Dr Oliver Mathematics
Advanced Subsidiary Paper 22: Mechanics
November 2021: Calculator
1 hour 15 minutes

The total number of marks available is 30.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.
(It goes with Paper 21: Statistics)

1. At time $t = 0$, a small stone is thrown vertically upwards with speed 14.7 m s^{-1} from a point A.

   At time $t = T$ seconds, the stone passes through A, moving downwards.

   The stone is modelled as a particle moving freely under gravity throughout its motion.

   Using the model,

   (a) find the value of T, (2)

   Solution

   $s = ?$, $u = 14.7$, $v = ?$, $a = -9.8$, and $t = T$: use $v = u + at$:

   $$\begin{aligned} 14.7 = -14.7 + 9.8T &\Rightarrow 29.4 = 9.8T \\ &\Rightarrow \underline{\underline{T = 3}}. \end{aligned}$$

   (b) find the total distance travelled by the stone in the first 4 seconds of its motion. (4)

   Solution

   The stone goes upwards for 1.5 seconds and then goes back down for 2.5 seconds. Now,

   $$\begin{aligned} \text{total distance} &= s_{0 \leqslant t \leqslant 1.5} + s_{1.5 \leqslant t \leqslant 4} \\ &= \left[\tfrac{1}{2}(1.5)(14.7 + 0)\right] + (0 + \tfrac{1}{2} \times 9.8 \times 2.5^2) \\ &= 11\tfrac{1}{40} + 30\tfrac{5}{6} \\ &= 41\tfrac{13}{20} \text{ (exact!)} \\ &= \underline{\underline{41.7 \text{ m (3 sf)}}}. \end{aligned}$$

(c) State one refinement that could be made to the model, apart from air resistance, that would make the model more realistic. (1)

Solution

E.g., take account of the dimensions of the stone, allow for spin, do not model the stone as a particle, use a more accurate value for g.

2. A particle P moves along a straight line.

At time t seconds, the velocity v m s^{-1} of P is modelled as

$$v = 10t - t^2 - k,\ k > 0,$$

where k is a constant.

(a) Find the acceleration of P at time t seconds. (2)

Solution

$$v = 10t - t^2 - k \Rightarrow \underline{\underline{a = (10 - 2t)\ \mathrm{m\,s^{-2}}}}.$$

The particle P is instantaneously at rest when $t = 6$.

(b) Find the other value of t when P is instantaneously at rest. (4)

Solution

$$\begin{aligned} t = 6, v = 0 &\Rightarrow 0 = 60 - 36 - k \\ &\Rightarrow k = 24; \end{aligned}$$

so,

$$\begin{aligned} v &= 10t - t^2 - 24 \\ &= -(t^2 - 10t + 24) \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -10 \\ \text{multiply to:} & +24 \end{array}\right\} -6,\ -4$$

$$= -(t-6)(t-4).$$

Finally,

$$\begin{aligned} v = 0 &\Rightarrow -(t-6)(t-4) = 0 \\ &\Rightarrow t = 4 \text{ or } t = 6; \end{aligned}$$

the other value is $\underline{\underline{t = 4}}$.

(c) Find the total distance travelled by P in the interval $0 \leqslant t \leqslant 6$. (4)

Solution

$$v = 10t - t^2 - 24 \Rightarrow s = 5t^2 - \tfrac{1}{3}t^3 - 24t + c,$$

for some constant$c$. Now,

$$\begin{aligned} s_1 &= \left[5t^2 - \tfrac{1}{3}t^3 - 24t\right]_{t=0}^{4} \\ &= (80 - 21\tfrac{1}{3} - 96) - (0 - 0 - 0 + c) \\ &= -37\tfrac{1}{3} \end{aligned}$$

and

$$\begin{aligned} s_2 &= \left[5t^2 - \tfrac{1}{3}t^3 - 24t\right]_{t=4}^{6} \\ &= (180 - 72 - 144) - (80 - 21\tfrac{1}{3} - 96) \\ &= 1\tfrac{1}{3}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{total distance} &= 37\tfrac{1}{3} + 1\tfrac{1}{3} \\ &= \underline{\underline{38\tfrac{2}{3} \text{ m}}}. \end{aligned}$$

3. A ball P of mass $2m$ is attached to one end of a string.

The other end of the string is attached to a ball Q of mass $5m$.

