# Dr Oliver Mathematics
# Mathematics: Higher
# 2013 Paper 1: Non-Calculator
# 1 hour 30 minutes

The total number of marks available is 70.
You must write down all the stages in your working.

## Section A

1. The functions f and g are defined by (2)

$$f(x) = x^2 + 1 \text{ and } g(x) = 3x - 4,$$

on the set of real numbers.

Find $g(f(x))$.

A. $3x^2 - 1$

B. $9x^2 - 15$

C. $9x^2 + 17$

D. $3x^3 - 4x^2 + 3x - 4$

**Solution**

**A**

$$\begin{aligned} g(f(x)) &= g(x^2 + 1) \\ &= 3(x^2 + 1) - 4 \\ &= 3x^2 - 1. \end{aligned}$$

2. The point $P(5, 12)$ lies on the curve with equation (2)

$$y = x^2 - 4x + 7.$$

What is the gradient of the tangent to this curve at $P$?

A. 2

B. 6

C. 12

D. 13

**Solution**

**B**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 4$$

and

$$x = 5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6.$$

3. Calculate the discriminant of the quadratic equation (2)

$$2x^2 + 4x + 5 = 0.$$

A. $-32$

B. $-24$

C. 48

D. 56

**Solution**

**B**

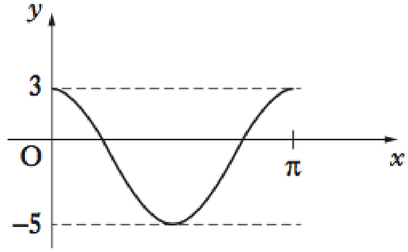
$a = 2$, $b = 4$, and $c = 5$:

$$b^2 - 4ac = 16 - 4 \times 2 \times 5 = -24.$$

4. Which of the following shows the graph of (2)

$$y = 4\cos 2x - 1, \text{ for } 0 \leqslant x \leqslant \pi?$$

A

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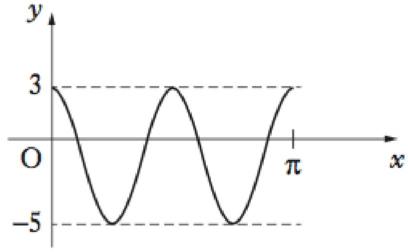


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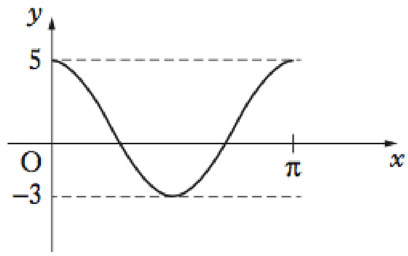
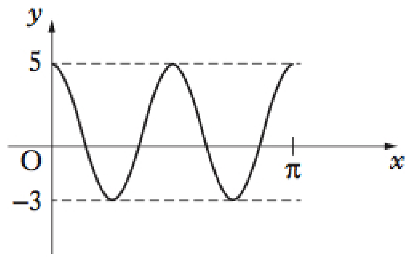



D




**Solution**

**A**

Well, $-5 \leqslant y \leqslant 3$ so it means either A or B. And is not B – the graph in that case is $y = 4\cos 4x - 1$.

5. The line $L$ passes through the point $(-2, -1)$ and is parallel to the line with equation (2)

$$5x + 3y - 6 = 0.$$

What is the equation of $L$?

A. $3x + 5y - 11 = 0$

B. $3x + 5y + 11 = 0$

C. $5x + 3y - 13 = 0$

D. $5x + 3y + 13 = 0$

**Solution**

**D**

$$5(-2) + 3(-1) = -13$$

and so it is

$$5x + 3y = -13 \Rightarrow 5x + 3y + 13 = 0.$$

6. What is the remainder when (2)

$$x^3 + 3x^2 - 5x - 6$$

is divided by $(x - 2)$?

A. 0

B. 3

C. 4

D. 8

**Solution**

**C**

We use synthetic division.

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | −5 | −6 |
| | ↓ | 2 | 10 | 10 |
| | 1 | 5 | 5 | 4 |

7. Find (2)

$$\int x(3x + 2)\,\mathrm{d}x.$$

A. $x^3 + c$

B. $x^3 + x^2 + c$

C. $\frac{1}{2}x^2(\frac{3}{2}x^2 + 2x) + c$

D. $3x^2 + 2x + c$

**Solution**

**B**

$$\begin{aligned}\int x(3x+2)\,\mathrm{d}x &= \int (3x^2+2x)\,\mathrm{d}x \\ &= x^3 + x^2 + c.\end{aligned}$$

8. A sequence is defined by the recurrence relation (2)

$$u_{n+1} = 0.1u_n + 8, \text{ with } u_1 = 11.$$

Here are two statements about this sequence:

(1) $u_0 = 9.1$;
(2) The sequence has a limit as $u \to \infty$.

Which of the following is true?

