Dr Oliver Mathematics
Further Mathematics
2×2 Matrices
Past Examination Questions

This booklet consists of 29 questions across a variety of examination topics.
The total number of marks available is 223.

1. $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are 2×2 matrices.

   (a) Given that $\mathbf{AB} = \mathbf{AC}$, and that $\mathbf{A}$ is not singular, prove that $\mathbf{B} = \mathbf{C}$. (2)

   (b) Given that $\mathbf{AB} = \mathbf{AC}$, where (3)

   $$\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix},$$

   find a matrix $\mathbf{C}$ whose elements are all non-zero.

2. A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

   $$\mathbf{A} = \begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix},$$

   where k is a constant. For the case $k = -4$,

   (a) find the image under T of the line with equation $y = 2x + 1$. (2)

   For the case $k = 2$, find

   (b) the two eigenvalues of $\mathbf{A}$, (4)

   (c) a cartesian equation for each of the two lines passing through the origin which are invariant under T. (3)

3. $$\mathbf{A} = \begin{pmatrix} a & -5 \\ 2 & a+4 \end{pmatrix},$$

   where a is real.

   (a) Find $\det \mathbf{A}$ in terms of a. (2)

   (b) Show that the matrix $\mathbf{A}$ is non-singular for all values of a. (3)

   Given that $a = 0$,

   (c) find $\mathbf{A}^{-1}$. (3)

4.
$$\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Describe fully the geometrical transformation represented by the matrix $\mathbf{M}$. (2)

The transformation represented by $\mathbf{M}$ maps the point A with coordinates (p, q) onto the point B with coordinates $(3\sqrt{2}, 4\sqrt{2})$.

(b) Find the value of p and the value of q. (4)

(c) Find, in its simplest surd form, the length OA, where O is the origin. (2)

(d) Find $\mathbf{M}^2$. (2)

The point B is mapped onto the point C by the transformation represented by $\mathbf{M}^2$.

(e) Find the coordinates of C. (2)

5.
$$\mathbf{M} = \begin{pmatrix} 2a & 3 \\ 6 & a \end{pmatrix},$$

where a is a real constant.

(a) Given that $a = 2$, find $\mathbf{M}^{-1}$. (3)

(b) Find the values of a for which $\mathbf{M}$ is singular. (2)

6. Write down the 2×2 matrix that represents

(a) an enlargement with centre $(0, 0)$ and scale factor 8, (1)

(b) a reflection in the x-axis. (1)

Hence, or otherwise,

(c) find the matrix $\mathbf{T}$ that represents an enlargement with centre $(0, 0)$ and scale factor 8, followed by a reflection in the x-axis. (2)

$$\mathbf{A} = \begin{pmatrix} 6 & 1 \\ 4 & 2 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} k & 1 \\ c & -6 \end{pmatrix},$$

where k and c are constants.

(d) Find $\mathbf{AB}$ (3)

Given that $\mathbf{AB}$ represents the same transformation as $\mathbf{T}$,

(e) find the value of k and the value of c. (2)

7.
$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}.$$

(a) Find $\mathbf{AB}$. (3)

Given that

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

(b) describe fully the geometrical transformation represented by $\mathbf{C}$, (2)

(c) write down $\mathbf{C}^{100}$. (1)

8.

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}.$$

(a) Find $\det \mathbf{A}$. (1)

(b) Find $\mathbf{A}^{-1}$. (2)

The triangle R is transformed to the triangle S by the matrix $\mathbf{A}$. Given that the area of triangle S is 72 square units,

(c) find the area of triangle R. (2)

The triangle S has vertices at the points $(0, 4)$, $(8, 16)$, and $(12, 4)$.

(d) Find the coordinates of the vertices of R. (4)

9. Given that

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix},$$

(a) (i) find $\mathbf{A}^2$, (4)

(ii) describe fully the geometrical transformation represented by $\mathbf{A}^2$.

(b) Given that (2)

$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

describe fully the geometrical transformation represented by $\mathbf{B}$.

(c) Given that (3)

$$\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix},$$

where k is a constant, find the value of k for which the matrix $\mathbf{C}$ is singular.

