Dr Oliver Mathematics
Mathematics: Higher
2010 Paper 1: Non-Calculator
1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

Section A

1. A line L is perpendicular to the line with equation (2)

$$2x - 3y - 6 = 0.$$

What is the gradient of the line L?

A. $-\frac{3}{2}$

B. $-\frac{1}{2}$

C. $\frac{2}{3}$

D. 2

Solution

A

$$\begin{aligned} 2x - 3y - 6 = 0 &\Rightarrow 3y = 2x - 6 \\ &\Rightarrow y = \tfrac{2}{3}x - 2 \end{aligned}$$

and so the gradient of the line L is

$$m = -\frac{1}{\frac{2}{3}} = -\tfrac{3}{2}.$$

2. A sequence is defined by the recurrence relation (2)

$$u_{n+1} = 2u_n + 3 \text{ and } u_0 = 1.$$

What is the value of u_2?

A. 7

B. 10

C. 13

D. 16

Solution

C

$$u_1 = 2 \times 1 + 3 = 5$$
$$u_2 = 2 \times 5 + 3 = 13.$$

3. Given that (2)

$$\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix},$$

find $3\mathbf{u} - 2\mathbf{v}$ in component form.

A. $\begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$

B. $\begin{pmatrix} 4 \\ -4 \\ 11 \end{pmatrix}$

C. $\begin{pmatrix} 8 \\ -1 \\ 5 \end{pmatrix}$

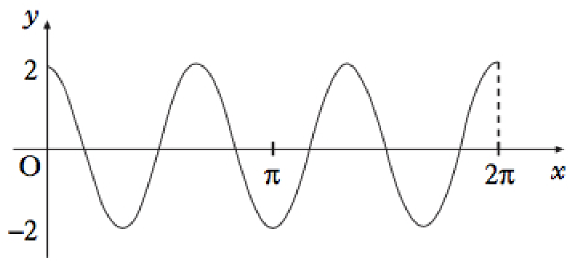
D. $\begin{pmatrix} 8 \\ -4 \\ -5 \end{pmatrix}$

Solution

D

$$\begin{aligned} 3\mathbf{u} - 2\mathbf{v} &= 3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ -4 \\ -5 \end{pmatrix}. \end{aligned}$$

4. The diagram shows the graph with equation of the form $y = a\cos bx$ for $0 \leqslant x \leqslant 2\pi$. (2)



What is the equation of this graph?

A. $y = 2\cos 3x$

B. $y = 2\cos 2x$

C. $y = 3\cos 2x$

D. $y = 4\cos 3x$

Solution

A

$a = 2$ and $b = 3$.

5. When (2)

$$x^2 + 8x + 3$$

is written in the form

$$(x + p)^2 + q,$$

what is the value of q?

A. -19

B. -13

C. -5

D. 19

Solution

B

$$\begin{aligned} x^2 + 8x + 3 &= (x^2 + 8x + 16) + 3 - 16 \\ &= (x + 4)^2 - 13. \end{aligned}$$

6. The roots of the equation (2)

$$kx^2 - 3x + 2 = 0$$

are equal. What is the value of k?

A. $-\frac{9}{8}$

B. $-\frac{8}{9}$

C. $\frac{8}{9}$

D. $\frac{9}{8}$

Solution

D

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (-3)^2 - 4 \times k \times 2 = 0 \\ &\Rightarrow 9 = 8k \\ &\Rightarrow k = \tfrac{9}{8}. \end{aligned}$$

7. A sequence is generated by the recurrence relation (2)

$$u_{n+1} = \tfrac{1}{4}u_n + 7 \text{ and } u_0 = -2.$$

What is the limit of this sequence as $n \to \infty$?

A. $\frac{1}{28}$

B. $\frac{28}{5}$

C. $\frac{28}{3}$

D. 28

Solution

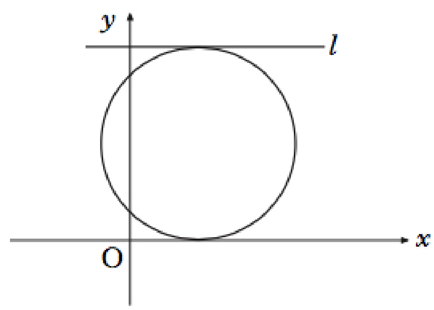
C

Let the sequence tend to u. Now,

$$\begin{aligned} u = \tfrac{1}{4}u + 7 &\Rightarrow \tfrac{3}{4}u = 7 \\ &\Rightarrow u = \tfrac{28}{3}. \end{aligned}$$

8. The equation of the circle shown in the diagram is (2)

$$x^2 + y^2 - 6x - 10y + 9 = 0.$$



The x-axis and the line l are parallel tangents to the circle.

