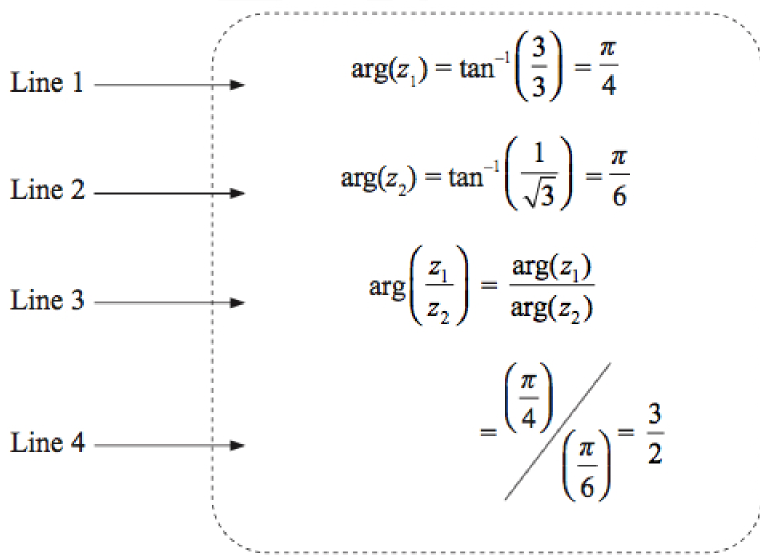
Dr Oliver Mathematics
Further Mathematics: Core Pure Mathematics 2
June 2022: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.
Inexact answers should be given to three significant figures unless otherwise stated.

1. A student was asked to answer the following:

   For the complex numbers $z_1 = 3 - 3\text{i}$ and $z_2 = \sqrt{3} + \text{i}$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$.

   The student's attempt is shown below.



   The student made errors in line 1 and line 3.

   Correct the error that the student made in

   (a) (i) line 1, (2)

Solution
It should have been

$$\tan^{-1}\left(\frac{-3}{3}\right),$$

or equivalent.

(ii) line 3.

Solution

It should have been

$$\arg\left(\frac{z_1}{z_2}\right) = \underline{\underline{\arg(z_1) - \arg(z_2)}}.$$

(b) Write down the correct value of $\arg\left(\frac{z_1}{z_2}\right)$. (1)

Solution

$$\begin{aligned}\arg\left(\frac{z_1}{z_2}\right) &\Rightarrow \arg(z_1) - \arg(z_2) \\ &\Rightarrow -\tfrac{1}{4}\pi - \tfrac{1}{6}\pi \\ &\Rightarrow \underline{\underline{-\tfrac{5}{12}\pi}}.\end{aligned}$$

2. **In this question you must show all stages of your working**.

A college offers only three courses: Construction, Design, and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1 110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction increased by 1.25%,
- Design increased by 2.5%, and
- Hospitality decreased by 2%.

In 2020, the total number of students at the college increased by 0.27%, to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019. (4)

Solution
Let

- x be the number of Construction students,
- y be the number of Design students, and
- z be the number of Hospitality students.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

Solution
Well, number of students in 2020 is

$$\begin{aligned} 1\,110 \times 1.002\,7 &= 1\,102.97 \\ &\approx 1\,103; \end{aligned}$$

the students increased by 3. So, e.g.,

$$\begin{aligned} \underline{\underline{x + y + z}} &= \underline{\underline{1\,100}} \\ \underline{\underline{x - z}} &= \underline{\underline{370}} \\ \underline{\underline{0.012\,5x + 0.025y - 0.02z}} &= \underline{\underline{3}}. \end{aligned}$$

or equivalent.

