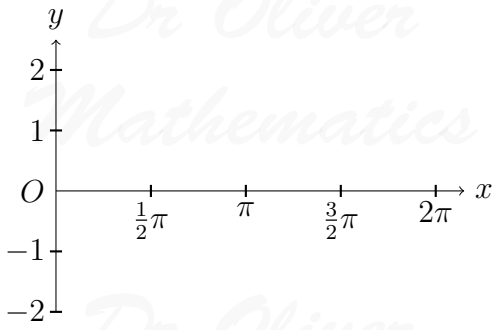
Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2013 June Paper 1 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

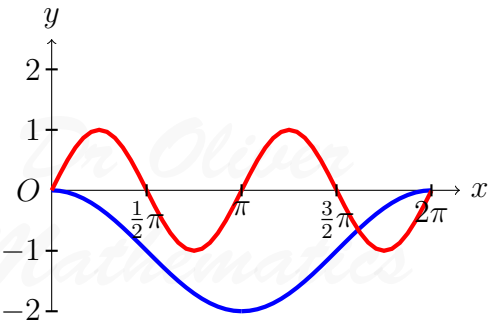
1. On the axes below, sketch, for $0 \leqslant x \leqslant 2\pi$, the graph of

   (a) $y = \cos x - 1$, (2)

   (b) $y = \sin 2x$. (2)



Solution



   (c) State the number of solutions of the equation (1)

$$\cos x - \sin 2x = 1 \text{ for } 0 \leqslant x \leqslant 2\pi.$$

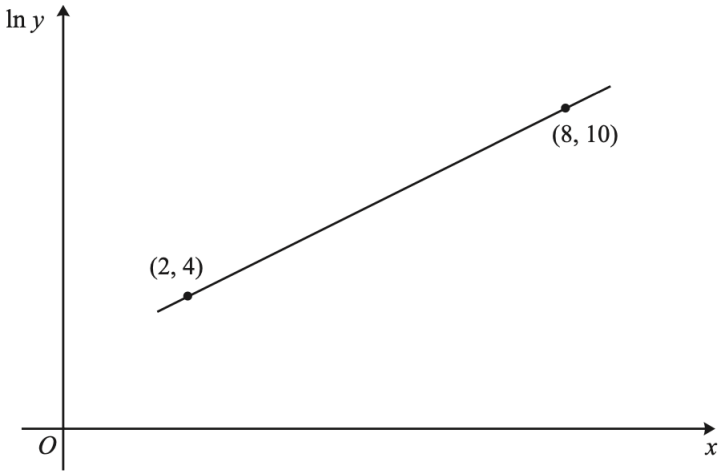
Solution

$\underline{\underline{3}}$.

2. Variables x and y are such that (5)

$$y = Ab^x,$$

where A and b are constants.



The diagram shows the graph of $\ln y$ against x, passing through the points $(2, 4)$ and $(8, 10)$.

Find the value of A and of b.

Solution

Well,

$$\begin{aligned} m &= \frac{10-4}{8-2} \\ &= \frac{6}{6} \\ &= 1 \end{aligned}$$

and the equation of the line is

$$\begin{aligned}\ln y - 10 = 1(x-8) &\Rightarrow \ln y - 10 = x - 8\\ &\Rightarrow \ln y = x + 2\\ &\Rightarrow y = \mathrm{e}^{x+2}\\ &\Rightarrow y = \mathrm{e}^2\mathrm{e}^x\\ &\Rightarrow \underline{\underline{y = \mathrm{e}^2(\mathrm{e})^x}};\end{aligned}$$

hence, $\underline{\underline{A = \mathrm{e}^2}}$ and $\underline{\underline{b = \mathrm{e}}}$.

