Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2004 November Paper 2: Calculator
2 hours

The total number of marks available is 80.
You must write down all the stages in your working.

1. Given that (4)

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -5 & 4 \end{pmatrix},$$

find $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$\begin{aligned} 2x + 3y + 4 &= 0 \\ -5x + 4y + 13 &= 0. \end{aligned}$$

Solution

Well,

$$\mathbf{A}^{-1} = \tfrac{1}{23}\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}$$

so

$$\begin{aligned} & \begin{pmatrix} 2 & 3 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -13 \end{pmatrix} \\ \Rightarrow\ & \tfrac{1}{23}\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{23}\begin{pmatrix} 4 & -3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -4 \\ -13 \end{pmatrix} \\ \Rightarrow\ & \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{23}\begin{pmatrix} 23 \\ -46 \end{pmatrix} \\ \Rightarrow\ & \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}; \end{aligned}$$

hence,

$$\underline{\underline{x = 1,\ y = -2}}.$$

2. Given that (4)

$$\sqrt{a + b\sqrt{3}} = \frac{13}{4 + \sqrt{3}},$$

where a and b are integers, find, without using a calculator, the value of a and b

Solution

$$\sqrt{a+b\sqrt{3}} = \frac{13}{4+\sqrt{3}} \Rightarrow a+b\sqrt{3} = \left(\frac{13}{4+\sqrt{3}}\right)^2$$

$\times$	4	$+\sqrt{3}$
4	16	$+4\sqrt{3}$
$+\sqrt{3}$	$+4\sqrt{3}$	$+3$

$$\Rightarrow a+b\sqrt{3} = \frac{169}{19+8\sqrt{3}}$$
$$\Rightarrow a+b\sqrt{3} = \frac{169}{19+8\sqrt{3}} \times \frac{19-8\sqrt{3}}{19-8\sqrt{3}}$$

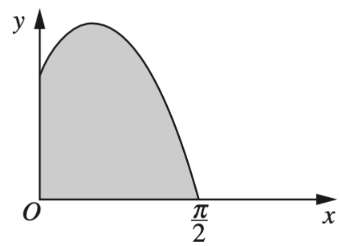
$\times$	19	$+8\sqrt{3}$
19	361	$+152\sqrt{3}$
$-8\sqrt{3}$	$-152\sqrt{3}$	-192

$$\Rightarrow a+b\sqrt{3} = \frac{169(19-8\sqrt{3})}{169}$$
$$\Rightarrow a+b\sqrt{3} = 19-8\sqrt{3};$$

so, $\underline{\underline{a=19}}$ and $\underline{\underline{b=-8}}$.

3. The diagram shows part of the curve (5)

$$y = 3\sin 2x + 4\cos x.$$



Find the area of the shaded region, bounded by the curve, and the coordinate axes.

Solution

$$\begin{aligned}\text{Area} &= \int_0^{\frac{1}{2}\pi} (3\sin 2x + 4\cos x)\,\mathrm{d}x \\ &= \left[-\tfrac{3}{2}\cos 2x + 4\sin x\right]_{x=0}^{\frac{1}{2}\pi} \\ &= \left(\tfrac{3}{2} + 4\right) - \left(-\tfrac{3}{2} + 0\right) \\ &= \underline{\underline{7}}.\end{aligned}$$

4. Find the values of k for which the line (5)

$$y = x + 2$$

meets the curve

$$y^2 + (x + k)^2 = 2.$$

Solution

$\times$	x	$+2$
x	x^2	$+2x$
$+2$	$+2x$	$+4$

and

$\times$	x	$+k$
x	x^2	$+kx$
$+2$	$+kx$	$+k^2$

Now,

$$\begin{aligned} y^2 + (x+k)^2 = 2 &\Rightarrow (x+2)^2 + (x+k)^2 = 2 \\ &\Rightarrow (x^2 + 4x + 4) + (x^2 + 2kx + k^2) = 2 \\ &\Rightarrow 2x^2 + (4+2k)x + (2+k^2) = 0. \end{aligned}$$

Next,

$$\begin{aligned} b^2 - 4ac = 0 &\Rightarrow (4+2k)^2 - 4 \times 2 \times (2+k^2) = 0 \\ &\Rightarrow 16 + 16k + 4k^2 - (16 + 4k^2) = 0 \\ &\Rightarrow -4k^2 + 16k = 0 \\ &\Rightarrow -4k(-k+4) = 0 \\ &\Rightarrow \underline{\underline{k = 0 \text{ or } k = 4}}. \end{aligned}$$