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019. (4)

Solution
Now,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.012\,5 & 0.025 & -0.02 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1\,100 \\ 370 \\ 3 \end{pmatrix} \\ \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0.012\,5 & 0.025 & -0.02 \end{pmatrix}^{-1} \begin{pmatrix} 1\,100 \\ 370 \\ 3 \end{pmatrix} \end{aligned}$$

Determinant:

$$1(0+0.025)-1(-0.02+0.0125)+1(0.025-0)=0.0575$$

Matrix of minors:

$$\begin{pmatrix} 0.025 & -0.0075 & 0.025 \\ -0.045 & -0.0325 & 0.0125 \\ -1 & -2 & -1 \end{pmatrix}$$

Matrix of cofactors:

$$\begin{pmatrix} 0.025 & 0.0075 & 0.025 \\ 0.045 & -0.0325 & -0.0125 \\ -1 & 2 & -1 \end{pmatrix}$$

Transpose:

$$\begin{pmatrix} 0.025 & 0.045 & -1 \\ 0.0075 & -0.0325 & 2 \\ 0.025 & -0.0125 & -1 \end{pmatrix}$$

Inverse:

$$\tfrac{1}{0.0575}\begin{pmatrix} 0.025 & 0.045 & -1 \\ 0.0075 & -0.0325 & 2 \\ 0.025 & -0.0125 & -1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 715\frac{15}{23} \\ 38\frac{15}{23} \\ 345\frac{15}{23} \end{pmatrix}.$$

So, rounding up, 720 students studied Construction, 40 students studied Design, and 350 students studied Hospitality ($720+40+350=1\,100$ ✓).

3.

$$\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}, \text{ where } a \text{ is a constant.}$$

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$, (6)

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{1}{2}a(3^n-1) \\ 0 & 1 \end{pmatrix}.$$

Solution

$\underline{n = 1:}$

$$\begin{aligned} \text{LHS} &= \mathbf{M}^1 \\ &= \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \text{RHS} &= \begin{pmatrix} 3^1 & \frac{1}{2}a(3^1 - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and so

$$\text{LHS} = \text{RHS}.$$

Now, suppose that it is true for $n = k$, i.e.,

$$\mathbf{M}^k = \begin{pmatrix} 3^k & \frac{1}{2}a(3^k - 1) \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} \mathbf{M}^{k+1} &= \mathbf{M}\mathbf{M}^k \\ &= \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3^k & \frac{1}{2}a(3^k - 1) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times 3^k & \frac{3}{2}a(3^k - 1) + a \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times 3^k & \frac{1}{2}a[3(3^k - 1) + 2] \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & \frac{1}{2}a[3^{k+1} - 3 + 2] \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3^{k+1} & \frac{1}{2}a(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and so the result is true for $n = k + 1$.

Hence, we have proved by mathematical induction that, for $n \in \mathbb{N}$,

$$\mathbf{M}^n = \underline{\underline{\begin{pmatrix} 3^n & \frac{1}{2}a(3^n - 1) \\ 0 & 1 \end{pmatrix}}}.$$

Triangle T has vertices A, B, and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$, where $n \in \mathbb{N}$.

Given that

- triangle T has an area of 5 cm^2,
- triangle T' has an area of 1 215 cm^2, and
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$,

(b) determine (5)

(i) the value of n,

Solution
Well,

$$\begin{aligned}\det \mathbf{M} &= (3 \times 1) - (a \times 0) \\ &= 3\end{aligned}$$

and

$$\begin{aligned}3^n &= |\det \mathbf{M}|^n \\ &= |\det \mathbf{M}^n| \\ &= \frac{\text{area of triangle } T'}{\text{area of triangle } T} \\ &= \frac{1\,215}{5} \\ &= 243 \\ &= 3^5\end{aligned}$$

so $\underline{\underline{n = 5}}$.

(ii) the value of a.

