Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2004 November Paper 1: Calculator
2 hours

The total number of marks available is 80.
You must write down all the stages in your working.

1. The position vectors of points A, B, and C, relative to an origin O, are $\mathbf{i} + 9\mathbf{j}$, $5\mathbf{i} - 3\mathbf{j}$, and $k(\mathbf{i} + 3\mathbf{j})$ respectively, where k is a constant. (4)

   Given that C lies on the line AB, find the value of k.

Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= \begin{pmatrix} 5 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -12 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{AC} &= \begin{pmatrix} k \\ 3k \end{pmatrix} - \begin{pmatrix} 1 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} k-1 \\ 3k-9 \end{pmatrix}\end{aligned}$$

so

$$\begin{pmatrix} k-1 \\ 3k-9 \end{pmatrix} = n \begin{pmatrix} 4 \\ -12 \end{pmatrix},$$

for some constant n. Now,

$$k - 1 = 4n \Rightarrow 3k - 3 = 12n \quad (1)$$

and

$$3k - 9 = -12n \quad (2).$$

Add (1) + (2):

$$\begin{aligned}6k - 12 = 0 &\Rightarrow 6k = 12 \\ &\Rightarrow \underline{\underline{k = 2}}.\end{aligned}$$

2. A youth club has facilities for members to play pool, darts, and table-tennis.

Every member plays at least one of the three games.

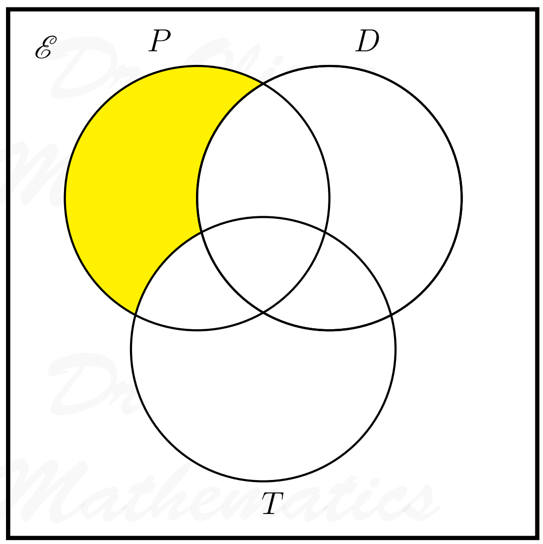
P, D, and T represent the sets of members who play pool, darts, and table-tennis respectively.

Express each of the following in set language and illustrate each by means of a Venn diagram.

(a) The set of members who only play pool. (2)

Solution

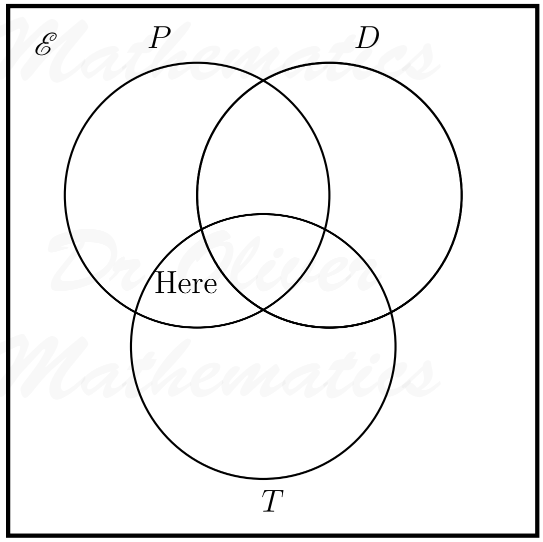
It means $P \cap D' \cap T'$:



(b) The set of members who play exactly 2 games, neither of which is darts. (2)

Solution

It means $P \cap D' \cap T$:



3. Without using a calculator, solve, for x and y, the simultaneous equations (5)

$$8^x \div 2^y = 64,$$
$$3^{4x} \times (\tfrac{1}{9})^{y-1} = 81.$$

Solution

Well,

$$\begin{aligned} 8^x \div 2^y = 64 &\Rightarrow \frac{(2^3)^x}{2^y} = 2^6 \\ &\Rightarrow \frac{2^{3x}}{2^y} = 2^6 \\ &\Rightarrow 2^{3x-y} = 2^6 \\ &\Rightarrow 3x - y = 6 \quad (1) \end{aligned}$$

and

$$\begin{aligned} 3^{4x} \times (\tfrac{1}{9})^{y-1} = 81 &\Rightarrow 3^{4x} \times (3^{-2})^{y-1} = 3^4 \\ &\Rightarrow 3^{4x} \times 3^{-2(y-1)} = 3^4 \\ &\Rightarrow 3^{4x-2(y-1)} = 3^4 \\ &\Rightarrow 4x - 2y + 2 = 4 \\ &\Rightarrow 4x - 2y = 2 \\ &\Rightarrow 2x - y = 1 \quad (2). \end{aligned}$$

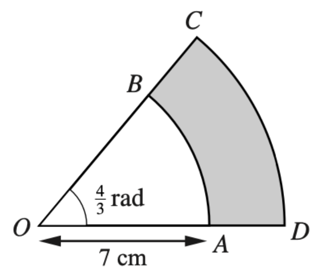
Subtract (1) − (2):

$$\underline{\underline{x = 5}}.$$

Insert $x = 5$ into (1):

$$\begin{aligned} 3(5) - y = 6 &\Rightarrow 15 - y = 6 \\ &\Rightarrow \underline{\underline{y = 9}}. \end{aligned}$$

4. The diagram shows a sector COD of a circle, centre O, in which angle $COD = \frac{4}{3}$ radians. (6)



The points A and B lie on OD and OC respectively, and AB is an arc of a circle, centre O, of radius 7 cm.

Given that the area of the shaded region $ABCD$ is 48 cm^2, find the perimeter of this shaded region.

Solution

Let $x = AD$ cm. Then

$$\begin{aligned} \text{sector } OAB &= \tfrac{1}{2} \times 7^2 \times \tfrac{4}{3} \\ &= 33\tfrac{2}{3} \text{ cm}^2 \end{aligned}$$

and

$$\begin{aligned} \text{sector } ODC &= \tfrac{1}{2} \times (7 + x)^2 \times \tfrac{4}{3} \\ &= \tfrac{2}{3}(7 + x)^2 \text{ cm}^2. \end{aligned}$$

Now,

$$\begin{aligned} & \text{sector } ODC - \text{sector } OAB = 48 \\ \Rightarrow\ & \tfrac{2}{3}(7+x)^2 - 33\tfrac{2}{3} = 48 \\ \Rightarrow\ & \tfrac{2}{3}(7+x)^2 = 80\tfrac{2}{3} \\ \Rightarrow\ & (7+x)^2 = 121 \\ \Rightarrow\ & 7+x = 11 \\ \Rightarrow\ & x = 4. \end{aligned}$$

Next,

$$\begin{aligned} \text{arc } AB &= \tfrac{4}{3} \times 7 \\ &= 9\tfrac{1}{3} \end{aligned}$$

and

$$\begin{aligned} \text{arc } CD &= \tfrac{4}{3} \times 11 \\ &= 14\tfrac{2}{3}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{perimeter} &= 9\tfrac{1}{3} + 14\tfrac{2}{3} + (2 \times 4) \\ &= \underline{\underline{32 \text{ cm}}}. \end{aligned}$$

5. Given that the expansion of (6)

$$(a+x)(1-2x)^n$$

in ascending powers of x is

$$3 - 41x + bx^2 + \ldots,$$

find the values of the constants a, n, and b.

Solution

$$\begin{aligned} (a+x)(1-2x)^n &= (a+x)\left[1 + (-2)(n)x + \tfrac{1}{2}n(n-1)(-2)^2x^2 + \ldots\right] \\ &= (a+x)\left[1 - 2nx + 2n(n-1)x^2 + \ldots\right] \end{aligned}$$

$\times$	1	$-2nx$	$+2n(n-1)x^2$
a	a	$-2anx$	$+2an(n-1)x^2$
$+x$	$+x$	$-2nx^2$	$\ldots$

$$= a + (1 - 2an)x + [2an(n-1) - 2n]x^2 + \ldots$$

So, $\underline{\underline{a = 3}}$,

$$\begin{aligned} 1 - 2(3)n = -41 &\Rightarrow 1 - 6n = -41 \\ &\Rightarrow -6n = -42 \\ &\Rightarrow \underline{\underline{n = 7}}, \end{aligned}$$

and

$$\begin{aligned} b &= 2(3)(7)(6) - 2(7) \\ &= 252 - 14 \\ &= \underline{\underline{238}}. \end{aligned}$$