5. Solve the equation (6)

$$\log_{16}(3x-1) = \log_4(3x) + \log_4(0.5).$$

Solution

$$\begin{aligned} & \log_{16}(3x-1) = \log_4(3x) + \log_4(0.5) \\ \Rightarrow\quad & \frac{\log_4(3x-1)}{\log_4 16} = \log_4[3x \times 0.5] \\ \Rightarrow\quad & \frac{\log_4(3x-1)}{2} = \log_4(\tfrac{3}{2}x) \\ \Rightarrow\quad & \log_4(3x-1) = 2\log_4(\tfrac{3}{2}x) \\ \Rightarrow\quad & \log_4(3x-1) = \log_4(\tfrac{3}{2}x)^2 \\ \Rightarrow\quad & 3x - 1 = (\tfrac{3}{2}x)^2 \\ \Rightarrow\quad & 3x - 1 = \tfrac{9}{4}x^2 \\ \Rightarrow\quad & 12x - 4 = 9x^2 \\ \Rightarrow\quad & 9x^2 - 12x + 4 = 0 \end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -12 \\ \text{multiply to:} & (+9)\times(+4) = +36\end{array}\right\} -3 \text{ (repeated)}$$

$$\begin{aligned} &\Rightarrow\quad (3x-2)^2 = 0 \\ &\Rightarrow\quad \underline{\underline{x = \tfrac{2}{3}}}. \end{aligned}$$

6. Given that

$$x = 3\sin\theta - 2\cos\theta \text{ and } y = 3\cos\theta + 2\sin\theta,$$

(a) find the value of the acute angle θ for which $x = y$, (3)

Solution

Now,

$$\begin{aligned} x = y &\Rightarrow 3\sin\theta - 2\cos\theta = 3\cos\theta + 2\sin\theta \\ &\Rightarrow \sin\theta = 5\cos\theta \\ &\Rightarrow \tan\theta = 5 \\ &\Rightarrow \theta = 1.373\,400\,767 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 1.37 \text{ (3 sf)}}}. \end{aligned}$$

(b) show that (3)

$$x^2 + y^2$$

is constant for all values of θ.

Solution

Now,

$\times$	$3\sin\theta$	$-2\cos\theta$
$3\sin\theta$	$9\sin^2\theta$	$-6\sin\theta\cos\theta$
$-2\cos\theta$	$-6\sin\theta\cos\theta$	$+4\cos^2\theta$

and

×	$3\cos\theta$	$+2\sin\theta$
$3\cos\theta$	$9\cos^2\theta$	$+6\sin\theta\cos\theta$
$+2\sin\theta$	$+6\sin\theta\cos\theta$	$+4\sin^2\theta$

Finally,

$$\begin{aligned} & x^2 + y^2 \\ = \; & (3\sin\theta - 2\cos\theta)^2 + (3\cos\theta + 2\sin\theta)^2 \\ = \; & (9\sin^2\theta - 12\sin\theta\cos\theta + 4\cos^2\theta) + (9\cos^2\theta + 12\sin\theta\cos\theta + 4\sin^2\theta) \\ = \; & 13\sin^2\theta + 13\cos^2\theta \\ = \; & 13(\sin^2\theta + \cos^2\theta) \\ = \; & \underline{\underline{13}}. \end{aligned}$$

7. Given that (7)

$$6x^3 + 5ax - 12a$$

leaves a remainder of -4 when divided by $(x - a)$, find the possible values of a.

Solution

We use synthetic division:

a	6	0	$5a$	$-12a$
	↓	$6a$	$6a^2$	$6a^3 + 5a^2$
	6	$6a$	$6a^2 + 5a$	$6a^3 + 5a^2 - 12a$

Now,

$$6a^3 + 5a^2 - 12a = -4 \Rightarrow 6a^3 + 5a^2 - 12a + 4 = 0.$$

Next, let

$$\mathrm{f}(a) = 6a^3 + 5a^2 - 12a + 4:$$

$$\begin{aligned} \mathrm{f}(1) &= 6 + 5 - 12 + 4 = 3 \\ \mathrm{f}(-1) &= -6 + 5 + 12 + 4 = 15 \\ \mathrm{f}(2) &= 48 + 20 - 24 + 4 = 47 \\ \mathrm{f}(-2) &= -48 + 20 + 24 + 4 = 0; \end{aligned}$$

well, $(a+2)$ is a root.

