Dr Oliver Mathematics
Mathematics: Advanced Higher
2024 Paper 1: Non-Calculator
1 hour

The total number of marks available is 35.
You must write down all the stages in your working.

1. Differentiate the following with respect to x:

   (a) $y = \cot 3x$, (2)

   Solution

   $$y = \cot 3x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\operatorname{cosec}^2 3x \times 3$$
   $$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -3\operatorname{cosec}^2 3x}}.$$

   (b) $\mathrm{f}(x) = 5x(4x-7)^{\frac{1}{2}}$. (2)

   Solution

   Product rule:

   $$u = 5x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 5$$
   $$v = (4x-7)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2(2x-7)^{-\frac{1}{2}}$$

   and

   $$\mathrm{f}'(x) = (5x)[2(2x-7)^{-\frac{1}{2}}] + (5)[(4x-7)^{\frac{1}{2}}]$$
   $$= \underline{\underline{10x(2x-7)^{-\frac{1}{2}} + 5(4x-7)^{\frac{1}{2}}}}.$$

2. A complex number is defined by

   $$z = 1 + \mathrm{i}.$$

   (a) Express z in polar form. (2)

Solution

Well,

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and

$$\begin{aligned} z &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}\right) \\ &= \underline{\underline{\sqrt{2}(\cos \tfrac{1}{4}\pi + \mathrm{i}\sin \tfrac{1}{4}\pi)}}. \end{aligned}$$

(b) Use de Moivre's theorem to evaluate z^8. (2)

Solution

$$\begin{aligned} z^8 &= \left(\sqrt{2}(\cos \tfrac{1}{4}\pi + \mathrm{i}\sin \tfrac{1}{4}\pi)\right)^8 \\ &= (\sqrt{2})^8(\cos 2\pi + \mathrm{i}\sin 2\pi) \\ &= 16 \times 1 \\ &= \underline{\underline{16}}. \end{aligned}$$

3. A geometric sequence of positive terms has third term 36 and fifth term 16.

(a) Calculate the value of the common ratio. (2)

Solution

Well,

$$ar^2 = 36 \text{ and } ar^4 = 16.$$

Dividing:

$$\begin{aligned} r^2 = \frac{ar^4}{ar^2} &\Rightarrow r^2 = \tfrac{16}{36} \\ &\Rightarrow r^2 = \tfrac{4}{9} \\ &\Rightarrow \underline{\underline{r = \tfrac{2}{3}}}. \end{aligned}$$

(b) Calculate the value of the first term. (1)

Solution

$$\begin{aligned} a &= \frac{36}{(\frac{2}{3})^2} \\ &= \frac{36}{\frac{4}{9}} \\ &= 36 \times \tfrac{9}{4} \\ &= 9 \times 9 \\ &= \underline{\underline{81}}. \end{aligned}$$

(c) State why the associated geometric series has a sum to infinity. (1)

Solution

The associated geometric series has a sum to infinity as $\underline{\underline{|\frac{2}{3}| < 1}}$.

(d) Find the value of this sum to infinity. (1)

Solution

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{81}{1-\frac{2}{3}} \\ &= \frac{81}{\frac{1}{3}} \\ &= \underline{\underline{243}}. \end{aligned}$$

4. Matrix $\mathbf{A}$ is defined by

$$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 11 & 3 \end{pmatrix}.$$

(a) Find $\mathbf{A}^{-1}$, the inverse of matrix $\mathbf{A}$. (2)

Solution

Well,

$$\det \mathbf{A} = 18 - 11 = 7$$

and the

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{7}\begin{pmatrix} 3 & -1 \\ -11 & 6 \end{pmatrix}}}.$$

Matrix **B** is defined by

$$\mathbf{B} = \begin{pmatrix} -4 & 3 \\ -5 & 2 \end{pmatrix}.$$

(b) Find the matrix **M** such that (2)

$$\mathbf{AM} = \mathbf{B}.$$

Solution

$$\begin{aligned} \mathbf{AM} = \mathbf{B} \Rightarrow \mathbf{M} &= \mathbf{A}^{-1}\mathbf{B} \\ \Rightarrow \mathbf{M} &= \tfrac{1}{7}\begin{pmatrix} 3 & -1 \\ -11 & 6 \end{pmatrix}\begin{pmatrix} -4 & 3 \\ -5 & 2 \end{pmatrix} \\ \Rightarrow \mathbf{M} &= \tfrac{1}{7}\begin{pmatrix} -7 & 7 \\ 14 & -21 \end{pmatrix} \\ \Rightarrow \mathbf{M} &= \underline{\underline{\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}}}. \end{aligned}$$

5. The function f is defined by

$$\mathrm{f}(x) = x^3 - x,\ x \in \mathbb{R}.$$

(a) Determine whether $\mathrm{f}(x)$ is even, odd, or neither. (2)

Solution

Well,

$$\begin{aligned} \mathrm{f}(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x. \end{aligned}$$

Now, $\mathrm{f}(x) \neq \mathrm{f}(-x)$ but $\mathrm{f}(x) = -\mathrm{f}(-x)$. Hence, it is $\underline{\underline{\text{odd}}}$.