The string passes over a fixed pulley.

The system is held at rest with the balls hanging freely and the string taut.

The hanging parts of the string are vertical with P at a height $2h$ above horizontal ground and with Q at a height h above the ground, as shown in Figure 1.

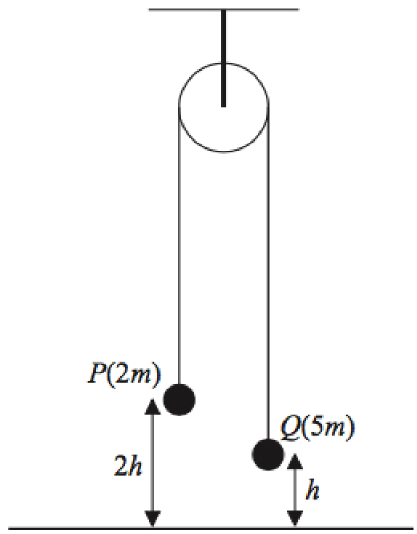

Figure 1: two masses

The system is released from rest.

In the subsequent motion, Q does not rebound when it hits the ground and P does not hit the pulley.

The balls are modelled as particles.

The string is modelled as being light and inextensible.

The pulley is modelled as being small and smooth.

Air resistance is modelled as being negligible.

Using this model,

(a) (i) write down an equation of motion for P, (4)

Solution
Let T be the tension and a be the acceleration (up the LHS and down the RHS). Then

$$\underline{\underline{T - 2mg = 2ma \quad (1).}}$$

(ii) write down an equation of motion for Q,

Solution

$$\underline{\underline{5mg - T = 5ma \quad (2).}}$$

(b) find, in terms of h only, the height above the ground at which P first comes to instantaneous rest. (7)

Solution

Adding (1) + (2):

$$\begin{aligned} 3mg = 7ma &\Rightarrow a = \tfrac{3}{7}g \\ &\Rightarrow T - 2mg = 2m(\tfrac{3}{7}g) \\ &\Rightarrow T - 2mg = \tfrac{6}{7}mg \\ &\Rightarrow T = \tfrac{20}{7}mg. \end{aligned}$$

Now, $s = h$, $u = 0$, $v = ?$, $a = \frac{3}{7}g$, and $t = ?$:
use $v^2 = u^2 + 2as$:

$$\begin{aligned} v^2 = u^2 + 2as &\Rightarrow v^2 = 0 + 2 \times \tfrac{3}{7}g \times h \\ &\Rightarrow v^2 = \tfrac{6}{7}gh \\ &\Rightarrow v = \sqrt{\tfrac{6}{7}gh}. \end{aligned}$$

Now, the string goes slack and let the the total time for this stage the the journey be H:
$s = ?$, $u = 0$, $v = \sqrt{\frac{6}{7}gh}$, $a = -g$, and $t = H$:
use $v^2 = u^2 + 2as$:

$$\begin{aligned} v^2 = u^2 + 2as \Rightarrow 0 &= \left(\sqrt{\tfrac{6}{7}gh}\right)^2 - 2 \times g \times H \\ &\Rightarrow \tfrac{6}{7}gh = 2gH \\ &\Rightarrow H = \tfrac{3}{7}h. \end{aligned}$$

Finally,

$$\begin{aligned} \text{total height} &= 2h + h + \tfrac{3}{7}h \\ &= \underline{\underline{3\tfrac{3}{7}h \text{ m}}}. \end{aligned}$$

(c) State one limitation of modelling the balls as particles that could affect your answer to part (b). (1)

Solution

E.g., the distance that Q falls to the ground would not be exactly h.

In reality, the string will not be inextensible.

(d) State how this would affect the accelerations of the particles. (1)

Solution

E.g., the accelerations of the balls would not have equal magnitude.