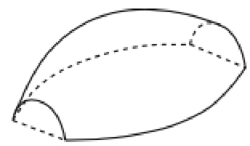

Figure 1: solid paperweight with a flat base

Figure 2 shows the curve with equation

$$y = H\cos^3\left(\tfrac{x}{4}\right),$$

where H is a positive constant and x is in radians.

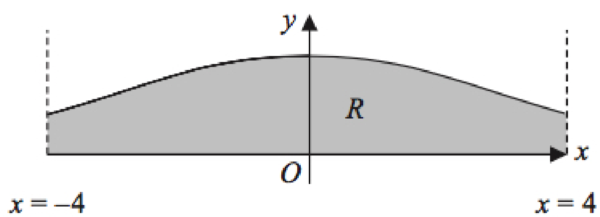

Figure 2: $y = H\cos^3\left(\tfrac{x}{4}\right)$

The region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = -4$, the line with equation $x = 4$, and the x-axis.

The paperweight is modelled by the solid of revolution formed when R is rotated 180° about the x-axis.

Given that the maximum height of the paperweight is 2 cm,

(b) write down the value of H. (1)

(c) Using algebraic integration and the result in part (a), determine, in cm^3, the volume of the paperweight, according to the model. (5)
Give your answer to 2 decimal places.
(Solutions based entirely on calculator technology are not acceptable.)

(d) State a limitation of the model. (1)

9. (a) Explain why (1)

$$\int_0^\infty \cosh x \, \mathrm{d}x$$

is an improper integral.

(b) Show that (3)

$$\int_0^\infty \cosh x \, \mathrm{d}x$$

is divergent.

$$4 \sinh x = p \cosh x,$$

where p is a real constant.

(c) Given that this equation has real solutions, determine the range of possible values for p. (2)

10. The motion of a pendulum, shown in Figure 3, is modelled by the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 9\theta = \tfrac{1}{2}\cos 3t,$$

where θ is the angle, in radians, that the pendulum makes with the downward vertical, t seconds after it begins to move.

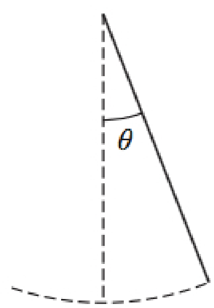

Figure 3: the motion of a pendulum

(a) (i) Show that a particular solution of the differential equation is (4)

$$\theta = \tfrac{1}{12} t \sin 3t.$$

(ii) Hence, find the general solution of the differential equation. (4)

Initially, the pendulum

- makes an angle of $\frac{1}{3}\pi$ radians with the downward vertical and
- is at rest.

Given that, 10 seconds after it begins to move, the pendulum makes an angle of α radians with the downward vertical,

(b) determine, according to the model, the value of α to 3 significant figures. (4)

Given that the true value of α is 0.62,

(c) evaluate the model. (1)

The differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 9\theta = \tfrac{1}{2}\cos 3t$$

models the motion of the pendulum as moving with forced harmonic motion.

(d) Refine the differential equation so that the motion of the pendulum is simple harmonic motion. (1)