3. A committee of 6 members is to be selected from 5 men and 9 women.

   Find the number of different committees that could be selected if

   (a) there are no restrictions, (1)

Solution

Well,

$$5 + 9 = 14$$

and

$$\binom{14}{6} = \underline{\underline{3\,003}}.$$

   (b) there are exactly 3 men and 3 women on the committee, (2)

Solution

Now,

$$\begin{aligned}\binom{5}{3} \times \binom{9}{3} &= 10 \times 84\\ &= \underline{\underline{840}}.\end{aligned}$$

   (c) there is at least 1 man on the committee. (3)

Solution

Well,

$$\text{all women} = \binom{9}{6}$$
$$= 84$$

so

$$\text{at least 1 man} = 3\,003 - 84$$
$$= \underline{\underline{2\,919}}.$$

4. (a) Given that (1)

$$\log_4 x = \tfrac{1}{2},$$

find the value of x.

Solution

Well,

$$\log_4 x = \tfrac{1}{2} \Rightarrow x = 4^{\frac{1}{2}}$$
$$\Rightarrow \underline{\underline{x = 2}}.$$

(b) Solve (4)

$$2\log_4 y - \log_4(5y - 12) = \tfrac{1}{2}.$$

Solution

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Now,

$$\begin{aligned}
2\log_4 y - \log_4(5y-12) = \tfrac{1}{2} &\Rightarrow \log_4 y^2 - \log_4(5y-12) = \tfrac{1}{2} \\
&\Rightarrow \log_4\left(\frac{y^2}{5y-12}\right) = \tfrac{1}{2} \\
&\Rightarrow \frac{y^2}{5y-12} = 4^{\frac{1}{2}} \\
&\Rightarrow \frac{y^2}{5y-12} = 2 \\
&\Rightarrow y^2 = 2(5y-12) \\
&\Rightarrow y^2 = 10y - 24 \\
&\Rightarrow y^2 - 10y + 24 = 0
\end{aligned}$$

add to: -10
multiply to: $+24$ $\Big\}$ $-6,\ -4$

$$\begin{aligned}
&\Rightarrow (y-6)(y-4) = 0 \\
&\Rightarrow y - 6 = 0 \text{ or } y - 4 = 0 \\
&\Rightarrow \underline{\underline{y = 6 \text{ or } y = 4}}.
\end{aligned}$$

5. (a) Find (2)

$$\int \left(1 - \frac{6}{x^2}\right) \mathrm{d}x.$$

Solution

Well,

$$\begin{aligned}
\int \left(1 - \frac{6}{x^2}\right) \mathrm{d}x &= \int \left(1 - 6x^{-2}\right) \mathrm{d}x \\
&= \underline{\underline{x + 6x^{-1} + c}}.
\end{aligned}$$

(b) Hence find the value of the positive constant k for which (4)

$$\int_k^{3k} \left(1 - \frac{6}{x^2}\right) \mathrm{d}x = 2.$$

Solution

Now,

$$\int_k^{3k} \left(1 - \frac{6}{x^2}\right) \mathrm{d}x = 2$$

$$\Rightarrow \quad \left[x + 6x^{-1}\right]_{x=k}^{3k} = 2$$

$$\Rightarrow \quad \left(3k + \frac{6}{3k}\right) - \left(k + \frac{6}{k}\right) = 2$$

$$\Rightarrow \quad 2k + \frac{6}{3k} - \frac{18}{3k} = 2$$

$$\Rightarrow \quad 2k - \frac{12}{3k} = 2$$

$$\Rightarrow \quad 6k^2 - 12 = 6k$$

$$\Rightarrow \quad 6k^2 - 6k - 12 = 0$$

$$\Rightarrow \quad 6(k^2 - k - 2) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -1 \\ \text{multiply to:} & -2 \end{array}\right\} -2,\ +1$$

$$\Rightarrow \quad 6(k-2)(k+1) = 0$$

$$\Rightarrow \quad k - 2 = 0 \text{ or } k + 1 = 0$$

$$\Rightarrow \quad k = 2 \text{ or } k = -1.$$

But k is positive so $\underline{\underline{k = 2}}$.

6. (a) Given that (2)

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix},$$

find $\mathbf{A}^{-1}$.