Solution
Well,

$$\begin{aligned} &\begin{pmatrix} 3^5 & \frac{1}{2}a(3^5 - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix} \\ \Rightarrow &\begin{pmatrix} 2(3^5) - a(3^5 - 1) \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix};\end{aligned}$$

so,

$$\begin{aligned} 2(3^5) - a(3^5 - 1) = 123 &\Rightarrow 486 - 242a = 123 \\ &\Rightarrow 242a = 363 \\ &\Rightarrow \underline{\underline{a = 1\tfrac{1}{2}}}. \end{aligned}$$

4. (a) Given that (3)

$$z_1 = 6\mathrm{e}^{\frac{1}{3}\pi \mathrm{i}} \text{ and } z_2 = 6\sqrt{3}\mathrm{e}^{\frac{5}{6}\pi \mathrm{i}},$$

show that

$$z_1 + z_2 = 12\mathrm{e}^{\frac{2}{3}\pi \mathrm{i}}.$$

Solution

$$\begin{aligned} z_1 + z_2 &= 6\left[\cos(\tfrac{1}{3}\pi) + \sin(\tfrac{1}{3}\pi)\mathrm{i}\right] + 6\sqrt{3}\left[\cos(\tfrac{5}{6}\pi) + \sin(\tfrac{5}{6}\pi)\mathrm{i}\right] \\ &= \left[6\cos(\tfrac{1}{3}\pi) + 6\sqrt{3}\cos(\tfrac{5}{6}\pi)\right] + \left[6\sin(\tfrac{1}{3}\pi) + 6\sqrt{3}\sin(\tfrac{5}{6}\pi)\right]\mathrm{i} \\ &= -6 + 6\sqrt{3}\mathrm{i}. \end{aligned}$$

Now,

$$r = \sqrt{(-6)^2 + (6\sqrt{3})^2} = 12$$

and

$$\begin{aligned} \tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) &= \tan^{-1}(-\sqrt{3}) \\ &= \tfrac{2}{3}\pi. \end{aligned}$$

Hence,

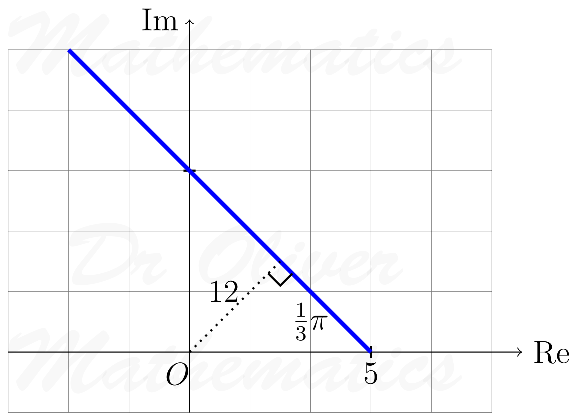
$$z_1 + z_2 = \underline{\underline{12\mathrm{e}^{\frac{2}{3}\pi \mathrm{i}}}}.$$

(b) Given that (3)

$$\arg(z - 5) = \tfrac{2}{3}\pi,$$

determine the least value of $|z|$ as z varies.

Solution



Well,

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin(\tfrac{1}{3}\pi) = \frac{|z|}{5} \\ &\Rightarrow |z| = 5\sin(\tfrac{1}{3}\pi) \\ &\Rightarrow \underline{\underline{|z| = \tfrac{5}{2}\sqrt{3}}}. \end{aligned}$$

5. (a) Given that show that (3)

$$y = \arcsin x, \ -1 \leqslant x \leqslant 1,$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 - x^2}}.$$

Solution

Now, we proceed by implicit differentiation:

$$\begin{aligned} y = \arcsin x \Rightarrow \sin y &= x \\ \Rightarrow \cos y \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1-\sin^2 y}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1-(\sin y)^2}} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1-x^2}}, \end{aligned}$$

as required

$$\mathrm{f}(x) = \arcsin(\mathrm{e}^x),\ x \leqslant 0.$$

(b) Prove that $\mathrm{f}(x)$ has no stationary points. (3)

Solution

Now,

$$\begin{aligned} \mathrm{f}'(x) &= \frac{1}{\sqrt{1-(\mathrm{e}^x)^2}} \times \mathrm{e}^x \\ &= \frac{\mathrm{e}^x}{\sqrt{1-\mathrm{e}^{2x}}}. \end{aligned}$$

Now, $\mathrm{e}^x > 0$ for all x. Hence, $\mathrm{f}(x)$ has no stationary points.