-2	6	5	-12	4
	$\downarrow$	-12	14	-4
	6	-7	2	0

so

$$6a^3 + 5a^2 - 12a + 4 = 0 \Rightarrow (a+2)(6a^2 - 7a + 2) = 0$$

$$\left.\begin{array}{ll} \text{add to:} & -7 \\ \text{multiply to:} & (+6) \times (+2) = +12 \end{array}\right\} -4, -3$$

e.g.,

$$\begin{aligned} &\Rightarrow (a+2)[6a^2 - 4a - 3a + 2] = 0 \\ &\Rightarrow (a+2)[2a(3a-2) - 1(3a+2)] = 0 \\ &\Rightarrow (a+2)(2a-1)(3a-2) = 0 \\ &\Rightarrow \underline{\underline{a = -2,\ a = \tfrac{1}{2},\ \text{or } a = \tfrac{2}{3}}}. \end{aligned}$$

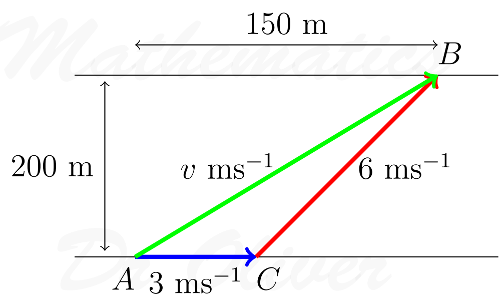
8. A motor boat travels in a straight line across a river which flows at 3 ms^{-1} between straight parallel banks 200 m apart. (7)

The motor boat, which has a top speed of 6 ms^{-1} in still water, travels directly from a point A on one bank to a point B, 150 m downstream of A, on the opposite bank.

Assuming that the motor boat is travelling at top speed, find, to the nearest second, the time it takes to travel from A to B.

Solution

Let v ms^{-1} be the speed of motor boat.



Now,

$$\tan = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan BAC = \frac{200}{150}$$
$$\Rightarrow \angle BAC = 53.130\,102\,235 \text{ (FCD)}.$$

Next, we apply the sine rule:

$$\frac{\sin ABC}{AC} = \frac{\sin BAC}{BC} \Rightarrow \frac{\sin ABC}{3} = \frac{\sin BAC}{6}$$
$$\Rightarrow \frac{\sin ABC}{3} = \frac{0.8}{6}$$
$$\Rightarrow \sin ABC = \tfrac{2}{5}$$
$$\Rightarrow \angle ABC = 23.578\,178\,48 \text{ (FCD)}$$

and

$$\angle ACB = 180 - (53.130\ldots + 23.578\ldots)$$
$$= 103.291\,719\,2 \text{ (FCD)}.$$

Now,

$$\frac{v}{\sin 103.291\ldots} = \frac{6}{\sin 53.130\ldots} \Rightarrow v = \frac{6 \sin 103.291\ldots}{\sin 53.130\ldots}$$
$$\Rightarrow v = 7.299\,090\,834 \text{ (FCD)}.$$

Finally,

$$\text{distance} = \sqrt{150^2 + 200^2}$$
$$= 250$$

and

$$\text{time taken} = \frac{250}{7.299\ldots}$$
$$= 34.250\,841\,06 \text{ (FCD)}$$
$$= \underline{\underline{34 \text{ s (nearest second)}}}.$$

9. In order that each of the equations (7)

$$y = ab^x \quad y = Ax^k \quad px + qy = xy,$$

where a, b, A, k, p, and q are unknown constants, may be represented by a straight line, they each need to be expressed in the form

$$Y = mX + c,$$

where X and Y are each functions of x and/or y, and m and c are constants.

	Y	X	m	c
$y = ab^x$				
$y = Ax^k$				
$px + qy = xy$				

Complete the following table and insert in it an expression for Y, X, m, and c for each case.

Solution

Well,

$$\begin{aligned} y = ab^x &\Rightarrow \log y = \log(ab^x) \\ &\Rightarrow \log y = \log a + \log b^x \\ &\Rightarrow \log y = \log a + x \log b, \end{aligned}$$

$$\begin{aligned} y = Ax^k &\Rightarrow \log y = \log(Ax^k) \\ &\Rightarrow \log y = \log A + \log x^k \\ &\Rightarrow \log y = \log A + k \log x, \end{aligned}$$

and

$$\begin{aligned} px + qy = xy &\Rightarrow \frac{px + qy}{xy} = 1 \\ &\Rightarrow \frac{px}{xy} + \frac{qy}{xy} = 1 \\ &\Rightarrow \frac{p}{y} + \frac{q}{x} = 1, \end{aligned}$$