Solution

Well,

$$\det \mathbf{A} = 10 - (-3) = 13$$

so

$$\mathbf{A}^{-1} = \underline{\underline{\tfrac{1}{13} \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}}}.$$

(b) Using your answer from part (a), or otherwise, find the values of a, b, c, and d such that (5)

$$\mathbf{A}\begin{pmatrix} a & b \\ c & -1 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 17 & d \end{pmatrix}.$$

Solution

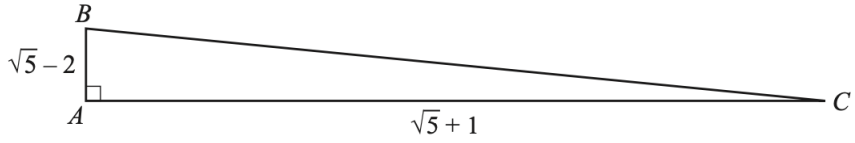
Now,

$$\mathbf{A}\begin{pmatrix} a & b \\ c & -1 \end{pmatrix} = \begin{pmatrix} 7 & 5 \\ 17 & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & -1 \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 17 & d \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & b \\ c & -1 \end{pmatrix} = \tfrac{1}{13}\begin{pmatrix} 52 & 25+d \\ 13 & -15+2d \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} a & b \\ c & -1 \end{pmatrix} = \begin{pmatrix} 4 & \frac{25+d}{13} \\ 1 & \frac{-15+2d}{13} \end{pmatrix}.$$

Now,

$$\frac{-15+2d}{13} = -1 \Rightarrow -15+2d = -13$$
$$\Rightarrow 2d = 2$$
$$\Rightarrow \underline{\underline{d = 1}}$$
$$\Rightarrow \underline{\underline{a = 4}}$$
$$\Rightarrow \underline{\underline{c = -1}}$$
$$\Rightarrow b = \tfrac{25+1}{13}$$
$$\Rightarrow b = \tfrac{26}{13}$$
$$\Rightarrow \underline{\underline{b = 2}}.$$

7. **Calculators must not be used in this question.**



The diagram shows a triangle ABC in which angle $BAC = 90°$.

Sides AB and AC are $(\sqrt{5}-2)$ and$(\sqrt{5}+1)$ respectively.

Find

(a) $\tan ABC$ in the form (3)

$$a + b\sqrt{5},$$

where a and b are integers,

Solution

Well,

$$\begin{aligned}\tan ABC &= \frac{\sqrt{5}+1}{\sqrt{5}-2}\\ &= \frac{\sqrt{5}+1}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}\end{aligned}$$

$\times$	$\sqrt{5}$	$+2$
$\sqrt{5}$	5	$+2\sqrt{5}$
$+1$	$+\sqrt{5}$	$+2$

$\times$	$\sqrt{5}$	$+2$
$\sqrt{5}$	5	$+2\sqrt{5}$
-2	$-2\sqrt{5}$	-1

$$\begin{aligned}&= \frac{7+3\sqrt{5}}{1}\\ &= \underline{\underline{7+3\sqrt{5}}};\end{aligned}$$

hence, $\underline{\underline{a=7}}$ and $\underline{\underline{b=3}}$.

(b) $\sec^2 ABC$ in the form (4)

$$c + d\sqrt{5},$$

where c and d are integers.

Solution

Now,

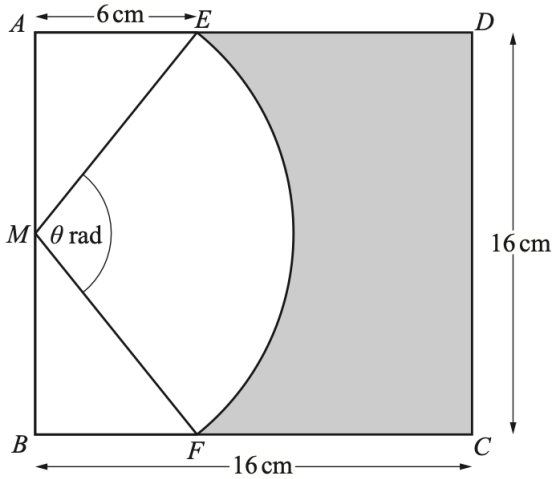
$$\begin{aligned}\sec^2 ABC &= 1 + \tan^2 ABC \\ &= 1 + (7 + 3\sqrt{5})^2\end{aligned}$$

