Dr Oliver Mathematics
Worked Examples
Length 7

From: Cambridge O Level Additional Mathematics, 2015 November Paper 1 Variant 3

1. The line (10)

$$x - y + 2 = 0$$

intersects the curve

$$2x^2 - y^2 + 2x + 1 = 0$$

at the points A and B.

The perpendicular bisector of the line AB intersects the curve at the points C and D.

Find the length of the line CD in the form $a\sqrt{5}$, where a is an integer.

Solution

Well, we need to

- decide where the points A and B are,
- construct the perpendicular bisector of the line AB,
- decide where the points C and D are, and
- the length of the line CD.

Points A and B

Now,

$$x - y + 2 = 0 \Rightarrow y = x + 2$$

and

$\times$	x	$+2$
x	x^2	$+2x$
$+2$	$+2x$	$+4$

so and

$$\begin{aligned} 2x^2 - y^2 + 2x + 1 = 0 &\Rightarrow 2x^2 - (x+2)^2 + 2x + 1 = 0 \\ &\Rightarrow 2x^2 - (x^2 + 4x + 4) + 2x + 1 = 0 \\ &\Rightarrow x^2 - 2x - 3 = 0 \end{aligned}$$

$$\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & -3 \end{array} \Bigg\} +1, -3$$

$$\begin{aligned}
&\Rightarrow (x+1)(x-3) = 0 \\
&\Rightarrow x+1 = 0 \text{ or } x-3 = 0 \\
&\Rightarrow x = -1 \text{ or } x = 3 \\
&\Rightarrow y = 1 \text{ or } y = 5;
\end{aligned}$$

hence, $A(-1, 1)$ and $B(3, 5)$.

Construct the perpendicular bisector of the line AB
Now,

$$\begin{aligned}
m_{AB} &= \frac{5-1}{3-(-1)} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

and so

$$m_{\text{normal}} = -1.$$

Next, the midpoint of AB is

$$\left(\frac{-1+3}{2}, \frac{1+5}{2}\right) = (1, 3)$$

and so the perpendicular bisector of the line AB is

$$\begin{aligned}
y - 3 = -1(x-1) &\Rightarrow y - 3 = -x + 1 \\
&\Rightarrow y = -x + 4.
\end{aligned}$$

Points C and D
The perpendicular bisector of the line AB intersects the curve at the points

$$2x^2 - y^2 + 2x + 1 = 0 \Rightarrow 2x^2 - (-x+4)^2 + 2x + 1 = 0$$

$\times$	$-x$	$+4$
$-x$	x^2	$-4x$
$+4$	$-4x$	$+16$

$$\begin{aligned}
&\Rightarrow 2x^2 - (x^2 - 8x + 16) + 2x + 1 = 0 \\
&\Rightarrow x^2 + 10x = 15
\end{aligned}$$

coefficient of x :	$+10$
half it:	$+5$
square it:	$(+5)^2 = +25$

$$\begin{aligned}
&\Rightarrow x^2 + 10x + 25 = 15 + 25 \\
&\Rightarrow (x+5)^2 = 40 \\
&\Rightarrow x + 5 = -2\sqrt{10} \text{ or } x + 5 = 2\sqrt{10} \\
&\Rightarrow x = -5 - 2\sqrt{10} \text{ or } x = -5 + 2\sqrt{10} \\
&\Rightarrow y = 9 + 2\sqrt{10} \text{ or } y = 9 - 2\sqrt{10};
\end{aligned}$$

so, $C(-5 - 2\sqrt{10}, 9 + 2\sqrt{10})$ and $D(-5 + 2\sqrt{10}, 9 - 2\sqrt{10})$.

Length of the line CD
Finally,

$$\begin{aligned}
& CD^2 = [(-5 + 2\sqrt{10}) - (-5 - 2\sqrt{10})]^2 + [(9 - 2\sqrt{10}) - (9 + 2\sqrt{10})]^2 \\
\Rightarrow\quad & CD^2 = (4\sqrt{10})^2 + (-4\sqrt{10})^2 \\
\Rightarrow\quad & CD^2 = 160 + 160 \\
\Rightarrow\quad & CD^2 = 320 \\
\Rightarrow\quad & CD = \sqrt{320} \\
\Rightarrow\quad & CD = \sqrt{64 \times 5} \\
\Rightarrow\quad & CD = \sqrt{64} \times \sqrt{5} \\
\Rightarrow\quad & \underline{\underline{CD = 8\sqrt{5}}};
\end{aligned}$$

hence, $\underline{\underline{a = 8}}$.