10.

$$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix},$$

where a and b are constants. Given that the matrix $\mathbf{A}$ maps the point with coordinates $(4, 6)$ onto the point with coordinates $(2, -8)$,

(a) find the value of a and the value of b. (4)

A quadrilateral R has area 30 square units. It is transformed into another quadrilateral S by the matrix $\mathbf{A}$. Using your values of a and b,

(b) find the area of quadrilateral S. (4)

11. A right angled triangle T has vertices $A(1, 1)$, $B(2, 1)$, and $C(2, 4)$. When T is transformed by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the image is T'.

(a) Find the coordinates of the vertices of T'. (2)

(b) Describe fully the transformation represented by $\mathbf{P}$. (2)

The matrices

$$\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

represent two transformations. When T is transformed by the matrix $\mathbf{QR}$, the image is T''.

(c) Find $\mathbf{QR}$. (2)

(d) Find the determinant of $\mathbf{QR}$. (2)

(e) Using your answer to part (d), find the area of T''. (3)

12.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix},$$

(a) Show that $\mathbf{A}$ is non-singular. (2)

(b) Find $\mathbf{B}$ such that $\mathbf{BA}^2 = \mathbf{A}$. (4)

13. Given that (4)

$$\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix},$$

where k is a constant and

$$\mathbf{E} = \mathbf{C} + \mathbf{D},$$

find the value of k for which $\mathbf{E}$ has no inverse.

14.

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}.$$

(a) Find det $\mathbf{M}$. (1)

The transformation represented by $\mathbf{M}$ maps the point $S(2a-7, a-1)$, where a is a constant, onto the point $S'(25, -14)$.

(b) Find the value of a. (3)

The point R has coordinates $(6, 0)$. Given that O is the origin,

(c) find the area of triangle ORS. (2)

Triangle ORS is mapped onto triangle $OR'S'$ by the transformation represented by $\mathbf{M}$.

(d) Find the area of triangle $OR'S'$. (2)

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

(e) describe fully the single geometrical transformation represented by $\mathbf{A}$. (2)

The transformation represented by $\mathbf{A}$ followed by the transformation represented by $\mathbf{B}$ is equivalent to the transformation represented by $\mathbf{M}$.

(f) Find $\mathbf{B}$ (4)

15. The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation through 90° anticlockwise about the origin.

(a) Write down the matrix $\mathbf{P}$. (1)

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y = -x$.

(b) Write down the matrix $\mathbf{Q}$. (1)

Given that U followed by V is transformation T, which is represented by the matrix $\mathbf{R}$,

(c) express $\mathbf{R}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$, (1)

(d) find the matrix $\mathbf{R}$, (2)

(e) give a full geometrical description of T as a single transformation. (2)

16.

$$\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix},$$

where a is a constant.

(a) Find the value of a for which the matrix $\mathbf{X}$ is singular. (2)

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

(b) Find $\mathbf{Y}^{-1}$. (2)

The transformation represented by $\mathbf{Y}$ maps the point A onto the point B. Given that B has coordinates $(1-\lambda, 7\lambda-2)$, where λ is a constant,

(c) find, in terms of λ, the coordinates of point A. (4)

17. (4)

$$\mathbf{M} = \begin{pmatrix} x & x-2 \\ 3x-6 & 4x-11 \end{pmatrix}.$$

Given that the matrix $\mathbf{M}$ is singular, find the possible values of x.

18.

$$\mathbf{A} = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}$$

and $\mathbf{I}$ is the 2×2 identity matrix.

(a) Prove that (2)

$$\mathbf{A}^2 = 7\mathbf{A} + 2\mathbf{I}.$$

(b) Hence show that (2)

$$\mathbf{A}^{-1} = \tfrac{1}{2}(\mathbf{A} - 7\mathbf{I}).$$

The transformation represented by $\mathbf{A}$ maps the point P onto the point Q. Given that Q has coordinates $(2k+8, -2k-5)$, where k is a constant,

(c) find, in terms of k, the coordinates of P. (4)

19.

$$\mathbf{A} = \begin{pmatrix} 2k+1 & k \\ -3 & -5 \end{pmatrix},$$

where k is a constant. Given that

$$\mathbf{B} = \mathbf{A} + 3\mathbf{I},$$

where $\mathbf{I}$ is the 2×2 identity matrix, find

(a) $\mathbf{B}$ in terms of k, (2)

(b) the value of k for which $\mathbf{B}$ is singular. (2)

20.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

The transformation represented by $\mathbf{B}$ followed by the transformation represented by $\mathbf{A}$ is equivalent to the transformation represented by $\mathbf{P}$.