A. $y = 5$

B. $y = 10$

C. $y = 18$

D. $y = 20$

Solution

B

$$x^2 + y^2 - 6x - 10y + 9 = 0 \Rightarrow (x^2 - 6x + 9) + (y^2 - 10y + 25) = -9 + 9 + 25$$
$$\Rightarrow (x-3)^2 + (y-5)^2 = 25.$$

So the centre is at $(3, 5)$ and you double that.

9. Find (2)

$$\int (2x^{-4} + \cos 5x)\,\mathrm{d}x.$$

A. $-\frac{2}{5}x^{-5} - 5\sin 5x + c$

B. $-\frac{2}{5}x^{-5} + \frac{1}{5}\sin 5x + c$

C. $-\frac{2}{3}x^{-3} + \frac{1}{5}\sin 5x + c$

D. $-\frac{2}{3}x^{-3} - 5\sin 5x + c$

Solution

C

$$\int (2x^{-4} + \cos 5x)\,\mathrm{d}x = -\frac{2}{3}x^{-3} + \frac{1}{5}\sin 5x + c.$$

10. The vectors (2)

$$x\mathbf{i} + 5\mathbf{j} + 7\mathbf{k} \text{ and } -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

are perpendicular.
What is the value of x?

A. 0

B. 1

C. $\frac{4}{3}$

D. $\frac{10}{3}$

Solution

B

$$\begin{aligned}(x\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}).(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0 &\Rightarrow -13x + 10 - 7 = 0 \\ &\Rightarrow 3x = 3 \\ &\Rightarrow x = 1.\end{aligned}$$

11. Functions f and g are defined on suitable domains by (2)

$$\mathrm{f}(x) = \cos x \text{ and } \mathrm{g}(x) = x + \tfrac{\pi}{6}.$$

What is the value of $\mathrm{f}\left(\mathrm{g}\left(\frac{\pi}{6}\right)\right)$?

A. $\frac{1}{2} + \frac{\pi}{6}$

B. $\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{1}{2}$

Solution

D

$$\begin{aligned}\mathrm{f}\left(\mathrm{g}\left(\tfrac{\pi}{6}\right)\right) &= \mathrm{f}\left(\tfrac{\pi}{6} + \tfrac{\pi}{6}\right) \\ &= \mathrm{f}\left(\tfrac{\pi}{3}\right) \\ &= \tfrac{1}{2}.\end{aligned}$$

12. If (2)

$$f(x) = \frac{1}{\sqrt[5]{x}},\ x \neq 0,$$

what is $f'(x)$?

A. $-\frac{1}{5}x^{-\frac{6}{5}}$

B. $-\frac{1}{5}x^{-\frac{4}{5}}$

C. $-\frac{5}{2}x^{-\frac{7}{2}}$

D. $-\frac{5}{2}x^{-\frac{3}{5}}$

Solution

A

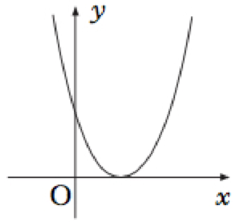
$$f(x) = \frac{1}{\sqrt[5]{x}} \Rightarrow f(x) = x^{-\frac{1}{5}}$$
$$\Rightarrow f'(x) = -\tfrac{1}{5}x^{-\frac{6}{5}}.$$

13. Which of the following diagrams shows a parabola with equation (2)

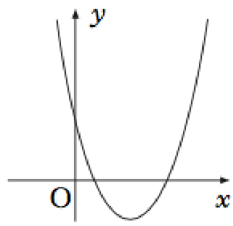
$$y = ax^2 + bx + c,$$

where $a > 0$ and $b^2 - 4ac > 0$?

