Dr Oliver Mathematics
Mathematics: Higher
2013 Paper 2: Calculator
1 hour 10 minutes

The total number of marks available is 60.
You must write down all the stages in your working.

1. The first three terms of a sequence are 4, 7, and 16. (4)

   The sequence is generated by the recurrence relation

   $$u_{n+1} = mu_n + c, \text{ with } u_1 = 4.$$

   Find the values of m and c.

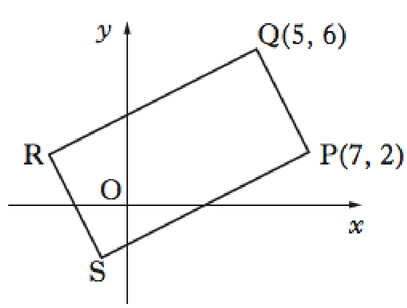
Solution

$$7 = 4m + c \quad (1)$$
$$16 = 7m + c \quad (2).$$

Do (2) − (1):

$$9 = 3m \Rightarrow \underline{\underline{m = 3}}$$
$$\Rightarrow 7 = 12 + c$$
$$\Rightarrow \underline{\underline{c = -5}}.$$

2. The diagram shows rectangle $PQRS$ with $P(7, 2)$ and $Q(5, 6)$.



(a) Find the equation of QR. (3)

Solution

The gradient of PQ is

$$\frac{6-2}{5-7} = \frac{4}{-2} = -2$$

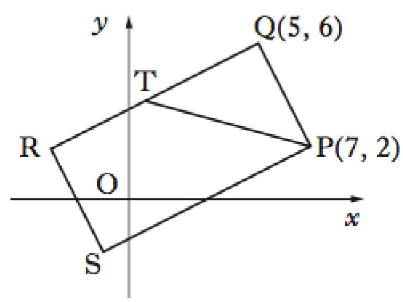
and the gradient of the normal is $\frac{1}{2}$. Hence, the equation of QR is

$$y - 6 = \tfrac{1}{2}(x-5) \Rightarrow y - 6 = \tfrac{1}{2}x - \tfrac{5}{2} \Rightarrow \underline{\underline{y = \tfrac{1}{2}x + \tfrac{7}{2}}}.$$

The line from P with the equation

$$x + 3y = 13$$

intersects QR at T.



(b) Find the coordinates of T. (3)

Solution

$$\begin{aligned} x + 3y = 13 &\Rightarrow x + 3(\tfrac{1}{2}x + \tfrac{7}{2}) = 13 \\ &\Rightarrow \tfrac{5}{2}x + \tfrac{21}{2} = 13 \\ &\Rightarrow \tfrac{5}{2}x = \tfrac{5}{2} \\ &\Rightarrow x = 1 \\ &\Rightarrow y = \tfrac{1}{2}(1) + \tfrac{7}{2} \\ &\Rightarrow y = 4; \end{aligned}$$

hence, $\underline{\underline{T(1,4)}}$.

(c) Given that T is the midpoint of QR, find the coordinates of R and S. (3)

Solution
$\underline{\underline{R(-3,2)}}$ and $\underline{\underline{S(-1,-2)}}$.

3. (a) Given that $(x-1)$ is a factor of (4)

$$x^3 + 3x^2 + x - 5,$$

factorise this cubic fully.

Solution
We use synthetic division.

$$\begin{array}{c|cccc} 1 & 1 & 3 & 1 & -5 \\ & \downarrow & 1 & 4 & 5 \\ \hline & 1 & 4 & 5 & 0 \end{array}$$

Hence,

$$x^3 + 3x^2 + x - 5 = \underline{\underline{(x-1)(x^2+4x+5)}}.$$

[The quadratic does not factorise: $b^2 - 4ac = 4^2 - 4 \times 1 \times 5 = -4$.]

(b) Show that the curve with equation (5)

$$y = x^4 + 4x^3 + 2x^2 - 20x + 3$$

has only one stationary point.

Find the x-coordinate and determine the nature of this point.