(a) Find the matrix **P**. (2)

Triangle T is transformed to the triangle T' by the transformation represented by **P**. Given that the area of triangle T' is 24 square units,

(b) find the area of triangle T. (3)

Triangle T' is transformed to the original triangle T by the the matrix represented by **Q**.

(c) Find the matrix **Q**. (2)

21. Given that (3)

$$\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix},$$

where k is a real number, find $\mathbf{C}^{-1}$, giving your answer in terms of k.

22. In each of the following cases, find a 2×2 matrix that represents

(a) (i) a reflection in the line $y = -x$, (4)

(ii) a rotation of 135° anticlockwise about $(0, 0)$,

(iii) a reflection in the line $y = -x$ followed by a rotation of 135° anticlockwise about $(0, 0)$.

The triangle T has vertices at the points $(1, k)$, $(3, 0)$, and $(11, 0)$, where k is a constant. Triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}.$$

(b) Given that the area of triangle T' is 364 square units, find the value of k. (6)

23.

$$\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) (i) Describe fully the single transformation represented by the matrix **A**. (2)

The matrix **B** represents an enlargement, scale factor -2, with centre the origin.

(ii) Write down the matrix **B**. (1)

$$\mathbf{M} = \begin{pmatrix} 3 & k \\ -2 & 3 \end{pmatrix},$$

where k is a positive constant. Triangle T has an area of 16 square units. Triangle T is transformed onto the triangle T' by the transformation represented by the matrix **M**.

(b) Given that the area of the triangle T' is 224 square units, find the value of k. (3)

24.
$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Given that $\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$,

(a) calculate the matrix $\mathbf{M}$, (6)

(b) find the matrix $\mathbf{C}$ such that $\mathbf{MC} = \mathbf{A}$. (4)

25. (a) (4)
$$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix},$$

where k is a real constant. Given that $\mathbf{A}$ is a singular matrix, find the possible values of k.

(b)
$$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}.$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix $\mathbf{B}$. The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$, and $(10c, 6c)$, where c is a positive constant. The area of triangle T' is 135 square units.

(i) Find the matrix $\mathbf{B}^{-1}$. (2)

(ii) Find the coordinates of the vertices of the triangle T, in terms of c where necessary. (3)

(iii) Find the value of c. (3)

26. Given that k is a real number and that (3)
$$\mathbf{A} = \begin{pmatrix} 1+k & k \\ k & 1-k \end{pmatrix},$$

find the exact values of k for which $\mathbf{A}$ is a singular matrix. Give your answers in their simplest form.

27.
$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(a) Describe fully the single geometrical transformation U represented by the matrix $\mathbf{P}$. (2)

The transformation U maps the point A, with coordinates (p, q), onto the point B, with $(6\sqrt{2}, 3\sqrt{2})$.

(b) Find the value of p and the value of q. (3)

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line with equation $y = x$.

(c) Write down the matrix $\mathbf{Q}$. (1)

The transformation U followed by the transformation V is the transformation T. The transformation T is represented by the matrix $\mathbf{R}$.

(d) Find the matrix $\mathbf{R}$. (3)

(e) Deduce that the transformation T is self-inverse. (1)

28.

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \text{ and } \mathbf{P} = \begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}.$$

(a) Find $\mathbf{A}^{-1}$. (2)

The transformation represented by the matrix $\mathbf{B}$ followed by the transformation represented by the matrix $\mathbf{A}$ is equivalent to the transformation represented by the matrix $\mathbf{P}$.

(b) Find $\mathbf{B}$, giving your answer in its simplest form. (3)

29.

$$\mathbf{A} = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix},$$

where p is a constant.

(a) (i) Find, in terms of p, the matrix $\mathbf{AB}$. (2)

Given that

$$\mathbf{AB} + 2\mathbf{A} = k\mathbf{I},$$

where k is a constant and $\mathbf{I}$ is the 2×2 identity matrix,

(ii) find the value of p and the value of k. (4)

$$\mathbf{M} = \begin{pmatrix} a & -9 \\ 1 & 2 \end{pmatrix},$$

where a is a real constant. Triangle T has an area of 15 square units. Triangle T is transformed to the triangle T' by the transformation represented by the matrix $\mathbf{M}$.

(b) Given that the area of triangle T' is 270 square units, find the possible values of a. (5)