6. The function f is defined, for $0 < x < \pi$,by

$$\mathrm{f}(x) = 5 + 3\cos 4x.$$

Find

(a) the amplitude and the period of f, (2)

Solution

The amplitude is $\underline{\underline{3}}$ and the period is

$$\frac{2\pi}{4} = \underline{\underline{\tfrac{1}{2}\pi}}.$$

(b) the coordinates of the maximum and minimum points of the curve $y = \mathrm{f}(x)$. (4)

Solution

Now,

$$\cos 4x = 1 \Rightarrow 4x = 2\pi$$
$$\Rightarrow x = \tfrac{1}{2}\pi$$

and

$$x = 0 \Rightarrow \mathrm{f}(x) = 5 + 3 = 8.$$

Next,

$$\cos 4x = -1 \Rightarrow 4x = \pi,\ 3\pi$$
$$\Rightarrow x = \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi$$

and

$$x = \tfrac{1}{2}\pi \Rightarrow \mathrm{f}(x) = 5 - 3 = 2.$$

Hence, there are one maximum point of the curve — $\underline{\underline{(\frac{1}{2}\pi, 8)}}$ — and two minimum points of the curve — $\underline{\underline{(\frac{1}{2}\pi, 2) \text{ and } (\frac{3}{2}\pi, 2)}}$.

7. (a) Find the number of different arrangements of the 9 letters of the word SINGAPORE in which S does not occur as the first letter. (2)

Solution

No letters are the same:

$$8 \times 8! = \underline{\underline{322\,560 \text{ ways}}}.$$

3 students are selected to form a chess team from a group of 5 girls and 3 boys.

(b) Find the number of possible teams that can be selected in which there are more girls than boys. (4)

Solution

<u>All girls:</u>

$$\binom{5}{3} = 10.$$

<u>1 boys, 2 girls:</u>

$$\binom{5}{2} \times \binom{3}{1} = 30.$$

Hence,

$$10 + 30 = \underline{\underline{40 \text{ ways}}}.$$

8. The function f is defined, for $x \in \mathbb{R}$, by

$$\mathrm{f} : x \mapsto \frac{3x + 11}{x - 3},\ x \neq 3.$$

(a) Find f^{-1} in terms of x and explain what this implies about the symmetry of the graph of $y = \mathrm{f}(x)$. (3)

Solution

$$\begin{aligned} y = \frac{3x+11}{x-3} &\Rightarrow y(x-3) = 3x + 11 \\ &\Rightarrow xy - 3y = 3x + 11 \\ &\Rightarrow xy - 3x = 3y + 11 \\ &\Rightarrow x(y-3) = 3y + 11 \\ &\Rightarrow x = \frac{3y+11}{y-3} \end{aligned}$$

and, hence,

$$\underline{\underline{\mathrm{f}^{-1}(x) = \frac{3x+11}{x-3}}}.$$

Now, $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ and the graph has the line $y = x$ as line of symmetry.

The function g is defined, for $x \in \mathbb{R}$, by

$$\mathrm{g} : x \mapsto \frac{x-3}{2}.$$

(b) Find the values of x for which $\mathrm{f}(x) = \mathrm{g}^{-1}(x)$. (3)

Solution

$$\begin{aligned} y = \frac{x-3}{2} &\Rightarrow 2y = x - 3 \\ &\Rightarrow 2y + 3 = x \end{aligned}$$

and

$$\mathrm{g}^{-1}(x) = 2x + 3.$$

Finally,

$$\begin{aligned} f(x) = g^{-1}(x) &\Rightarrow \frac{3x+11}{x-3} = 2x+3 \\ &\Rightarrow 3x+11 = (2x+3)(x-3) \end{aligned}$$

×	$2x$	$+3$
x	$2x^2$	$+3x$
-3	$-6x$	-9

$$\begin{aligned} &\Rightarrow 3x+11 = 2x^2-3x-9 \\ &\Rightarrow 2x^2-6x-20 = 0 \\ &\Rightarrow 2(x^2-3x-10) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -3 \\ \text{multiply to:} & -10 \end{array}\right\} -5, +2$$

$$\begin{aligned} &\Rightarrow 2(x-5)(x+2) = 0 \\ &\Rightarrow \underline{\underline{x = 5 \text{ or } x = -2}}. \end{aligned}$$