A. Neither statement is correct.

B. Only statement (1) is correct.

C. Only statement (2) is correct.

D. Both statements are correct.

**Solution**

**C**

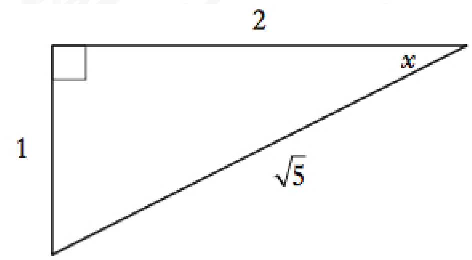
$$\begin{aligned}9.1 = 0.1u_0 + 8 &\Rightarrow 0.1u_0 = 1.1 \\ &\Rightarrow u_0 = 11\end{aligned}$$

so (1) is not true. Now, let the limit (if it exists) be $u$. Then,

$$\begin{aligned}u = 0.1u + 8 &\Rightarrow 0.9u = 8 \\ &\Rightarrow u = \frac{8}{0.9}\end{aligned}$$

so (2) is true.

9. The diagram shows a right-angled triangle with sides and angles as marked. (2)




Find the value of $\sin 2x$.

A. $\frac{4}{5}$

B. $\frac{2}{5}$

C. $\frac{2}{\sqrt{5}}$

D. $\frac{1}{\sqrt{5}}$

**Solution**

**A**

$$\begin{aligned}\sin 2x &= 2\sin x\cos x\\ &= 2\times\tfrac{1}{\sqrt{5}}\times\tfrac{2}{\sqrt{5}}\\ &= \tfrac{4}{5}.\end{aligned}$$

10. If $0 < a < 90$, which of the following is equivalent to $\cos(270 - a)^\circ$? (2)

A. $\cos a^\circ$

B. $\sin a^\circ$

C. $-\cos a^\circ$

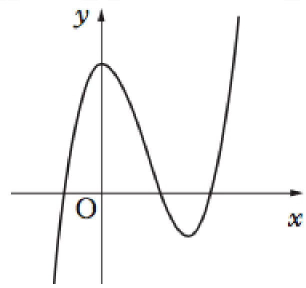
D. $-\sin a^\circ$

**Solution**

**D**

$$\begin{aligned}\cos(270 - a)^\circ &= \cos 270^\circ \cos a^\circ + \sin 270^\circ \sin a^\circ\\ &= 0 - \sin a^\circ\\ &= \sin a^\circ.\end{aligned}$$

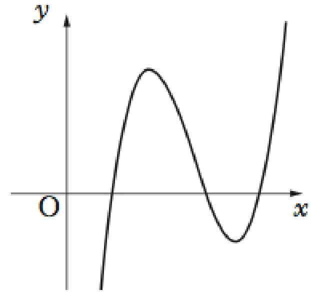
11. The diagram shows a cubic curve with equation $y = \mathrm{f}(x)$. (2)



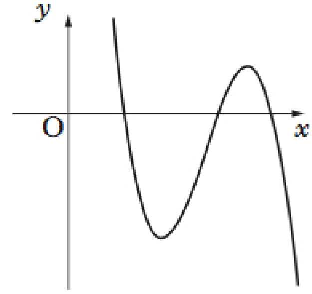

Which of the following diagrams could show the curve with equation

$$y = -\mathrm{f}(x - k),\ k > 0?$$

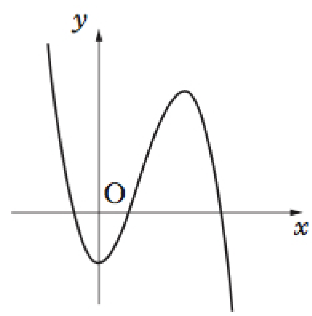
A



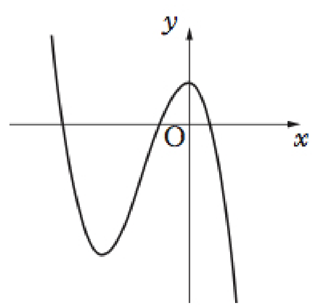

B




C




D




**Solution**

**B**

$A$ is a translation; $C$ is a reflection in the $x$-axis; $D$ is a reflection in the $x$-axis and then a translation by $-c$.

12. If (2)

$$\mathbf{f} = 3\mathbf{i} + 2\mathbf{k} \text{ and } \mathbf{g} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k},$$

find

$$|\mathbf{f} + \mathbf{g}|.$$

A. $\sqrt{14}$ units
B. $\sqrt{42}$ units
C. $\sqrt{66}$ units
D. $\sqrt{70}$ units

**Solution**

**C**

$$\begin{aligned}|\mathbf{f} + \mathbf{g}| &= |(3\mathbf{i} + 2\mathbf{k}) + (2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})| \\ &= |5\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}| \\ &= \sqrt{5^2 + 4^2 + 5^2} \\ &= \sqrt{66}.\end{aligned}$$

13. A function f is defined on a suitable domain by (2)

$$\mathrm{f}(x) = \frac{x+2}{x^2 - 7x + 12}.$$

What value(s) of $x$ cannot be in this domain?