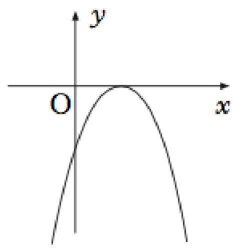
A



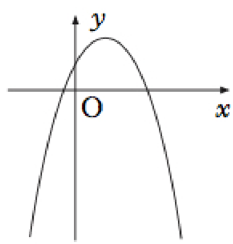
B



C



D

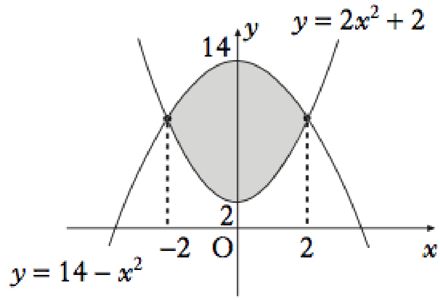


Solution

B

14. The diagram shows graphs with equations (2)

$$y = 14 - x^2 \text{ and } y = 2x^2 + 2.$$



Which of the following represents the shaded area?

A. $\displaystyle\int_{2}^{14} (12 - 3x^2)\,\mathrm{d}x$

B. $\displaystyle\int_{2}^{14} (3x^2 - 12)\,\mathrm{d}x$

C. $\displaystyle\int_{-2}^{2} (12 - 3x^2)\,\mathrm{d}x$

D. $\displaystyle\int_{-2}^{2} (3x^2 - 12)\,\mathrm{d}x$

Solution

C

$$\int_{-2}^{2} \left[(14 - x^2) - (2x^2 + 2)\right]\,\mathrm{d}x = \int_{-2}^{2} (12 - 3x^2)\,\mathrm{d}x.$$

15. The derivative of a function f is given by (2)

$$\mathrm{f}'(x) = x^2 - 9.$$

Here are two statements about f:
(1) f is increasing at $x = 1$;
(2) f is stationary at $x = -3$.

Which of the following is true?

A. Neither statement is correct.

B. Only statement (1) is correct.

C. Only statement (2) is correct.

D. Both statements are correct.

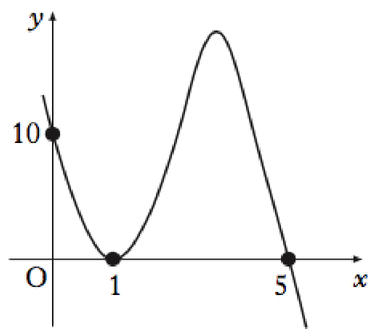
Solution

C

$f'(1) = -8$ so statement (1) is wrong.
$f'(-3) = 0$ so statement (2) is correct.

16. The diagram shows the graph with equation (2)

$$y = k(x-1)^2(x+t).$$



What are the values of k and t?

A. $k = -2$ and $t = -5$

B. $k = -2$ and $t = 5$

C. $k = 2$ and $t = -5$

D. $k = 2$ and $t = 5$

Solution

A

Well, $t = -5$ and

$$(0-1)^2(0-5) = -5$$

which means

$$k = \frac{10}{-5} = -2.$$

17. If (2)

$$s(t) = t^2 - 5t + 8,$$

what is the rate of change of s with respect to t when $t = 3$?

A. -5
B. 1
C. 2
D. 9

Solution

B

$$\mathrm{s}(t) = t^2 - 5t + 8 \Rightarrow \mathrm{s}'(t) = 2t - 5.$$

Finally,

$$t = 3 \Rightarrow \mathrm{s}'(3) = 1.$$

18. What is the solution of (2)

$$x^2 + 4x > 0,$$

where x is a real number?

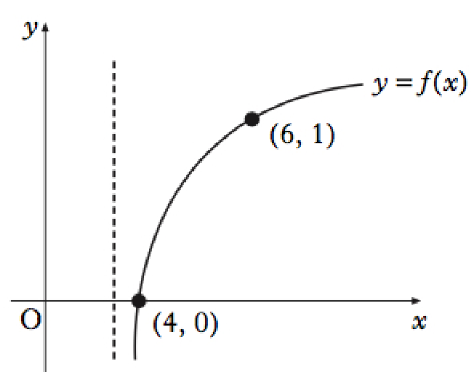
A. $-4 < x < 0$
B. $x < -4$ or $x > 0$
C. $0 < x < 4$
D. $x < 0$ or $x > 4$

Solution

B

$$x^2 + 4x = x(x + 4).$$

19. The diagram shows the graph of $y = \mathrm{f}(x)$ where f is a logarithmic function. (2)



What is $\mathrm{f}(x)$?