(c) State the value of x for which $g\,f(x) = -2$. (1)

Solution
Well,

$$\begin{aligned} g\,f(x) = -2 &\Rightarrow f(x) = g^{-1}(-2) \\ &\Rightarrow \underline{\underline{x = -2}}. \end{aligned}$$

9. (a) Solve, for $0^\circ \leqslant x \leqslant 360^\circ$, the equation (4)

$$\sin^2 x = 3\cos^2 x + 4\sin x.$$

Solution

Now,

$$\begin{aligned}\sin^2 x = 3\cos^2 x + 4\sin x &\Rightarrow \sin^2 x = 3(1-\sin^2 x) + 4\sin x\\ &\Rightarrow \sin^2 x = 3 - 3\sin^2 x + 4\sin x\\ &\Rightarrow 4\sin^2 x - 4\sin x - 3 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -4\\ \text{multiply to:} & (+4)\times(-3) = -12\end{array}\right\} -6,\ +2$$

e.g.,

$$\begin{aligned}&\Rightarrow 4\sin^2 x - 6\sin x + 2\sin x - 3 = 0\\ &\Rightarrow 2\sin x(2\sin x + 3) + 1(2\sin x - 3) = 0\\ &\Rightarrow (2\sin x + 1)(2\sin x + 3) = 0\\ &\Rightarrow \sin x = -\tfrac{1}{2} \text{ or } \sin x = -\tfrac{3}{2} \text{ (no!).}\end{aligned}$$

Hence,

$$\begin{aligned}\sin x = -\tfrac{1}{2} &\Rightarrow x = -30 \text{ (not in range)}\\ &\Rightarrow \underline{\underline{x = 210,\ 330}}\end{aligned}$$

(b) Solve, for $0 < y < 4$, the equation (4)

$$\cot 2y = 0.25,$$

giving your answer radians correct to 2 decimal places.

Solution

$$\begin{aligned}&\cot 2y = 0.25\\ \Rightarrow\ &\tan 2y = 4\\ \Rightarrow\ &2y = 1.325\,817\,664,\ 4.467\,410\,317,\ 7.609\,002\,971 \text{ (FCD)}\\ \Rightarrow\ &y = 0.662\,908\,831\,8,\ 2.233\,705\,159,\ 3.804\,450\,485 \text{ (FCD)}\\ \Rightarrow\ &\underline{\underline{y = 0.663,\ 2.23,\ \text{or } 3.80 \text{ (3 sf)}}}\end{aligned}$$

10. A curve has the equation

$$y = x^3 \ln x,$$

where $x > 0$.

(a) Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (2)

Solution

Product rule:

$$u = x^3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$$
$$v = \ln x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x}$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x^3)\left(\frac{1}{x}\right) + (\ln x)(3x^2) \\ &= x^2 + 3x^2 \ln x \\ &= \underline{\underline{x^2(1 + 3\ln x)}}.\end{aligned}$$

Hence

(b) calculate the value of $\ln x$ at the stationary point of the curve, (2)

Solution

As $x > 0$,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow x^2(1 + 3\ln x) = 0 \\ &\Rightarrow 1 + 3\ln x = 0 \\ &\Rightarrow 3\ln x = -1 \\ &\Rightarrow \underline{\underline{\ln x = -\tfrac{1}{3}}}.\end{aligned}$$

(c) find the approximate increase in y as x increases from e to $e + p$, where p is small, (2)

Solution

$$\begin{aligned}\delta y &= \frac{\mathrm{d}y}{\mathrm{d}x} \times \delta x \\ &= (x^2 + 3x^2 \ln x)_{x=p} \times p \\ &= (\mathrm{e}^2 + 3\mathrm{e}^2) \times p \\ &= \underline{\underline{4\mathrm{e}^2 p}}.\end{aligned}$$