$\times$	7	$+3\sqrt{5}$
7	49	$+21\sqrt{5}$
$+3\sqrt{5}$	$+21\sqrt{5}$	$+45$

$$= \underline{\underline{94 + 42\sqrt{5}}};$$

hence, $\underline{\underline{c = 94}}$ and $\underline{\underline{d = 42}}$.

8. The diagram shows a square $ABCD$ of side 16 cm.



- M is the mid-point of AB.
- The points E and F are on AD and BC respectively such that

$$AE = BF = 6 \text{ cm}.$$

- EF is an arc of the circle centre M, such that angle EMF is θ radians.

(a) Show that $\theta = 1.855$ radians, correct to 3 decimal places. (2)

Solution
Well,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan EMA = \tfrac{6}{8}$$
$$\Rightarrow \angle EMA = 0.643\,501\,108\,8 \text{ (FCD)}$$

and

$$\begin{aligned}\theta &= \pi - 2 \times 0.643\ldots \\ &= 1.854\,590\,436 \text{ (FCD)} \\ &= \underline{\underline{1.855 \text{ (3 dp)}}}.\end{aligned}$$

(b) Calculate the perimeter of the shaded region. (4)

Solution
Now,

$$\begin{aligned}EM &= \sqrt{EA^2 + AM^2} \\ &= \sqrt{6^2 + 8^2} \\ &= 10\end{aligned}$$

and

$$\begin{aligned}\text{perimeter} &= ED + DC + CF + \text{arc } FE \\ &= 10 + 16 + 10 + (10 \times 1.854\ldots) \\ &= 54.545\,904\,36 \text{ (FCD)} \\ &= \underline{\underline{54.5 \text{ cm (3 sf)}}}.\end{aligned}$$

(c) Calculate the area of the shaded region. (3)

Solution

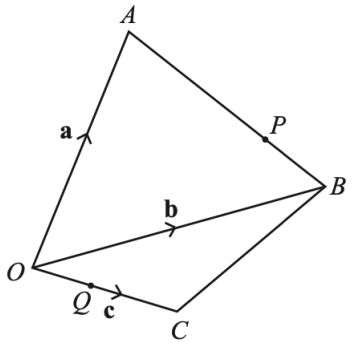
Now,

$$\begin{aligned}\text{unshaded region} &= \text{triangle } EAM + \text{triangle } FBM + \text{sector } EMF \\ &= 2 \times \text{triangle } EAM + \text{sector } EMF \\ &= (2 \times \tfrac{1}{2} \times 6 \times 8) + (\tfrac{1}{2} \times 10^2 \times 1.854\ldots) \\ &= 140.729\,521\,8 \text{ (FCD)}\end{aligned}$$

and

$$\begin{aligned}\text{shaded region} &= \text{square } ABCD - \text{unshaded region} \\ &= 256 - 140.729\ldots \\ &= 115.270\,478\,2 \text{ (FCD)} \\ &= \underline{\underline{115 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

9. The figure shows points A, B, and C with position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively, relative to an origin O.