A. $f(x) = \log_6(x-3)$

B. $f(x) = \log_3(x+3)$

C. $f(x) = \log_3(x-3)$

D. $f(x) = \log_6(x+3)$

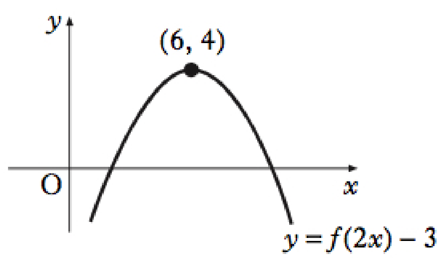
Solution

C

It has been shifted right by 3 units.

20. The diagram shows the graph of (2)

$$y = f(2x) - 3.$$



What are the coordinates of the turning point on the graph of $y = f(x)$?

A. $(12, 7)$

B. $(12, 1)$

C. $(3, 7)$

D. $(3, 1)$

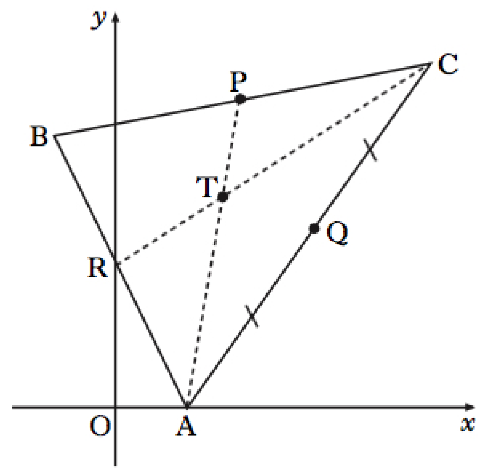
Solution

A

Equation	Vertex
$y = f(2x) - 3$	$(6, 4)$
$y = f(2x)$	$(6, 7)$
$y = f(x)$	$(12, 7)$

Section B

21. Triangle ABC has vertices $A(4,0)$, $B(-4,16)$, and $C(18,20)$, as shown in the diagram below.



Medians AP and CR intersect at the point $T(6,12)$.

(a) Find the equation of median BQ. (3)

Solution
$Q(11,10)$ and the gradient of BQ is

$$\frac{16-10}{-4-11} = -\tfrac{2}{5}$$

and the equation is

$$y - 16 = -\tfrac{2}{5}(x+4) \Rightarrow y - 16 = -\tfrac{2}{5}x - \tfrac{8}{5}$$
$$\Rightarrow \underline{\underline{y = -\tfrac{2}{5}x + \tfrac{72}{5}}}.$$

(b) Verify that T lies on BQ. (1)

Solution

$$-\tfrac{2}{5} \times 6 + \tfrac{72}{5} = 12$$

and so $\underline{\underline{T \text{ lies on } BQ}}$.

(c) Find the ratio in which T divides BQ. (2)

Solution

$$\begin{aligned} BT : TQ &= \sqrt{10^2 + 4^2} : \sqrt{5^2 + 2^2} \\ &= \sqrt{116} : \sqrt{29} \\ &= \sqrt{4 \times 29} : \sqrt{29} \\ &= 2\sqrt{29} : \sqrt{29} \\ &= \underline{\underline{2 : 1}}. \end{aligned}$$

22. (a) (i) Show that $(x-1)$ is a factor of $\mathrm{f}(x) = 2x^3 + x^2 - 8x + 5$. (5)

Solution

1	2	1	−8	5
	↓	2	3	−5
	2	3	−5	0

Since the remainder is zero, $(x-1)$ is a $\underline{\underline{\text{factor}}}$ of $\mathrm{f}(x)$.

(ii) Hence factorise $\mathrm{f}(x)$ fully.

Solution

$$\begin{aligned} 2x^3 + x^2 - 8x + 5 &= (x-1)(2x^2 + 3x - 5) \\ &= (x-1)(2x+5)(x-1) \\ &= \underline{\underline{(x-1)^2(2x+5)}}. \end{aligned}$$

(b) Solve (1)

$$2x^3 + x^2 - 8x + 5 = 0.$$

Solution

$\underline{\underline{x = 1 \text{ (repeated twice) or } x = -2\tfrac{1}{2}}}$.