(b) Show that the graph of $\mathrm{f}(x)$ has a point of inflection. (2)

Solution
Now,

$$\begin{aligned} \mathrm{f}(x) = x^3 - x \Rightarrow \mathrm{f}'(x) &= 3x^2 - 1 \\ \Rightarrow \mathrm{f}''(x) &= 6x. \end{aligned}$$

Next,

$$\begin{aligned} \mathrm{f}''(x) = 0 \Rightarrow 6x &= 0 \\ \Rightarrow x &= 0. \end{aligned}$$

Now,

$$x > 0 \Rightarrow \mathrm{f}''(x) > 0$$

but

$$x < 0 \Rightarrow \mathrm{f}''(x) < 0;$$

hence, the graph of $\mathrm{f}(x)$ has a point of inflection.

6. (a) Find the 2×2 matrix, $\mathbf{A}$, associated with a reflection in the x-axis. (1)

Solution

$$\underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}.$$

(b) Describe the transformation associated with the matrix (1)

$$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution
The transformation is a reflection in the line $y = x$.

(c) Find the 2×2 matrix, $\mathbf{C}$, associated with a reflection in the x-axis followed by the transformation associated with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (2)

Solution

$$\mathbf{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}}.$$

7. A curve is defined by the equation

$$x^2y + 4xy^2 = -32,\ y > 0.$$

(a) Use implicit differentiation to find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. (3)

Solution

Implicit differentiation:

$$\begin{aligned} 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4y^2 + 8xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 8xy\frac{\mathrm{d}y}{\mathrm{d}x} = -2xy - 4y^2 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(x^2 + 8xy) = -2xy - 4y^2 \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2xy - 4y^2}{x^2 + 8xy}}}. \end{aligned}$$

The curve has only one stationary point.

(b) Find the coordinates of the stationary point. (3)

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{-2xy - 4y^2}{x^2 + 8xy} = 0 \\ &\Rightarrow -2xy - 4y^2 = 0 \\ &\Rightarrow -2y(x + 2y) = 0 \end{aligned}$$

we know that $y > 0$:

$$\Rightarrow x = -2y.$$

Insert that into the equation of the curve:

$$\begin{aligned} x^2y + 4xy^2 = -32 &\Rightarrow (-2y)^2y + 4(-2y)y^2 = -32 \\ &\Rightarrow 4y^3 - 8y^3 = -32 \\ &\Rightarrow -4y^3 = -32 \\ &\Rightarrow y^3 = 8 \\ &\Rightarrow y = 2 \\ &\Rightarrow x = -4; \end{aligned}$$

so, the coordinates of the stationary point are $\underline{\underline{(-4, 2)}}$.

8. Use the substitution (4)

$$u = \tan 2x$$

to evaluate

$$\int_0^{\frac{1}{8}\pi} \frac{\sqrt{\tan 2x}}{\cos^2 2x}\,\mathrm{d}x.$$

Solution

Now,

$$\begin{aligned} u = \tan 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} &= 2\sec^2 2x \\ \Rightarrow \mathrm{d}u &= 2\sec^2 2x\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} x = 0 &\Rightarrow u = 0 \\ x = \tfrac{1}{8}\pi &\Rightarrow u = 1. \end{aligned}$$

Next,

$$
\begin{aligned}
\int_0^{\frac{1}{8}\pi} \frac{\sqrt{\tan 2x}}{\cos^2 2x}\,\mathrm{d}x &= \tfrac{1}{2}\int_0^{\frac{1}{8}\pi} \sqrt{\tan 2x}\,2\sec^2 2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_0^1 \sqrt{u}\,\mathrm{d}u \\
&= \tfrac{1}{2}\left[\tfrac{2}{3}u^{\frac{3}{2}}\right]_{u=0}^{1} \\
&= \tfrac{1}{2}\left(\tfrac{2}{3}-0\right) \\
&= \underline{\underline{\tfrac{1}{3}}}.
\end{aligned}
$$