- The point P lies on AB such that

$$AP : AB = 3 : 4.$$

- The point Q lies on OC such that

$$OQ : QC = 2 : 3.$$

(a) Express $\overrightarrow{AP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and hence show that (3)

$$\overrightarrow{OP} = \tfrac{1}{4}(\mathbf{a} + 3\mathbf{b}).$$

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Solution

Well,

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -\mathbf{a} + \mathbf{b}.\end{aligned}$$

Now,

$$AP : AB = 3 : 4 \Rightarrow \overrightarrow{AP} = \tfrac{3}{4}\overrightarrow{AB}$$

and

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \overrightarrow{AP} \\ &= \overrightarrow{OA} + \tfrac{3}{4}\overrightarrow{AB} \\ &= \mathbf{a} + \tfrac{3}{4}(-\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} - \tfrac{3}{4}\mathbf{a} + \tfrac{3}{4}\mathbf{b} \\ &= \tfrac{1}{4}\mathbf{a} + \tfrac{3}{4}\mathbf{b} \\ &= \underline{\underline{\tfrac{1}{4}(\mathbf{a} + 3\mathbf{b})}},\end{aligned}$$

as required.

(b) Find $\overrightarrow{PQ}$ in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. (3)

Solution

Well,

$$\overrightarrow{OQ} = \tfrac{2}{5}\overrightarrow{OC}$$

and

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA} + \overrightarrow{AO} + \overrightarrow{OQ} \\ &= -\overrightarrow{AP} - \overrightarrow{OA} + \overrightarrow{OQ} \\ &= -\tfrac{3}{4}(-\mathbf{a} + \mathbf{b}) - \mathbf{a} + \tfrac{2}{5}\mathbf{c} \\ &= \tfrac{3}{4}\mathbf{a} - \tfrac{3}{4}\mathbf{b} - \mathbf{a} + \tfrac{2}{5}\mathbf{c} \\ &= \underline{\underline{-\tfrac{1}{4}\mathbf{a} - \tfrac{3}{4}\mathbf{b} + \tfrac{2}{5}\mathbf{c}}}.\end{aligned}$$

(c) Given that (2)

$$5\overrightarrow{PQ} = 6\overrightarrow{BC},$$

find $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Solution
Well,

$$\begin{aligned} 5\overrightarrow{PQ} = 6\overrightarrow{BC} &\Rightarrow \overrightarrow{PQ} = \tfrac{6}{5}\overrightarrow{BC} \\ &\Rightarrow \overrightarrow{PQ} = \tfrac{6}{5}(\overrightarrow{BO} + \overrightarrow{OC}) \\ &\Rightarrow \overrightarrow{PQ} = \tfrac{6}{5}(-\overrightarrow{OB} + \overrightarrow{OC}) \\ &\Rightarrow \overrightarrow{PQ} = \tfrac{6}{5}(-\mathbf{b} + \mathbf{c}) \\ &\Rightarrow \overrightarrow{PQ} = -\tfrac{6}{5}\mathbf{b} + \tfrac{6}{5}\mathbf{c}. \end{aligned}$$

Now, the two expressions for $\overrightarrow{PQ}$ must be equal:

$$\begin{aligned} & -\tfrac{1}{4}\mathbf{a} - \tfrac{3}{4}\mathbf{b} + \tfrac{2}{5}\mathbf{c} = -\tfrac{6}{5}\mathbf{b} + \tfrac{6}{5}\mathbf{c} \\ \Rightarrow \quad & -\tfrac{1}{4}\mathbf{a} + \tfrac{9}{20}\mathbf{b} = \tfrac{4}{5}\mathbf{c} \\ \Rightarrow \quad & \underline{\underline{\mathbf{c} = -\tfrac{5}{16}\mathbf{a} + \tfrac{9}{16}\mathbf{b}}}. \end{aligned}$$

10. The point A, whose x-coordinate is 2, lies on the curve with equation

$$y = x^3 - 4x^2 + x + 1.$$

(a) Find the equation of the tangent to the curve at A. (4)

Solution
Well,

$$y = x^3 - 4x^2 + x + 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 8x + 1$$

and

$$\begin{aligned} x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= 12 - 16 + 1 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= -3. \end{aligned}$$

Now,

$$\begin{aligned} x = 2 &\Rightarrow y = 8 - 16 + 2 + 1 \\ &\Rightarrow y = -5 \end{aligned}$$

— so $A(2, -5)$ — and the equation of the tangent to the curve at A is

$$\begin{aligned} y + 5 = -3(x - 2) &\Rightarrow y + 5 = -3x + 6 \\ &\Rightarrow \underline{\underline{y = -3x + 1}}. \end{aligned}$$

This tangent meets the curve again at the point B.

