Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2005 June Paper 1: Calculator
2 hours

The total number of marks available is 80.
You must write down all the stages in your working.

1. Given that (4)

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix},$$

find $(\mathbf{A}^2)^{-1}$.

Solution

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 \\ -3 & 0 \end{pmatrix} \end{aligned}$$

and

$$(\mathbf{A}^2)^{-1} = \underline{\underline{\tfrac{1}{9} \begin{pmatrix} 0 & -3 \\ 3 & 3 \end{pmatrix}}}.$$

2. A student has a collection of 9 CDs, of which 4 are by the Beatles, 3 are by Abba, and 2 are by the Rolling Stones.

She selects 4 of the CDs from her collection.

Calculate the number of ways in which she can make her selection if

(a) her selection must contain her favourite Beatles CD, (2)

Solution

$$\binom{8}{3} = \underline{\underline{56}}.$$

(b) her selection must contain 2 CDs by one group and 2 CDs by another. (3)

Solution

$$\begin{aligned}
&\text{Number of different ways}\\
=\ &\text{2 Beatles, 2 Abba + 2 Beatles, 2 RS + 2 Abba, 2 RS}\\
=\ &\left(\binom{4}{2}\times\binom{3}{2}\right)+\left(\binom{4}{2}\times\binom{2}{2}\right)+\left(\binom{3}{2}\times\binom{2}{2}\right)\\
&= 18+6+3\\
&= \underline{\underline{27}}.
\end{aligned}$$

3. Given that θ is acute and that (5)

$$\sin\theta = \tfrac{1}{\sqrt{3}},$$

express, without using a calculator,

$$\frac{\sin\theta}{\cos\theta-\sin\theta},$$

in the form $a+\sqrt{b}$, where a and b are integers.

Solution

Pythagoras' theorem:

$$\begin{aligned}
\text{opp}^2+\text{adj}^2=\text{hyp}^2 &\Rightarrow 1^2+\text{adj}^2=(\sqrt{3})^2\\
&\Rightarrow 1+\text{adj}^2=3\\
&\Rightarrow \text{adj}^2=2\\
&\Rightarrow \text{adj}=\sqrt{2}
\end{aligned}$$

and

$$\frac{\sin\theta}{\cos\theta-\sin\theta}=\frac{\frac{1}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}}-\frac{1}{\sqrt{3}}}$$

multiply by $\dfrac{\sqrt{3}}{\sqrt{3}}$:

$$\begin{aligned}
&=\frac{1}{\sqrt{2}-1}\\
&=\frac{1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}
\end{aligned}$$

$$\begin{array}{c|cc}
\hline
\times & \sqrt{2} & +1 \\
\hline
\sqrt{2} & 2 & +\sqrt{2} \\
-1 & -\sqrt{2} & -1 \\
\hline
\end{array}$$

$$\begin{aligned}
&= \frac{\sqrt{2}+1}{1} \\
&= \underline{\underline{\sqrt{2}+1}}.
\end{aligned}$$

4. The position vectors of points A and B relative to an origin O are $-3\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$ respectively. (6)

The point C lies on AB and is such that $\overrightarrow{AC} = \frac{3}{5}\overrightarrow{AB}$.

Find the position vector of C and show that it is a unit vector.

Solution

$$\begin{aligned}
\overrightarrow{AB} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ 3 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\overrightarrow{AC} &= \tfrac{3}{5}\overrightarrow{AB} \\
&= \tfrac{3}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 2.4 \\ 1.8 \end{pmatrix}.
\end{aligned}$$

Now,

$$\begin{aligned}
\overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\
&= \begin{pmatrix} -3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2.4 \\ 1.8 \end{pmatrix} \\
&= \underline{\underline{\begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}}}.
\end{aligned}$$

Next,

$$\begin{aligned} OC &= \sqrt{(-0.6)^2 + 0.8^2} \\ &= \sqrt{0.36 + 0.64} \\ &= \sqrt{1} \\ &= 1; \end{aligned}$$

hence, it is a $\underline{\underline{\text{unit vector}}}$.

5. The function f is defined, for $0^\circ \leqslant x \leqslant 180^\circ$, by

$$\mathrm{f}(x) = A + 5\cos Bx,$$

where A and B are constants.