and so

	Y	X	m	c
$y = ab^x$	$\underline{\underline{\log y}}$	$\underline{\underline{x}}$	$\underline{\underline{\log b}}$	$\underline{\underline{\log a}}$
$y = Ax^k$	$\underline{\underline{\log y}}$	$\underline{\underline{\log x}}$	$\underline{\underline{k}}$	$\underline{\underline{\log A}}$
$px + qy = xy$	$\underline{\underline{\frac{1}{y}}}$	$\underline{\underline{\frac{1}{x}}}$	$\underline{\underline{-\frac{q}{p}}}$	$\underline{\underline{\frac{1}{p}}}$

10. The function f is defined by

$$\mathrm{f} : x \mapsto |x^2 - 8x + 7|$$

for the domain $3 \leqslant x \leqslant 8$.

(a) By first considering the stationary value of the function (4)

$$x \mapsto x^2 - 8x + 7,$$

show that the graph of $y = \mathrm{f}(x)$ has a stationary point at $x = 4$ and determine the nature of this stationary point.

Solution

Now,

$$y = x^2 - 8x + 7 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 8$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2.$$

Now,

$$x = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

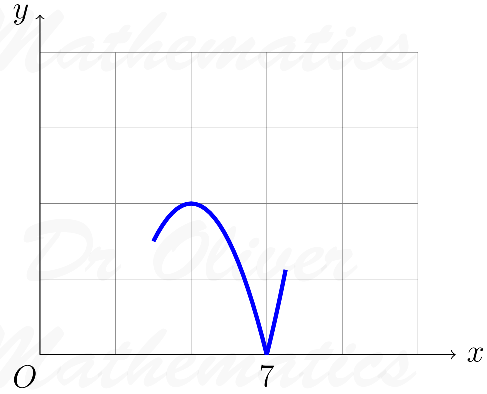
and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 > 0;$$

Hence, it will have a stationary point at $\underline{\underline{x = 4}}$ and this is a $\underline{\underline{\text{minimum}}}$.

(b) Sketch the graph of $y = \mathrm{f}(x)$. (2)

Solution



(c) Find the range of f. (2)

Solution

Well,

$$x = 4 \Rightarrow |4^2 - 8(4) + 7| = 9$$

so

$$\underline{\underline{0 \leqslant \mathrm{f}(x) \leqslant 9}}.$$

The function g is defined by

$$\mathrm{g} : x \mapsto |x^2 - 8x + 7|$$

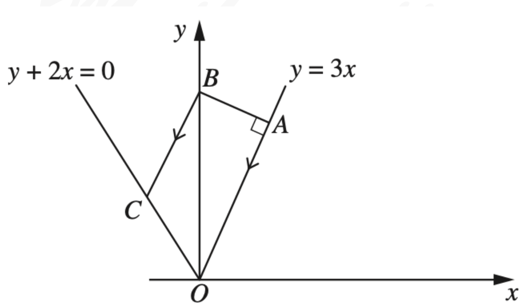
for the domain $3 \leqslant x \leqslant k$.

(d) Determine the largest value of k for which g^{-1} exists. (1)

Solution

$\underline{\underline{k = 4}}$.

11. The diagram shows a trapezium $OABC$, where O is the origin. (10)



The equation of OA is

$$y = 3x$$

and the equation of OC is

$$y + 2x = 0.$$

The line through A perpendicular to OA meets the y-axis at B and BC is parallel to AO.

Given that the length of OA is $\sqrt{250}$ units, calculate the coordinates of A, of B, and of C.

Solution

Let $A(x, 3x)$. Then

$$\begin{aligned} OA = \sqrt{250} &\Rightarrow OA^2 = 250 \\ &\Rightarrow x^2 + (3x)^2 = 250 \\ &\Rightarrow 10x^2 = 250 \\ &\Rightarrow x^2 = 25 \\ &\Rightarrow x = 5, \end{aligned}$$

so $\underline{\underline{A(5, 15)}}$.

Now,

$$m_{OA} = 3 \Rightarrow m_{AB} = -\tfrac{1}{3}$$

and, so, the equation of AB is

$$y - 15 = -\tfrac{1}{3}(x - 5).$$

Next,

$$\begin{aligned} x = 0 &\Rightarrow y - 15 = -\tfrac{1}{3}(0 - 5) \\ &\Rightarrow y - 15 = 1\tfrac{2}{3} \\ &\Rightarrow y = 16\tfrac{2}{3}, \end{aligned}$$

so $\underline{\underline{B(0, 16\tfrac{2}{3})}}$.

