Dr Oliver Mathematics
Mathematics: Advanced Higher
2013 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Write down the binomial expansion of (4)

$$\left(3x - \frac{2}{x^2}\right)^4$$

and simplify your answer.

2. Differentiate (3)

$$f(x) = e^{\cos x} \sin^2 x.$$

3. Matrices $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} = \begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}.$$

   (a) Find $\mathbf{A}^2$. (1)

   (b) Find the value of p for which $\mathbf{A}^2$ is singular. (2)

   (c) Find the values of p and x if $\mathbf{B} = 3\mathbf{A}^T$. (2)

4. The velocity, v, of a particle P at time t is given by

$$v = e^{3t} + 2e^t.$$

   (a) Find the acceleration of P at time t. (2)

   (b) Find the distance covered by P between $t = 0$ and $t = \ln 3$. (3)

5. Use the Euclidean algorithm to obtain the greatest common divisor of 1 204 and 833, expressing it in the form (4)

$$1\,204a + 833b,$$

where a and b are integers.

6. Integrate (4)

$$\int \frac{\sec^2 3x}{1 + \tan 3x} \, dx$$

with respect to x.

7. Given that (4)

$$z = 1 - \sqrt{3}\mathrm{i},$$

write down $\bar{z}$ and express $\bar{z}^2$ in polar form.

8. Use integration by parts to obtain (5)

$$\int x^2 \cos 3x \, \mathrm{d}x.$$

9. Prove by induction that, for all positive integers n, (6)

$$\sum_{r=1}^{n} (4r^3 + 3r^2 + r) = n(n+1)^3.$$

10. Describe the loci in the complex plane given by:

(a) $|z + \mathrm{i}| = 1$, (2)

(b) $|z - 1| = |z + 5|$. (3)

11. A curve has equation (6)

$$x^2 + 4xy + y^2 + 11 = 0.$$

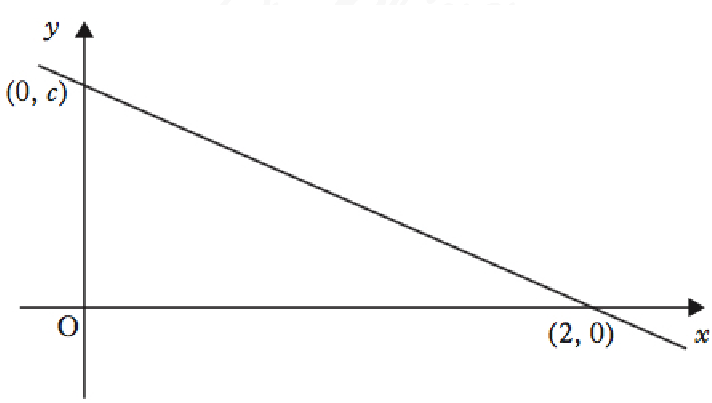
Find the values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point $(-2, 3)$.

12. Let n be a natural number.
For each of the following statements, decide whether it is true or false.
If true, give a proof; if false, give a counterexample.

(a) If n is a multiple of 9, then so is n^2. (3)

(b) If n^2 is a multiple of 9, then so is n. (1)

13. Part of the straight line graph of a function $\mathrm{f}(x)$ is shown.



(a) Sketch the graph of $\mathrm{f}^{-1}(x)$, showing points of intersection with the axes. (2)

(b) State the value of k for which $\mathrm{f}(x) + k$ is an odd function. (1)

(c) Find the value of h for which $|\,\mathrm{f}(x+h)|$ is an even function. (2)

14. Solve the differential equation (11)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 4\mathrm{e}^{3x},$$

given that $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1$ when $x = 0$.

15. (a) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$, and $C(0, 0, 5)$. (4)

π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.

(b) Find an equation of the plane π_2. (2)

(c) Determine the acute angle between planes π_1 and π_2. (3)

16. In an environment without enough resources to support a population greater than 1 000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(1\,000 - P).$$

(a) Show that (4)

$$\ln\left(\frac{P}{1\,000 - P}\right) = 1\,000t + C,$$

for some constant C.

(b) Hence show that (3)

$$P(t) = \frac{1\,000K}{K + \mathrm{e}^{-1\,000t}},$$

for some constant K.

(c) Given that (3)

$$P(0) = 200,$$

determine at what time t, $P(t) = 900$.

17. (a) Write down the sums to infinity of the geometric series (2)

$$1 + x + x^2 + x^3 + \ldots$$

and

$$1 - x + x^2 - x^3 + \ldots$$

valid for $|x| < 1$.

(b) Assuming that it is permitted to integrate an infinite series term by term, show that, for $|x| < 1$, (5)

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots\right).$$

(c) Show how this series can be used to evaluate $\ln 2$. (2)

(d) Hence determine the value of $\ln 2$ correct to 3 decimal places. (1)