6. The cubic equation

$$4x^3 + px^2 - 14x + q = 0,$$

where p and q are real positive constants, has roots α, β, and γ.

Given that

$$\alpha^2 + \beta^2 + \gamma^2 = 16,$$

(a) show that $p = 12$. (3)

Solution

$$4x^3 + px^2 - 14x + q = 0 \Rightarrow x^3 + \tfrac{1}{4}px^2 - \tfrac{14}{4}x + \tfrac{1}{4}q = 0$$

and

$$\begin{aligned} \alpha + \beta + \gamma &= -\tfrac{1}{4}p \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -\tfrac{7}{2} \\ \alpha\beta\gamma &= -\tfrac{1}{4}q. \end{aligned}$$

Now,

$$\begin{aligned} & (\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma) \\ \Rightarrow \quad & (-\tfrac{1}{4}p)^2 = 16 + 2(-\tfrac{7}{2}) \\ \Rightarrow \quad & \tfrac{1}{16}p^2 = 9 \\ \Rightarrow \quad & p^2 = 144 \\ \Rightarrow \quad & \underline{\underline{p = 12}}, \end{aligned}$$

since p is a real positive constant.

Given that

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \tfrac{14}{3},$$

(b) determine the value of q (3)

Solution

Well,

$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \tfrac{14}{3} &\Rightarrow \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \tfrac{14}{3} \\ &\Rightarrow \frac{-\frac{7}{2}}{-\frac{1}{4}q} = \tfrac{14}{3} \\ &\Rightarrow \frac{-\frac{7}{2}}{\frac{14}{3}} = -\tfrac{1}{4}q \\ &\Rightarrow -\tfrac{3}{4} = -\tfrac{1}{4}q \\ &\Rightarrow \underline{\underline{q = 3}}. \end{aligned}$$

Without solving the cubic equation,

(c) determine the value of

$$(\alpha - 1)(\beta - 1)(\gamma - 1).$$

(4)

Solution

Now,

$$\begin{aligned}(\alpha - 1)(\beta - 1)(\gamma - 1) &= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1 \\ &= -\tfrac{3}{4} - (-\tfrac{7}{2}) + (-3) - 1 \\ &= \underline{\underline{-1\tfrac{1}{4}}}.\end{aligned}$$

7. Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan\theta,\ 0 \leqslant \theta < \tfrac{1}{3}\pi.$$

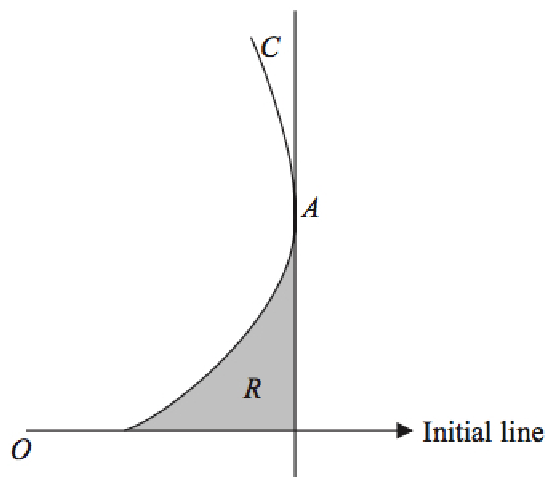


Figure 1: $r = 1 + \tan\theta$

Figure 1 also shows the tangent to C at the point A.
This tangent is perpendicular to the initial line.

(a) Use differentiation to prove that the polar coordinates of A are $(2, \frac{1}{4}\pi)$. (4)

Solution

Well,

$$\begin{aligned} x &= r\cos\theta \\ &= (1+\tan\theta)\cos\theta \\ &= \cos\theta + \sin\theta \end{aligned}$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta + \cos\theta.$$

Now,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\theta} = 0 &\Rightarrow -\sin\theta + \cos\theta = 0 \\ &\Rightarrow \sin\theta = \cos\theta \\ &\Rightarrow \tan\theta = 1 \\ &\Rightarrow \theta = \tfrac{1}{4}\pi \end{aligned}$$

and

$$\begin{aligned} r &= 1 + \tan(\tfrac{1}{4}\pi) \\ &= 2. \end{aligned}$$

Hence, the polar coordinates of A are $\underline{\underline{(2, \tfrac{1}{4}\pi)}}$, as required.