The line with equation $y = 2x - 3$ is a tangent to the curve with equation $y = 2x^3 + x^2 - 6x + 2$ at the point G.

(c) Find the coordinates of G. (5)

Solution

$$2x^3 + x^2 - 6x + 2 = 2x - 3 \Rightarrow 2x^3 + x^2 - 8x + 5 = 0$$
$$\Rightarrow (x-1)^2(2x+5) = 0.$$

Now, the tangent occurs where have equal roots:

$$x = 1 \Rightarrow y = -1$$

and so $\underline{\underline{G(1, -1)}}$.

This tangent meets the curve again at the point H.

(d) Write down the coordinates of H. (1)

Solution

$$x = -2\tfrac{1}{2} \Rightarrow y = -8$$

and so $\underline{\underline{H(-2\tfrac{1}{2}, -8)}}$.

23. Diagram 1 shows a right-angled triangle, where the line OA has equation $3x - 2y = 0$.

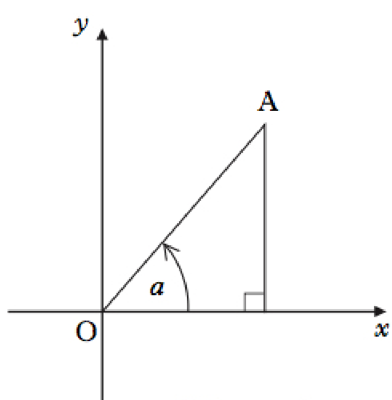


Diagram 1

(a) (i) Show that $\tan a = \frac{3}{2}$. (4)

Solution

$$3x - 2y = 0 \Rightarrow 2y = 3x$$
$$\Rightarrow y = \tfrac{3}{2}x.$$

Finally,

$$\tan\theta = m \Rightarrow \tan a = \tfrac{3}{2}.$$

(ii) Find the value of $\sin a$.

Solution

$$\text{hypotenuse} = \sqrt{2^2 + 3^2} = \sqrt{13}$$

and so

$$\sin a = \tfrac{3}{\sqrt{13}} \text{ or } \tfrac{3\sqrt{13}}{13}.$$

A second right angled triangle is added as shown in Diagram 2.

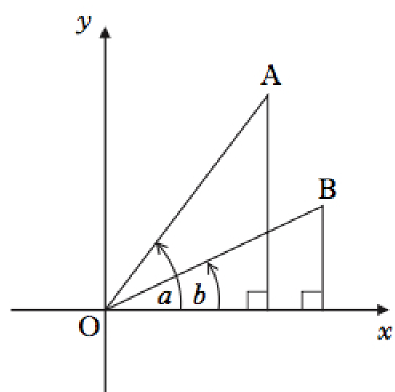


Diagram 2

The line OB has equation $3x - 4y = 0$.

(b) Find the values of $\sin b$ and $\cos b$. (4)

Solution

$$3x - 4y = 0 \Rightarrow 4y = 3x$$
$$\Rightarrow y = \tfrac{3}{4}x$$

and

$$\text{hypotenuse} = \sqrt{4^2 + 3^2} = 5.$$

Hence,

$$\sin b = \underline{\underline{\tfrac{3}{5}}} \text{ and } \cos b = \underline{\underline{\tfrac{4}{5}}}.$$

(c) (i) Find the value of $\sin(a - b)$. (4)

Solution

$$\begin{aligned}\sin(a-b) &= \sin a \cos b - \sin b \cos a \\ &= \left(\tfrac{3\sqrt{13}}{13} \times \tfrac{4}{5}\right) - \left(\tfrac{3}{5} \times \tfrac{2\sqrt{13}}{13}\right) \\ &= \tfrac{12\sqrt{13}}{13} - \tfrac{6\sqrt{13}}{13} \\ &= \underline{\underline{\tfrac{6\sqrt{13}}{65}}}.\end{aligned}$$

(ii) State the value of $\sin(b - a)$.

Solution

Use $a - b = -(b - a)$:

$$\sin(b - a) = \underline{\underline{-\tfrac{6\sqrt{13}}{65}}}.$$