(a) Given that the maximum value of f is 3, state the value of A. (1)

Solution

$$3 - 5 = \underline{\underline{-2}}.$$

(b) State the amplitude of f. (1)

Solution

$\underline{\underline{5}}$.

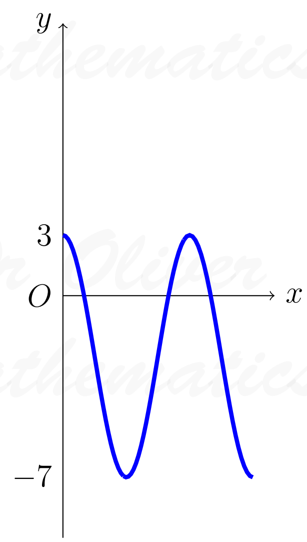
(c) Given that the period of f is 120°, state the value of B. (1)

Solution

$$\frac{360}{120} = \underline{\underline{3}}.$$

(d) Sketch the graph of f. (3)

Solution



6. Given that each of the following functions is defined for the domain $-2 \leqslant x \leqslant 3$, find the range of

   (a) $\mathrm{f} : x \mapsto 2 - 3x$, (1)

   Solution
   Well,
   $$\mathrm{f}(-2) = 8 \text{ and } \mathrm{f}(3) = -7$$
   and so
   $$\underline{\underline{-7 \leqslant \mathrm{f}(x) \leqslant 8}}.$$

   (b) $\mathrm{g} : x \mapsto |2 - 3x|$, (2)

   Solution
   Well, we have to 'flip' the part where $\mathrm{g}(x)$ is negative:
   $$\underline{\underline{0 \leqslant \mathrm{g}(x) \leqslant 8}}.$$

   (c) $\mathrm{h} : x \mapsto 2 - |3x|$. (2)

   Solution
   Well,
   $$\mathrm{h}(-2) = -4 \text{ and } \mathrm{f}(3) = -7$$

and so

$$\underline{\underline{-7 \leqslant \mathrm{h}(x) \leqslant 2}}.$$

(d) State which of the functions f, g, and h has an inverse. (2)

Solution

$\mathrm{f}(x)$ has an $\underline{\underline{\text{inverse}}}$ but $\mathrm{g}(x)$ and $\mathrm{h}(x)$ $\underline{\underline{\text{do not}}}$.

7. Variables l and t are related by the equation

$$l = l_0(1+\alpha)^t,$$

where l_0 and α are constants.

(a) Given that $l_0 = 0.64$ and $\alpha = 2.5 \times 10^{-3}$, find the value of t for which $l = 0.66$. (3)

Solution

Now,

$$\begin{aligned}0.66 = 0.64(1 + 2.5 \times 10^{-3})^t &\Rightarrow \tfrac{33}{32} = (1.002\,5)^t \\ &\Rightarrow \ln(\tfrac{33}{32}) = \ln(1.002\,5)^t \\ &\Rightarrow \ln(\tfrac{33}{32}) = t\ln(1.002\,5) \\ &\Rightarrow t = \frac{\ln(\tfrac{33}{32})}{\ln(1.002\,5)} \\ &\Rightarrow t = 12.324\,048\,9 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{t = 12.3 \text{ (3 sf)}}}.\end{aligned}$$

(b) Solve the equation (4)

$$1 + \log(8 - x) = \log(3x + 2).$$

Solution

$$
\begin{aligned}
1+\log(8-x)=\log(3x+2) &\Rightarrow 1=\log(3x+2)-\log(8-x)\\
&\Rightarrow \log_{10}10=\log\left(\frac{3x+2}{8-x}\right)\\
&\Rightarrow 10=\frac{3x+2}{8-x}\\
&\Rightarrow 10(8-x)=3x+2\\
&\Rightarrow 80-10x=3x+2\\
&\Rightarrow 78=13x\\
&\Rightarrow \underline{\underline{x=6}}.
\end{aligned}
$$

8. The table below shows experimental values of the variables x and y which are related by an equation of the form

$$y = kx^n,$$

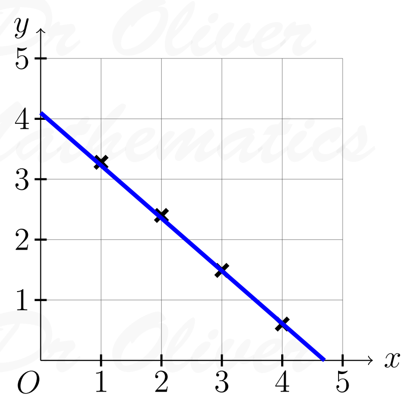
where k and n are constants.