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 4x^3 + 12x^2 + 4x - 20 = 0 \\ &\Rightarrow 4(x^3 + 3x^2 + x - 5) = 0 \\ &\Rightarrow x^3 + 3x^2 + x - 5 = 0 \\ &\Rightarrow (x-1)(x^2+4x+5) = 0. \end{aligned}$$

Now, $x = 1$ is a root and $y = -10$. Next,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 + 24x + 4$$

and

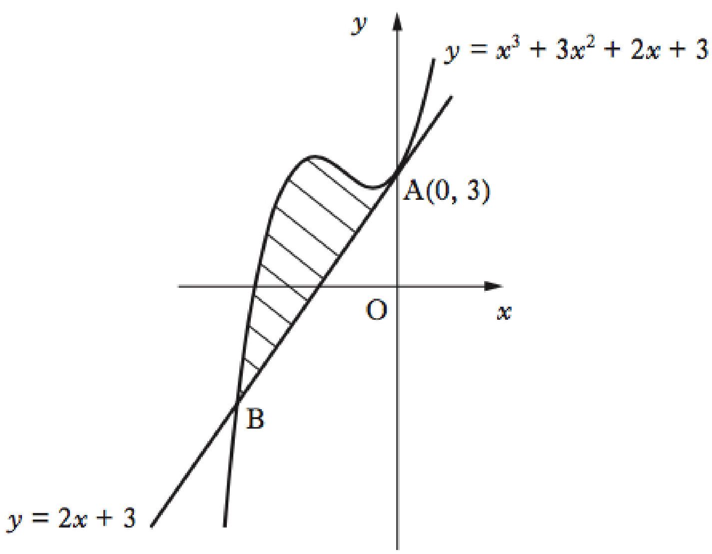
$$x = 1 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 40 > 0$$

which means that (1, −10) is a minimum point.

4. The line with equation $y = 2x + 3$ is a tangent to the curve with equation (6)

$$y = x^3 + 3x^2 + 2x + 3$$

at $A(0, 3)$, as shown in the diagram.



The line meets the curve again at B.

Show that B is the point $(-3, -3)$ and find the area enclosed by the line and the curve.

Solution

$$\begin{aligned} x^3 + 3x^2 + 2x + 3 = 2x + 3 &\Rightarrow x^3 + 3x^2 = 0 \\ &\Rightarrow x^2(x+3) = 0 \\ &\Rightarrow x = -3 \text{ or } x = 0 \text{ (repeated twice).} \end{aligned}$$

Now,

$$x = -3 \Rightarrow y = -3$$

and so $\underline{\underline{B(-3, -3)}}$ is the point.

Finally,

$$\begin{aligned} \int_{-3}^{0} (x^3 + 3x^2)\, \mathrm{d}x &= \left[\tfrac{1}{4}x^4 + x^3\right]_{x=-3}^{0} \\ &= (0+0) - \left(\tfrac{81}{4} - 27\right) \\ &= \underline{\underline{6\tfrac{3}{4}}}. \end{aligned}$$

5. Solve the equation (4)

$$\log_5(3 - 2x) + \log_5(2 + x) = 1,$$

where x is a real number.

Solution

$$\begin{aligned} \log_5(3-2x) + \log_5(2+x) = 1 &\Rightarrow \log_5[(3-2x)(2+x)] = 1 \\ &\Rightarrow \log_5(6 - x - 2x^2) = 1 \\ &\Rightarrow 6 - x - 2x^2 = 5 \\ &\Rightarrow 2x^2 + x - 1 = 0 \\ &\Rightarrow (2x-1)(x+1) = 0 \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2} \text{ or } x = -1}}. \end{aligned}$$

6. Given that (5)

$$\int_0^a 5 \sin 3x \, \mathrm{d}x = \tfrac{10}{3},\ 0 \leqslant a < \pi,$$

calculate the value of a.

Solution

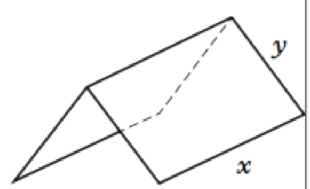
$$\int_0^a 5\sin 3x \,\mathrm{d}x = \tfrac{10}{3} \Rightarrow \left[-\tfrac{5}{3}\cos 3x\right]_{x=0}^{a} = \tfrac{10}{3}$$
$$\Rightarrow -\tfrac{5}{3}\cos 3a + \tfrac{5}{3} = \tfrac{10}{3}$$
$$\Rightarrow -\tfrac{5}{3}\cos 3a = \tfrac{5}{3}$$
$$\Rightarrow \cos 3a = -1$$
$$\Rightarrow 3a = \pi$$
$$\Rightarrow \underline{\underline{a = \tfrac{1}{3}\pi}}.$$

7. A manufacturer is asked to design an open-ended shelter, as shown, subject to the following conditions.

Condition 1

The frame of a shelter is to be made of rods of two different lengths:

- x metres for top and bottom edges;
- y metres for each sloping edge.