The finite region R, shown shaded in Figure 1, is bounded by C, the tangent at A, and the initial line.

(b) Use calculus to show that the exact area of R is (6)

$$\tfrac{1}{2}(1 - \ln 2).$$

Solution

Well,

$$\begin{aligned}
\text{area bounded by the curve} &= \tfrac{1}{2}\int_0^{\frac{1}{4}\pi} (1+\tan\theta)^2\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\int_0^{\frac{1}{4}\pi} (1+2\tan\theta+\tan^2\theta)\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\int_0^{\frac{1}{4}\pi} \left[1+2\tan\theta+(\sec^2\theta-1)\right]\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\int_0^{\frac{1}{4}\pi} (2\tan\theta+\sec^2\theta)\,\mathrm{d}\theta \\
&= \tfrac{1}{2}\left[2\ln|\sec\theta|+\tan\theta\right]_{x=0}^{\frac{1}{4}\pi} \\
&= \tfrac{1}{2}\left\{2\left(\ln|\sec(\tfrac{1}{4}\pi)|+\tan(\tfrac{1}{4}\pi)\right)-(0+0)\right\} \\
&= \tfrac{1}{2}(2\ln\sqrt{2}+1) \\
&= \ln\sqrt{2}+\tfrac{1}{2} \\
&= \tfrac{1}{2}\ln 2+\tfrac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\text{area of the triangle} &= \tfrac{1}{2}\times 2\cos(\tfrac{1}{4}\pi)\times 2\sin(\tfrac{1}{4}\pi) \\
&= \tfrac{1}{2}\times\sqrt{2}\times\sqrt{2} \\
&= 1.
\end{aligned}$$

Finally,

$$\begin{aligned}
\text{area of } R &= \text{area of the triangle} - \text{area bounded by the curve} \\
&= 1-(\tfrac{1}{2}\ln 2+\tfrac{1}{2}) \\
&= \tfrac{1}{2}-\tfrac{1}{2}\ln\sqrt{2} \\
&= \underline{\underline{\tfrac{1}{2}(1-\ln\sqrt{2})}},
\end{aligned}$$

as required

8. Two birds are flying towards their nest, which is in a tree.

   Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

where λ and μ are scalar parameters and a is a constant.

In the model, the angle between the birds' flight paths is 120°.

(a) Determine the value of a. (4)

Solution

Scalar product:

$$\begin{aligned}
\begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} &= \left|\begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right| \cos 120^\circ \\
\Rightarrow \quad & 0 + a + 0 = \sqrt{2^2 + a^2} \times \sqrt{1^2 + (-1)^2} \times (-\tfrac{1}{2}) \\
\Rightarrow \quad & a = \sqrt{4 + a^2} \times \sqrt{2} \times (-\tfrac{1}{2}) \\
\Rightarrow \quad & a^2 = (4 + a^2) \times 2 \times (\tfrac{1}{4}) \\
\Rightarrow \quad & a^2 = \tfrac{1}{2}(4 + a^2) \\
\Rightarrow \quad & 2a^2 = 4 + a^2 \\
\Rightarrow \quad & a^2 = 4 \\
\Rightarrow \quad & \underline{\underline{a = \pm 2}}.
\end{aligned}$$

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point. (5)

Solution

We make $\mathbf{r}_1 = \mathbf{r}_2$:

$$\begin{aligned}
-1 + 2\lambda &= 4 \quad (1) \\
5 - 2\lambda &= -1 + \mu \quad (2) \\
2 &= 3 - \mu \quad (3).
\end{aligned}$$