x	10	100	1 000	10 000
y	1 900	250	31	4

(a) Using draw the graph of $\log y$ against $\log x$. (3)

Solution
Use 3 sf for $\log y$:

$\log x$	1	2	3	4
$\log y$	3.28	2.40	1.49	0.602



(b) Use your graph to estimate the value of k and of n. (4)

Solution

$$\begin{aligned} y = kx^n &\Rightarrow \log y = \log kx^n \\ &\Rightarrow \log y = \log k + \log x^n \\ &\Rightarrow \log y = \log k + n \log x. \end{aligned}$$

Well,

$$\log k = 4.1 \Rightarrow k = 12\,589.254\,12 \text{ (FCD)}$$

and

$$\begin{aligned} n &= \frac{4.1 - 0}{0 - 4.7} \\ &= -\tfrac{41}{47}. \end{aligned}$$

Hence,

$$\underline{\underline{y = 12\,600\,x^{-0.872} \text{ (3 sf)}}}.$$

9. (a) Determine the set of values of k for which the equation (5)

$$x^2 + 2x + k = 3kx - 1$$

has no real roots.

Solution

Now,

$$x^2 + 2x + k = 3kx - 1 \Rightarrow x^2 + (2 - 3k)x + (k + 1) = 0$$

and

$$b^2 - 4ac < 0 \Rightarrow (2 - 3k)^2 - 4 \times 1 \times (k + 1) < 0$$

$\times$	2	$-3k$
2	4	$-6k$
$-3k$	$-6k$	$+9k^2$

$$\begin{aligned} &\Rightarrow 4 - 12k + 9k^2 - 4k - 4 < 0 \\ &\Rightarrow 9k^2 - 16k < 0 \\ &\Rightarrow k(9k - 16) < 0 \end{aligned}$$

	$k<0$	$k=0$	$0<k<\frac{16}{9}$	$k=\frac{16}{9}$	$k>\frac{16}{9}$
k	$-$	0	$+$	$+$	$+$
$9k-16$	$-$	$-$	$-$	0	$+$
$k(9k-16)$	$+$	0	$-$	0	$+$

$$\Rightarrow \underline{\underline{0<k<\tfrac{16}{9}}}.$$

(b) Hence state, giving a reason, what can be deduced about the curve $y=(x+1)^2$ and the line $y=3x-1$. (2)

Solution

Well, $k=1$ and the result from (a) shows that is that it has no real roots.

So, the graphs cross $\underline{\underline{\text{do not at all}}}$.

10. The remainder when

$$2x^3+2x^2-13x+12$$

is divided by $(x+a)$ is three times the remainder when it is divided by $(x-a)$.

(a) Show that (3)

$$2a^3+a^2-13a+6=0.$$

Solution

We use synthetic division twice:

a	2	2	-13	12
	$\downarrow$	$2a$	$2a^2+2a$	$2a^3+2a^2-13a$
	2	$2a+2$	$2a^2+2a-13$	$2a^3+2a^2-13a+12$

and

$-a$	2	2	-13	12
	$\downarrow$	$-2a$	$2a^2-2a$	$-2a^3+2a^2+13a$
	2	$-2a+2$	$2a^2-2a-13$	$-2a^3+2a^2+13a+12$

Now,

$$\begin{aligned}
& -2a^3 + 2a^2 + 13a + 12 = 3(2a^3 + 2a^2 - 13a + 12) \\
\Rightarrow\ & -2a^3 + 2a^2 + 13a + 12 = 6a^3 + 6a^2 - 39a + 36 \\
\Rightarrow\ & 0 = 8a^3 + 4a^2 - 52a + 24 \\
\Rightarrow\ & 0 = 4(2a^3 + a^2 - 13a + 6) \\
\Rightarrow\ & \underline{\underline{2a^3 + a^2 - 13a + 6 = 0}},
\end{aligned}$$

as required.