Condition 2

The frame is to be covered by a rectangular sheet of material.

The total area of the sheet is $24\,\mathrm{m}^2$.

(a) Show that the total length, L metres, of the rods used in a shelter is given by (3)

$$L = 3x + \frac{48}{x}.$$

Solution

$$24 = x(2y) \Rightarrow y = \frac{12}{x}$$

and

$$\begin{aligned} L &= 3x + 4y \text{ (where is the third fold?)} \\ &= \underline{\underline{3x + \frac{48}{x}}}, \end{aligned}$$

as required.

These rods cost £8.25 per metre.

To minimise production costs, the total length of rods used for a frame should be as small as possible.

(b) (i) Find the value of x for which L is a minimum. (7)

Solution

$$L = 3x + \frac{48}{x} \Rightarrow L = 3x + 48x^{-1}$$
$$\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}x} = 3 - 48x^{-2}$$

and

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}x} = 0 &\Rightarrow 3 - 48x^{-2} = 0 \\ &\Rightarrow 3 = 48x^{-2} \\ &\Rightarrow x^2 = 16 \\ &\Rightarrow \underline{\underline{x = 4}}. \end{aligned}$$

(ii) Calculate the minimum cost of a frame.

Solution

$$\text{Minimum cost} = 8.25\left(12 + \frac{48}{4}\right) = \underline{\underline{£198}}.$$

8. Solve algebraically the equation (6)

$$\sin 2x = 2\cos^2 x \text{ for } 0 \leqslant x < 2\pi.$$

Solution

$$\begin{aligned}\sin 2x = 2\cos^2 x &\Rightarrow 2\sin x\cos x = 2\cos^2 x\\ &\Rightarrow 2\cos^2 x - 2\sin x\cos x = 0\\ &\Rightarrow 2\cos x(\cos x - \sin x) = 0\\ &\Rightarrow \cos x = 0 \text{ or } \cos x - \sin x = 0\\ &\Rightarrow \cos x = 0 \text{ or } \cos x = \sin x\\ &\Rightarrow \cos x = 0 \text{ or } \tan x = 1\\ &\Rightarrow \underline{\underline{x = \tfrac{1}{2}\pi, \tfrac{3}{2}\pi \text{ or } x = \tfrac{1}{4}\pi, \tfrac{5}{4}\pi}}.\end{aligned}$$

9. The concentration of the pesticide, *Xpesto*, in soil can be modelled by the equation

$$P_t = P_0 e^{-kt},$$

where:

- P_0 is the initial concentration;
- P_t is the concentration at time t; and
- t is the time, in days, after the application of the pesticide.

Once in the soil, the half-life of a pesticide is the time taken for its concentration to be reduced to one half of its initial value.

(a) If the half-life of *Xpesto* is 25 days, find the value of k to 2 significant figures. (4)

Solution

$$\begin{aligned}e^{-25k} = \tfrac{1}{2} &\Rightarrow -25k = \ln\tfrac{1}{2}\\ &\Rightarrow k = -\tfrac{1}{25}\ln\tfrac{1}{2}\\ &\Rightarrow k = 0.027\,725\,887\,22 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{k = 0.028 \text{ (2 sf)}}}.\end{aligned}$$

(b) Eighty days after the initial application, what is the percentage decrease in concentration of *Xpesto*? (3)

Solution
Well,

$$\begin{aligned} P_{80} &= P_0 \mathrm{e}^{-80\times 0.027\ldots} \\ &= P_0 \times 0.108\,818\,820\,4 \text{ (FCD)} \end{aligned}$$

and so the percentage decrease is

$$\begin{aligned} 100\left[1 - \frac{P_0}{P_{80}}\right] \% &= 100\,[1 - 0.108\ldots]\ \% \\ &= 89.118\,117\,96 \text{ (FCD)} \\ &= \underline{\underline{89.1\% \ \ (3 \text{ sf})}}. \end{aligned}$$