(b) Find the coordinates of B. (4)

Solution

Well,

$$\begin{aligned} x^3 - 4x^2 + x + 1 = -3x + 1 &\Rightarrow 3x^2 - 4x^2 + 4x = 0 \\ &\Rightarrow x(x^2 - 4x + 4) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -4 \\ \text{multiply to:} & +4 \end{array}\right\} -2, -2$$

$$\begin{aligned} &\Rightarrow x(x-2)^2 = 0 \\ &\Rightarrow x = 0 \text{ or } x - 2 = 0 \text{ (repeated twice);} \end{aligned}$$

So,

$$\begin{aligned} x = 0 &\Rightarrow y = 0 - 0 + 0 + 1 \\ &\Rightarrow y = 1 \end{aligned}$$

and, hence, $\underline{\underline{B(0,1)}}$.

(c) Find the equation of the perpendicular bisector of the line AB. (4)

Solution

Let M be the midpoint of the AB. Then

$$\left(\frac{2+0}{2}, \frac{-5+1}{2}\right) = M(1, -2).$$

Now,

$$\begin{aligned} m_{AB} &= \frac{-5-1}{2-0} \\ &= \frac{-6}{2} \\ &= -3 \end{aligned}$$

so

$$m_{\text{normal}} = \tfrac{1}{3}.$$

Hence, the equation of the perpendicular bisector of the line AB is

$$y + 2 = \tfrac{1}{3}(x - 1) \Rightarrow y + 2 = \tfrac{1}{3}x - \tfrac{1}{3}$$
$$\Rightarrow \underline{\underline{y = \tfrac{1}{3}x - \tfrac{7}{3}}}.$$

11. (a) Solve (4)

$$2\sin(x + \tfrac{1}{3}\pi) = -1, \text{ for } 0 \leqslant x \leqslant 2\pi \text{ radians.}$$

Solution

Well,

$$0 \leqslant x \leqslant 2\pi \Rightarrow \tfrac{1}{3} \leqslant x + \tfrac{1}{3}\pi \leqslant \tfrac{7}{3}\pi$$

and

$$2\sin(x + \tfrac{1}{3}\pi) = -1 \Rightarrow \sin(x + \tfrac{1}{3}\pi) = -\tfrac{1}{2}$$
$$\Rightarrow x + \tfrac{1}{3}\pi = \tfrac{7}{6}\pi,\ \tfrac{11}{6}\pi$$
$$\Rightarrow \underline{\underline{x = \tfrac{5}{6}\pi,\ \tfrac{3}{2}\pi}}.$$

(b) Solve (6)

$$\tan y - 2 = \cot y, \text{ for } 0^\circ \leqslant y \leqslant 180^\circ.$$

Solution

Well,

$$\tan y - 2 = \cot y \Rightarrow \tan y - 2 = \frac{1}{\tan y}$$

multiply by $\tan y$:

$$\Rightarrow \tan^2 y - 2\tan y = 1$$
$$\Rightarrow \tan^2 y - 2\tan y + 1 = 1 + 1$$
$$\Rightarrow (\tan y - 1)^2 = 2$$
$$\Rightarrow \tan y - 1 = \pm\sqrt{2}$$
$$\Rightarrow \tan y = 1 \pm \sqrt{2}.$$

$\underline{\tan y = 1 - \sqrt{2}}$:

$$\tan y = 1 - \sqrt{2} \Rightarrow \underline{\underline{y = 157\tfrac{1}{2}}}.$$

$\underline{\tan y = 1 + \sqrt{2}}$:

$$\tan y = 1 + \sqrt{2} \Rightarrow \underline{\underline{y = 67\tfrac{1}{2}}}.$$

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