(b) Solve this equation completely. (5)

Solution
Let

$$f(a) = 2a^3 + a^2 - 13a + 6.$$

Then

$$\begin{aligned}
f(1) &= 2 + 1 - 13 + 6 = -4, \\
f(-1) &= -2 + 1 + 13 + 6 = 18, \\
f(2) &= 16 + 4 - 26 + 6 = 0,
\end{aligned}$$

and $(a - 2)$ is a root:

$$\begin{array}{c|cccc}
2 & 2 & 1 & -13 & 6 \\
 & \downarrow & 4 & 10 & -6 \\
\hline
 & 2 & 5 & -3 & 0
\end{array}$$

so

$$2a^3 + a^2 - 13a + 6 = 0 \Rightarrow (a - 2)(2a^2 + 5a - 3) = 0$$

$$\left.\begin{array}{ll}
\text{add to:} & +5 \\
\text{multiply to:} & (+2) \times (-3) = -6
\end{array}\right\} -1,\ +6$$

e.g.,

$$\begin{aligned}
&\Rightarrow (a - 2)[2a^2 + 6a - a - 3] = 0 \\
&\Rightarrow (a - 2)[2a(a + 3) - 1(a + 3)] = 0 \\
&\Rightarrow (a - 2)(2a - 1)(a + 3) = 0 \\
&\Rightarrow \underline{\underline{a = -3,\ a = \tfrac{1}{2},\ \text{or } a = 2}}.
\end{aligned}$$

11. A particle travels in a straight line so that, t seconds after passing a fixed point A on the line, its acceleration, a ms^{-2}, is given by

$$a = -2 - 2t.$$

It comes to rest at a point B when $t = 4$.

(a) Find the velocity of the particle at A. (4)

Solution
Integrate:

$$a = -2 - 2t \Rightarrow v = -2t - t^2 + c,$$

for some constant c. Now,

$$\begin{aligned} t = 4, v = 0 &\Rightarrow 0 = -8 - 16 + c \\ &\Rightarrow c = 24. \end{aligned}$$

So

$$v = -2t - t^2 + 24.$$

Hence,

$$t = 0 \Rightarrow v = 0 + 0 + 24 = \underline{\underline{24 \text{ ms}^{-1}}}.$$

(b) Find the distance AB. (3)

Solution
Well,

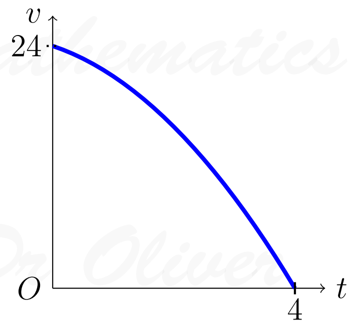
$$v = -2t - t^2 + 24 \Rightarrow s = -t^2 - \tfrac{1}{3}t^3 + 24t + c$$

and

$$\begin{aligned} AB &= \left[-t^2 - \tfrac{1}{3}t^3 + 24t + c\right]_{x=0}^{4} \\ &= \left(-16 - 21\tfrac{1}{3} + 96 + c\right) - (0 + 0 + 0 + c) \\ &= \underline{\underline{58\tfrac{2}{3} \text{ m.}}} \end{aligned}$$

(c) Sketch the velocity-time graph for the motion from A to B. (1)

Solution

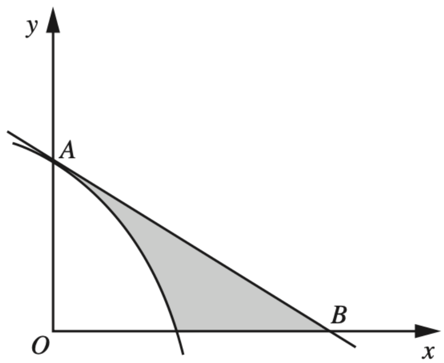


EITHER

12. The diagram, which is not drawn to scale, shows part of the graph of (10)

$$y = 8 - e^{2x},$$

crossing the y-axis at A.



The tangent to the curve at A crosses the x-axis at B.

Find the area of the shaded region bounded by the curve, the tangent, and the x-axis.