Finally, the equation of BC is

$$y = 3x + 16\tfrac{2}{3}$$

and this meets

$$y + 2x = 0 \Rightarrow y = -2x$$

at

$$-2x = 3x + 16\tfrac{2}{3} \Rightarrow 5x = -16\tfrac{2}{3}$$
$$\Rightarrow x = -3\tfrac{1}{3}$$
$$\Rightarrow y = 6\tfrac{2}{3},$$

so $\underline{\underline{C(-3\tfrac{1}{3}, 6\tfrac{2}{3})}}$

EITHER

12. A particle, travelling in a straight line, passes a fixed point O on the line with a speed of 0.5 ms^{-1}.

The acceleration, a ms^{-2}, of the particle, t s after passing O, is given by

$$a = 1.4 - 0.6t.$$

(a) Show that the particle comes instantaneously to rest when $t = 5$. (4)

Solution

Now,

$$a = 1.4 - 0.6t \Rightarrow v = 1.4t - 0.3t^2 + c,$$

where c is a constant. Next,

$$t = 0, v = 0.5 \Rightarrow 0.5 = 0 - 0 + c$$
$$\Rightarrow c = 0.5$$

and

$$v = 1.4t - 0.3t^2 + 0.5.$$

When $t = 5$,

$$v = 1.4(5) - 0.3(5^2) + 0.5$$
$$= 7 - 7.5 + 0.5$$
$$= 0;$$

hence, the particle comes <u>instantaneously to rest</u> when $t = 5$.

(b) Find the total distance travelled by the particle between $t = 0$ and $t = 10$. (6)

Solution

Well,

$$\begin{aligned} s_1 &= \int_0^5 (1.4t - 0.3t^2 + 0.5)\,\mathrm{d}t \\ &= \left[0.7t^2 - 0.1t^3 + 0.5t\right]_{x=0}^{5} \\ &= (17.5 - 12.5 + 2.5) - (0 - 0 + 0) \\ &= 7.5 \end{aligned}$$

and

$$\begin{aligned} s_2 &= \int_5^{10} (1.4t - 0.3t^2 + 0.5)\,\mathrm{d}t \\ &= \left[0.7t^2 - 0.1t^3 + 0.5t\right]_{x=5}^{10} \\ &= (70 - 100 + 5) - (17.5 - 12.5 + 2.5) \\ &= -25 - 7.5 \\ &= -32.5. \end{aligned}$$

Hence,

$$\begin{aligned} \text{distance} &= |s_1| + |s_2| \\ &= 7.5 + 32.5 \\ &= \underline{\underline{40 \text{ m}}}. \end{aligned}$$

OR

13. Each member of a set of curves has an equation of the form

$$y = ax + \frac{b}{x^2},$$

where a and b are integers.

(a) For the curve where $a = 3$ and $b = 2$, find the area bounded by the curve, the x-axis, and the lines $x = 2$ and $x = 4$. (4)

Solution

$$y = 3x + \frac{2}{x^2} = 3x + 2x^{-2}$$

and

$$\int_2^4 (3x + 2x^{-2})\,\mathrm{d}x = \left[\tfrac{3}{2}x^2 - 2x^{-1}\right]_{x=2}^4$$
$$= \left(24 - \tfrac{1}{2}\right) - (6 - 1)$$
$$= 23\tfrac{1}{2} - 5$$
$$= \underline{\underline{18\tfrac{1}{2}}}.$$

Another curve of this set has a stationary point at $(2, 3)$.

(b) Find the value of a and of b in this case and determine the nature of the stationary point. (6)

Solution

Well, $(2, 3)$ lies on the curve so

$$3 = 2a + \frac{b}{4}.$$

Now,

$$y = ax + bx^{-2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = a - 2bx^{-3}$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\Rightarrow a - 2b(2^{-3}) = 0$$
$$\Rightarrow a - \tfrac{1}{4}b = 0$$
$$\Rightarrow a = \tfrac{1}{4}b.$$

Next, let $\underline{\underline{a = 1}}$ and $\underline{\underline{b = 4}}$ (say). Then

$$y = x + 4x^{-2} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 8x^{-3}$$
$$\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 24x^{-4}.$$

Now,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 1 - \frac{8}{x^3} = 0$$
$$\Rightarrow x^3 = 8$$
$$\Rightarrow x = 2$$

and

$$x = 2 \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \tfrac{3}{2} > 0;$$

so, it is a minimum.