Dr Oliver Mathematics
Proof by Contrapositive

In this note, we will explore proof by contrapositive.

The contrapositive of the statement "if A, then B" is "if not B, then not A." A statement and its contrapositive are logically equivalent: if the statement is true, then its contrapositive is true, and vice versa.

Example 1
If $a^7 + 1$ is even, then a is odd.

Solution
We need to prove "if a is even, then $a^7 + 1$ is odd".
So, a is even and $a = 2k$ for some constant k. Then,

$$\begin{aligned} a^7 + 1 &= (2k)^7 + 1 \\ &= 128k^7 + 1 \\ &= 2(64k^7) + 1, \end{aligned}$$

and we get an odd number out.

So, we have proven the result by contraposition. ■

Example 2
Let a, b, and $n \in \mathbb{Z}$. If n does not divide ab, then n does not divide a **and** n does not divide b.

Solution
We need to prove "if $n|a$ **or** $n|b$, then $n|ab$".
Suppose that n divides a. Then $a = nc$ for some $c \in \mathbb{Z}$, and

$$ab = (nc)b = n(cb),$$

so $n|ab$.
Similarly, $b = nd$ for some $d \in \mathbb{Z}$, and

$$ab = a(nd) = n(ad),$$

so $n|ab$.
So, we have proven the result by contraposition. ■

Here are some examples for you to try.

1. Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then x is odd.

Solution

We need to prove "if x is even, then $x^2 - 6x + 5$ is odd".
Write $x = 2$a for some $a \in \mathbb{Z}$. Then

$$\begin{aligned} x^2 - 6x + 5 &= (2a)^2 - 6(2a) + 5 \\ &= 4a^2 - 12a + 5 \\ &= 2(2a^2 - 6a + 2) + 1, \end{aligned}$$

and we get an odd number out.

So, we have proven the result by contraposition.

2. Suppose $x, y \in \mathbb{R}$. If

$$y^3 + yx^2 \leqslant x^3 + xy^2,$$

then $y \leqslant x$.

Solution

We need to prove "if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$".
Then,

$$\begin{aligned} y > x &\Rightarrow y - x > 0 \\ &\Rightarrow (y - x)(x^2 + y^2) > 0(x^2 + y^2) \\ &\Rightarrow y^3 + yx^2 - x^3 - xy^2 > 0 \\ &\Rightarrow y^3 + yx^2 > x^3 + xy^2. \end{aligned}$$

So, we have proven the result by contraposition.

3. For any integers a and b, $a + b \geqslant 15$ implies that $a \geqslant 8$ or $b \geqslant 8$.

Solution

We need to prove "if $a < 8$ **and** $b < 8$, then $a + b < 15$".
So, suppose that a and b are integers such that $a < 8$ and $b < 8$. Since they are integers (not, e.g., real numbers), this implies that $a \leqslant 7$ and $b \leqslant 7$. Adding these two equations together, we find that $a + b \leqslant 14$. But this implies that $a + b < 15$.

So, we have proven the result by contraposition.