Solution

Well,

$$x = 0 \Rightarrow y = 7$$

and

$$\begin{aligned} y = 0 \Rightarrow 0 &= 8 - \mathrm{e}^{2x} \\ \Rightarrow \mathrm{e}^{2x} &= 8 \\ \Rightarrow 2x &= \ln 8 \\ \Rightarrow x &= \tfrac{1}{2} \ln 8; \end{aligned}$$

so, $A(0, 7)$ and the point where the line through $y = 8 - \mathrm{e}^{2x}$ crosses OB is $(\frac{1}{2} \ln 8, 0)$. Now,

$$y = 8 - \mathrm{e}^{2x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2\mathrm{e}^{2x}$$

and

$$x = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -2.$$

Now, the equation of the tangent is

$$y - 7 = -2(x - 0) \Rightarrow y = -2x + 7$$

and

$$\begin{aligned} y = 0 \Rightarrow 0 &= -2x + 7 \\ \Rightarrow 2x &= 7 \\ \Rightarrow x &= 3\tfrac{1}{2}; \end{aligned}$$

so, $B(3\frac{1}{2}, 0)$. Now,

$$\begin{aligned} \triangle OAB &= \tfrac{1}{2} \times 3\tfrac{1}{2} \times 7 \\ &= 12\tfrac{1}{4} \end{aligned}$$

and

$$\begin{aligned} \text{integral} &= \int_0^{\frac{1}{2}\ln 8} (8 - \mathrm{e}^{2x})\,\mathrm{d}x \\ &= \left[8x - \tfrac{1}{2}\mathrm{e}^{2x}\right]_{x=0}^{\frac{1}{2}\ln 8} \\ &= (4\ln 8 - 4) - \left(0 - \tfrac{1}{2}\right) \\ &= 4\ln 8 - 3\tfrac{1}{2}. \end{aligned}$$

Finally,

$$\begin{aligned} \text{shaded area} &= \text{triangle} - \text{integral} \\ &= 12\tfrac{1}{4} - (4\ln 8 - 3\tfrac{1}{2}) \\ &= \underline{\underline{(15\tfrac{3}{4} - 4\ln 8) \text{ or } 7.43 \ \ (3 \text{ sf})}}. \end{aligned}$$

OR

13. A piece of wire, of length 2 m, is divided into two pieces.

One piece is bent to form a square of side x m and the other is bent to form a circle of radius r m.

(a) Express r in terms of x and show that the total area, A m^2, of the two shapes is given by (4)

$$A = \frac{(\pi + 4)x^2 - 4x + 1}{\pi}.$$

Solution

Well,

$$\begin{aligned} 4x + 2\pi r = 2 &\Rightarrow 2\pi r = 2 - 4x \\ &\Rightarrow r = \frac{1 - 2x}{\pi} \end{aligned}$$

and

$$\begin{aligned} A &= x^2 + \pi \left(\frac{1 - 2x}{\pi} \right)^2 \\ &= x^2 + \frac{(1 - 2x)^2}{\pi} \\ &= \frac{\pi x^2 + (1 - 2x)^2}{\pi} \end{aligned}$$

$\times$	1	$-2x$
1	x^2	$-2x$
$-2x$	$-2x$	$+4x^2$

$$\begin{aligned} &= \frac{\pi x^2 + 1 - 4x + 4x^2}{\pi} \\ &= \underline{\underline{\frac{(\pi + 4)x^2 - 4x + 1}{\pi}}}, \end{aligned}$$

as required.

Given that x can vary, find

(b) the stationary value of A, (4)

Solution
Now,
$$A = \frac{(\pi+4)x^2 - 4x + 1}{\pi} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2(\pi+4)x - 4}{\pi}$$
and
$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}x} = 0 &\Rightarrow \frac{2(\pi+4)x-4}{\pi} = 0\\ &\Rightarrow 2(\pi+4)x - 4 = 0\\ &\Rightarrow 2(\pi+4)x = 4\\ &\Rightarrow (\pi+4)x = 2\\ &\Rightarrow x = \frac{2}{\pi+4}.\end{aligned}$$
Finally,
$$\begin{aligned}x = \frac{2}{\pi+4} &\Rightarrow A = 0.140\,024\,788\,4 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{A = 0.140 \text{ (3 sf)}}}.\end{aligned}$$

(c) the nature of this stationary value. (2)

Solution
Well,
$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = \frac{2(\pi+4)}{\pi} > 0$$
so this is a minimum.