Solve (1):

$$-1 + 2\lambda = 4 \Rightarrow 2\lambda = 5$$
$$\Rightarrow \lambda = \tfrac{5}{2}$$

and solve (3):

$$2 = 3 - \mu \Rightarrow \mu = 1.$$

CHECK in (2):

$$5 + (-2)(\tfrac{5}{2}) = 0$$

and

$$-1 + 1 = 0. \quad \checkmark$$

Hence, the coordinates of this common point is

$$\underline{\underline{(4, 0, 2)}}.$$

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2.$$

(c) Hence determine the shortest distance from the nest to the ground level of the park. (3)

Solution

$$\begin{aligned}\text{Shortest distance} &= \frac{\left|\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} - 2\right|}{\sqrt{2^2 + (-3)^2 + 1^2}} \\ &= \frac{|(8 + 0 + 2) - 2|}{\sqrt{4 + 9 + 1}} \\ &= \frac{8}{\sqrt{14}} \\ &= \underline{\underline{\tfrac{4\sqrt{14}}{7}}}.\end{aligned}$$

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer. (1)

Solution

E.g., Not reliable as the birds will not fly in a straight line, as angle between flights paths will not always be 120°.

9.

$$y = \cosh^n x,\ n \geqslant 5.$$

(a) (i) Show that (4)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2} x.$$

Solution

$$\begin{aligned}
& y = \cosh^n x \\
\Rightarrow\ & \frac{\mathrm{d}y}{\mathrm{d}x} = n\cosh^{n-1} x \times \sinh x \\
\Rightarrow\ & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n\sinh x\left[(n-1)\cosh^{n-2} x \times \sinh x\right] + n\cosh^{n-1} x \times \cosh x \\
\Rightarrow\ & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n(n-1)\cosh^{n-2} x\sinh^2 x + n\cosh^n x \\
\Rightarrow\ & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n(n-1)\cosh^{n-2} x(\cosh^2 x - 1) + n\cosh^n x \\
\Rightarrow\ & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n(n-1)\cosh^n x - n(n-1)\cosh^{n-2} x + n\cosh^n x \\
\Rightarrow\ & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = [n^2 - n + n]\cosh^n x - n(n-1)\cosh^{n-2} x \\
\Rightarrow\ & \underline{\underline{\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n^2\cosh^n x - n(n-1)\cosh^{n-2} x}},
\end{aligned}$$

as required.

(ii) Determine an expression for (2)

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}.$$

Solution

Well,

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = \underline{\underline{n^2\left[n^2\cosh^n x - n(n-1)\cosh^{n-2} x\right]}}$$
$$\underline{\underline{-n(n-1)\left[(n-2)^2\cosh^{n-2} x - (n-2)(n-3)\cosh^{n-4} x\right]}}.$$

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form. (2)

Solution

Well, $x = 0$ and

$$\begin{aligned}
y &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= n^2 - n(n-1) = n^2 - n^2 + n = n \\
\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} &= 0 \\
\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} &= n^4 - n^3(n-1) - n(n-1)(n-2)^2 + n(n-1)(n-2)(n-3) \\
&= n^4 - n^4 + n^3 - n(n-1)(n-2)[(n-2) - (n-3)] \\
&= n^3 - n(n-1)(n-2)[1] \\
&= n^3 - n(n-1)(n-2)
\end{aligned}$$

$\times$	n	-1
n	n^2	$-n$
-2	$-2n$	$+3$

$$\begin{aligned}
&= n^3 - n(n^2 - 3n + 2) \\
&= n^3 - n^3 + 3n^2 - 2n \\
&= 3n^2 - 2n \\
&= n(3n - 2).
\end{aligned}$$

Finally, the first three non-zero terms of the Maclaurin series for y are

$$y = 1 + \frac{1}{2!}(n)x^2 + \frac{1}{4!}[n(3n-2)](x^4) + \dots$$
$$\Rightarrow \quad \underline{\underline{y = 1 + \frac{1}{2}nx^2 + \frac{1}{24}n(3n-2)x^4 + \dots}}.$$