(d) find (3)

$$\int x^2 \ln x \,\mathrm{d}x.$$

Solution

Well,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + 3x^2 \ln x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} - x^2 &= 3x^2 \ln x \\ \Rightarrow \tfrac{1}{3}\frac{\mathrm{d}y}{\mathrm{d}x} - \tfrac{1}{3}x^2 &= x^2 \ln x\end{aligned}$$

and so

$$\begin{aligned}\int x^2 \ln x \,\mathrm{d}x &= \int \left(\tfrac{1}{3}\frac{\mathrm{d}y}{\mathrm{d}x} - \tfrac{1}{3}x^2\right) \mathrm{d}x \\ &= \tfrac{1}{3}y - \tfrac{1}{3}\int x^2 \,\mathrm{d}x \\ &= \underline{\underline{\tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + c}}.\end{aligned}$$

11. The line (9)

$$4y = 3x + 1$$

intersects the curve

$$xy = 28x - 27y$$

at the point $P(1, 1)$ and at the point Q.

The perpendicular bisector of PQ intersects the line $y = 4x$ at the point R.

Calculate the area of triangle PQR.

Solution

Well,

$$4y = 3x + 1 \Rightarrow y = \tfrac{1}{4}(3x + 1)$$

and insert this into the curve:

$$xy = 28x - 27y \Rightarrow x[\tfrac{1}{4}(3x + 1)] = 28x - 27[\tfrac{1}{4}(3x + 1)]$$

multiply by 4:

$$\begin{aligned}
&\Rightarrow x(3x + 1) = 112x - 27(3x + 1) \\
&\Rightarrow 3x^2 + x = 112x - 81x - 27 \\
&\Rightarrow 3x^2 - 30x + 27 = 0 \\
&\Rightarrow 3(x^2 - 10x + 9 = 0)
\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -10 \\ \text{multiply to:} & +9\end{array}\right\} -9,\ -1$$

$$\begin{aligned}
&\Rightarrow 3(x - 9)(x - 1) = 0 \\
&\Rightarrow x = 9 \text{ or } x = 1.
\end{aligned}$$

Now,

$$x = 9 \Rightarrow y = \tfrac{1}{4}(3 \times 9 + 1) = 7,$$

so $Q(9, 7)$.

Next,

$$\begin{aligned}
m_{PQ} &= \frac{7 - 1}{9 - 1} \\
&= \tfrac{3}{4}
\end{aligned}$$

which means

$$m_{\text{normal}} = -\frac{1}{\frac{3}{4}} = -\tfrac{4}{3}.$$

Let M be the midpoint of PQ:

$$\left(\frac{1 + 9}{2}, \frac{1 + 7}{2}\right) = M(5, 4).$$

The equation is

$$\begin{aligned}
y - 4 = -\tfrac{4}{3}(x - 5) &\Rightarrow y - 4 = -\tfrac{4}{3}x + \tfrac{20}{3} \\
&\Rightarrow y = -\tfrac{4}{3}x + \tfrac{32}{3}.
\end{aligned}$$

We do a subtraction:

$$\begin{aligned} 4x = -\tfrac{4}{3}x + \tfrac{32}{3} &\Rightarrow \tfrac{16}{3}x = \tfrac{32}{3} \\ &\Rightarrow x = 2 \\ &\Rightarrow y = 8; \end{aligned}$$

so, $R(2, 8)$. Now,

$$\begin{aligned} MR &= \sqrt{(2-5)^2 + (8-4)^2} \\ &= \sqrt{(-3)^2 + 4^2} \\ &= 5 \end{aligned}$$

and

$$\begin{aligned} PQ &= \sqrt{(9-1)^2 + (7-1)^2} \\ &= \sqrt{8^2 + 6^2} \\ &= 10. \end{aligned}$$

Finally,

$$\begin{aligned} \text{area of } PQR &\Rightarrow \tfrac{1}{2} \times MR \times PQ \\ &\Rightarrow \tfrac{1}{2} \times 5 \times 10 \\ &= \underline{\underline{25 \text{ units}^2}}. \end{aligned}$$

EITHER

12. At the beginning of 1960, the number of animals of a certain species was estimated at 20 000.

This number decreased so that, after a period of n years, the population was

$$20\,000\,\mathrm{e}^{-0.05n}.$$

Estimate

(a) (i) the population at the beginning of 1970, (1)

Solution

$$\begin{aligned}\text{Population} &= 20\,000\,\mathrm{e}^{-0.05(10)} \\ &= 20\,000\,\mathrm{e}^{-0.5} \\ &= 12\,130.613\,19 \text{ (FCD)} \\ &= \underline{\underline{12\,100 \text{ (3 sf)}}}.\end{aligned}$$

