Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2006 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. (a) Calculate $\mathbf{A}^{-1}$ where (3)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Solution

$$\begin{aligned} \det \mathbf{A} &= 1(3-2) - 1(2-2) + 0 \\ &= 1. \end{aligned}$$

Matrix of minors:

$$\begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}$$

Inverse:

$$\mathbf{A}^{-1} = \underline{\underline{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix}}}.$$

(b) Hence solve the system of equations (2)

$$\begin{aligned} x + y \quad\;\; &= 1 \\ 2x + 3y + z &= 2 \\ 2x + 2y + z &= 1. \end{aligned}$$

Solution

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix};$$

hence, $\underline{\underline{x = 0,\ y = 1,\ z = -1}}$.

2. Given that (5)

$$y = \ln(1 + \sin x),$$

where $0 < x < \pi$, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{-1}{1 + \sin x}.$$

Solution

$$y = \ln(1 + \sin x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{1 + \sin x}$$

$$u = \cos x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -\sin x$$
$$v = 1 + \sin x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$$

$$\begin{aligned}
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2} \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2} \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{-\sin x - 1}{(1+\sin x)^2} \\
\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{-(1+\sin x)}{(1+\sin x)^2} \\
\Rightarrow \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-1}{1+\sin x}}},
\end{aligned}$$

as required.

3. Define

$$S_n = \sum_{r=1}^{n} r^2,\ n \geqslant 1.$$

(a) Write down formulae for S_n and S_{2n+1}. (2)

Solution

$$S_n = \underline{\underline{\tfrac{1}{6}n(n+1)(2n+1)}}$$

$$\begin{aligned}
S_{2n+1} &= \tfrac{1}{6}(2n+1)[(2n+1)+1][2(2n+1)+1] \\
&= \tfrac{1}{6}(2n+1)(2n+2)(4n+3) \\
&= \underline{\underline{\tfrac{1}{3}(n+1)(2n+1)(4n+3)}}.
\end{aligned}$$

(b) Obtain a formula for (1)

$$2^2 + 4^2 + \ldots + (2n)^2.$$

Solution

$$\begin{aligned} 2^2 + 4^2 + \ldots + (2n)^2 &= (2 \cdot 1)^2 + (2 \cdot 2)^2 + \ldots + (2 \cdot n)^2 \\ &= (4 \cdot 1^2) + (4 \cdot 2^2) + \ldots + (4 \cdot n^2) \\ &= 4(1^2 + 2^2 + \ldots + n^2) \\ &= 4 \cdot \tfrac{1}{6} n(n+1)(2n+1) \\ &= \underline{\underline{\tfrac{2}{3} n(n+1)(2n+1)}}. \end{aligned}$$

4. Solve the differential equation (5)

$$\cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = y,$$

given that $y > 0$ and that $y = 2$ when $x = 0$.

Solution

$$\begin{aligned} \cos^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = y &\Rightarrow \frac{1}{y}\,\mathrm{d}y = \sec^2 x\,\mathrm{d}x \\ &\Rightarrow \int \frac{1}{y}\,\mathrm{d}y = \int \sec^2 x\,\mathrm{d}x \\ &\Rightarrow \ln y = \tan x + c \end{aligned}$$

as $y > 0$. Now,

$$x = 0, y = 2 \Rightarrow \ln 2 = 0 + c$$

and

$$\begin{aligned} \ln y = \tan x + \ln 2 &\Rightarrow \ln y - \ln 2 = \tan x \\ &\Rightarrow \ln\left(\frac{y}{2}\right) = \tan x \\ &\Rightarrow \frac{y}{2} = \mathrm{e}^{\tan x} \\ &\Rightarrow \underline{\underline{y = 2\mathrm{e}^{\tan x}}}. \end{aligned}$$

5. Use the substitution $u = 1 + x^2$ to obtain (5)

$$\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x.$$

Solution

$$u = 1 + x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$
$$\Rightarrow \mathrm{d}u = 2x\,\mathrm{d}x.$$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x &= \tfrac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}\,2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int \frac{(1+x^2)-1}{\sqrt{1+x^2}}\,2x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \\
&= \tfrac{1}{2}\int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\,\mathrm{d}u \\
&= \tfrac{1}{2}\left(\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right) + c \\
&= \tfrac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + c \\
&= \underline{\underline{\tfrac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + c.}}
\end{aligned}$$

6. (a) Evaluate (4)

$$\int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x.$$

Solution

$$u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}$$

Now,

$$\begin{aligned}\int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x &= \left[\tfrac{1}{2}x\mathrm{e}^{2x}\right]_{x=0}^{1} - \tfrac{1}{2}\int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x \\ &= \left(\tfrac{1}{2}\mathrm{e}^2 - 0\right) - \tfrac{1}{2}\left[\tfrac{1}{2}\mathrm{e}^{2x}\right]_{x=0}^{1} \\ &= \tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{2}\left(\tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{2}\right) \\ &= \tfrac{1}{2}\mathrm{e}^2 - \tfrac{1}{4}\tfrac{1}{2}\mathrm{e}^2 + \tfrac{1}{4} \\ &= \tfrac{1}{4}\mathrm{e}^2 + \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{1}{4}(\mathrm{e}^2 + 1)}}.\end{aligned}$$

(b) Use part (a) to evaluate (3)

$$\int_0^1 x^2\mathrm{e}^{2x}\,\mathrm{d}x.$$

Solution

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{2x} \Rightarrow v = \tfrac{1}{2}\mathrm{e}^{2x}$$

Now,

$$\begin{aligned}\int_0^1 x^2\mathrm{e}^{2x}\,\mathrm{d}x &= \left[\tfrac{1}{2}x^2\mathrm{e}^{2x}\right]_{x=0}^{1} - \int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \left(\tfrac{1}{2}\mathrm{e}^2 - 0\right) - \left(\tfrac{1}{4}\mathrm{e}^2 + \tfrac{1}{4}\right) \\ &= \tfrac{1}{4}\mathrm{e}^2 - \tfrac{1}{4} \\ &= \underline{\underline{\tfrac{1}{4}(\mathrm{e}^2 - 1)}}.\end{aligned}$$

(c) Hence obtain (2)

$$\int_0^1 (3x^2 + 2x)\mathrm{e}^{2x}\,\mathrm{d}x.$$

Solution

$$\begin{aligned}\int_0^1 (3x^2+2x)\mathrm{e}^{2x}\,\mathrm{d}x &= 3\int_0^1 x^2\mathrm{e}^{2x}\,\mathrm{d}x + 2\int_0^1 x\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \tfrac{3}{4}(\mathrm{e}^2-1)+\tfrac{1}{2}(\mathrm{e}^2+1) \\ &= (\tfrac{3}{4}\mathrm{e}^2-\tfrac{3}{4})+(\tfrac{1}{2}\mathrm{e}^2+\tfrac{1}{2}) \\ &= \tfrac{5}{4}\mathrm{e}^2-\tfrac{1}{4} \\ &= \underline{\underline{\tfrac{1}{4}(5\mathrm{e}^2-1)}}.\end{aligned}$$