(ii) the year in which the population would be expected to have first decreased to 2 000. (3)

Solution

$$\begin{aligned}20\,000\,\mathrm{e}^{-0.05n} < 2\,000 &\Rightarrow \mathrm{e}^{-0.05n} < \tfrac{1}{10} \\ &\Rightarrow -0.05n < \ln \tfrac{1}{10} \\ &\Rightarrow n > -20 \ln \tfrac{1}{10} \\ &\Rightarrow n > 46.051\,701\,86 \text{ (FCD)};\end{aligned}$$

hence, it will take

$$1960 + 46 = \underline{\underline{2006}}.$$

(b) Solve the equation (6)

$$3^{x+1} - 2 = 8 \times 3^{x-1}.$$

Solution

Well,

$$3^{x+1} = 3 \cdot 3^x$$

and

$$3^{x-1} = \tfrac{1}{3} \cdot 3^x$$

Now,

$$\begin{aligned}3^{x+1} - 2 = 8 \times 3^{x-1} &\Rightarrow 3 \cdot 3^x - 2 = 8 \times \tfrac{1}{3} \cdot 3^x \\ &\Rightarrow 3 \cdot 3^x - 2 = \tfrac{8}{3} \cdot 3^x \\ &\Rightarrow 3 \cdot 3^x - \tfrac{8}{3} \cdot 3^x = 2 \\ &\Rightarrow \tfrac{1}{3} \cdot 3^x = 2 \\ &\Rightarrow 3^x = 6 \\ &\Rightarrow \underline{\underline{x = \log_3 6 \text{ or } 1.63 \ \text{(3 sf)}}}.\end{aligned}$$

OR

13. A curve has the equation

$$y = e^{\frac{1}{2}x} + 3e^{-\frac{1}{2}x}.$$

(a) Show that the exact value of the y-coordinate of the stationary point of the curve is $2\sqrt{3}$. (4)

Solution
Well,

$$y = e^{\frac{1}{2}x} + 3e^{-\frac{1}{2}x} \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}e^{\frac{1}{2}x} - \tfrac{3}{2}e^{-\frac{1}{2}x}$$

and

$$\frac{dy}{dx} = 0 \Rightarrow \tfrac{1}{2}e^{\frac{1}{2}x} - \tfrac{3}{2}e^{-\frac{1}{2}x} = 0$$

multiply by $2e^{\frac{1}{2}x}$:

$$\begin{aligned} &\Rightarrow e^x - 3 = 0 \\ &\Rightarrow e^x = 3 \\ &\Rightarrow x = \ln 3 \end{aligned}$$

and

$$\begin{aligned} y &= e^{\frac{1}{2}\ln 3} + 3e^{-\frac{1}{2}\ln 3} \\ &= e^{\ln 3^{\frac{1}{2}}} + 3e^{\ln 3^{-\frac{1}{2}}} \\ &= 3^{\frac{1}{2}} + 3(3^{-\frac{1}{2}}) \\ &= \sqrt{3} + \frac{3}{\sqrt{3}} \\ &= \sqrt{3} + \sqrt{3} \\ &= \underline{\underline{2\sqrt{3}}}, \end{aligned}$$

as required.

(b) Determine whether the stationary point is a maximum or a minimum. (2)

Solution
Now,

$$\frac{d^2y}{dx^2} = \tfrac{1}{4}e^{\frac{1}{2}x} + \tfrac{3}{4}e^{-\frac{1}{2}x}$$

and

$$x = \ln 3 \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0.$$

Hence, it is a minimum.

(c) Calculate the area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 1$. (4)

Solution

$$\begin{aligned}\int_0^1 \left(\mathrm{e}^{\frac{1}{2}x} + 3\mathrm{e}^{-\frac{1}{2}x}\right) \mathrm{d}x &= \left[2\mathrm{e}^{\frac{1}{2}x} - 6\mathrm{e}^{-\frac{1}{2}x}\right]_{x=0}^{1} \\ &= \left(2\mathrm{e}^{\frac{1}{2}} - 6\mathrm{e}^{-\frac{1}{2}}\right) - (2 - 6) \\ &= 2\mathrm{e}^{\frac{1}{2}} - 6\mathrm{e}^{-\frac{1}{2}} + 4.\end{aligned}$$