A. 3 and 4
B. $-3$ and $-4$
C. $-2$
D. 0

**Solution**

**A**

$$\begin{aligned}x^2 - 7x + 12 = 0 &\Rightarrow (x-3)(x-4) = 0 \\ &\Rightarrow x = 3 \text{ or } x = 4.\end{aligned}$$

14. Given that (2)

$$|\mathbf{a}| = 3,\ |\mathbf{b}| = 2,\ \text{and}\ \mathbf{a}.\mathbf{b} = 5,$$

what is the value of

$$\mathbf{a}.(\mathbf{a}+\mathbf{b})?$$

A. 11
B. 14
C. 15
D. 21

**Solution**

**B**

$$\begin{aligned}\mathbf{a}.(\mathbf{a}+\mathbf{b}) &= \mathbf{a}.\mathbf{a}+\mathbf{a}.\mathbf{b}\\ &= 9+5\\ &= 14.\end{aligned}$$

15. Solve (2)

$$\tan\left(\frac{x}{2}\right) = -1$$

for $0 \leqslant x < 2\pi$.

A. $\frac{1}{2}\pi$
B. $\frac{7}{8}\pi$
C. $\frac{3}{2}\pi$
D. $\frac{15}{8}\pi$

**Solution**

**C**

$$\begin{aligned}\tan\left(\frac{x}{2}\right) = -1 &\Rightarrow \frac{x}{2} = \tfrac{3}{4}\pi\\ &\Rightarrow x = \tfrac{3}{2}\pi.\end{aligned}$$

16. Find (2)

$$\int (1-6x)^{-\frac{1}{2}}\,\mathrm{d}x,$$

where $x < \frac{1}{6}$.

A. $\frac{1}{9}(1-6x)^{-\frac{3}{2}} + c$

B. $3(1-6x)^{-\frac{3}{2}} + c$

C. $-\frac{1}{3}(1-6x)^{\frac{1}{2}} + c$

D. $-3(1-6x)^{\frac{1}{2}} + c$

**Solution**

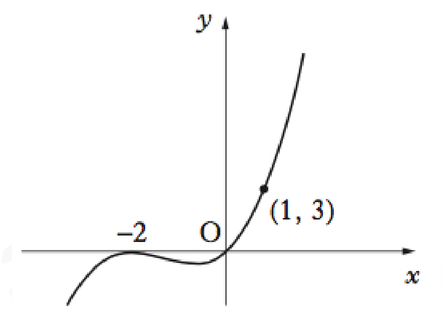
**C**

$$\begin{aligned}\int (1-6x)^{-\frac{1}{2}}\,\mathrm{d}x &= \frac{2(1-6x)^{\frac{1}{2}}}{-6} + c \\ &= -\tfrac{1}{3}(1-6x)^{\frac{1}{2}} + c.\end{aligned}$$

17. The diagram shows a curve with equation of the form (2)

$$y = kx(x+a)^2,$$

which passes through the points $(-2, 0)$, $(0, 0)$, and $(1, 3)$.




What are the values of $a$ and $k$?

A. $a = -2$ and $k = \frac{1}{3}$

B. $a = -2$ and $k = 3$

C. $a = 2$ and $k = \frac{1}{3}$

D. $a = 2$ and $k = 3$

**Solution**

**C**

$$\begin{aligned} x = -2, y = 0 &\Rightarrow 0 = -2k(-2+a)^2 \\ &\Rightarrow -2 + a = 0 \\ &\Rightarrow a = 2 \\ x = 1, y = 3 &\Rightarrow 3 = k(1+2)^2 \\ &\Rightarrow 3 = 9k \\ &\Rightarrow k = \tfrac{1}{3}. \end{aligned}$$

18. Given that (2)

$$y = \sin(x^2 - 3),$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

A. $\sin 2x$

B. $\cos 2x$

C. $2x \sin(x^2 - 3)$

D. $2x \cos(x^2 - 3)$

**Solution**

**D**

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \cos(x^2 - 3) \times 2x \\ &= 2x \cos(x^2 - 3). \end{aligned}$$

19. Solve (2)

$$1 - 2x - 3x^2 > 0,$$

where $x$ is a real number.

A. $x < -1$ or $x > \frac{1}{3}$

B. $-1 < x < \frac{1}{3}$

C. $x < -\frac{1}{3}$ or $x > 1$

D. $-\frac{1}{3} < x < 1$

**Solution**

**B**

$$\left.\begin{array}{ll} \text{add to:} & -2 \\ \text{multiply to:} & (+1)\times(-3) = -3 \end{array}\right\} -3,\ +1$$

$$\begin{aligned} 1 - 2x - 3x^2 = 0 &\Rightarrow 1 - 3x + x - 3x^2 = 0 \\ &\Rightarrow (1 - 3x) + x(1 - 3x) = 0 \\ &\Rightarrow (1 + x)(1 - 3x) = 0 \\ &\Rightarrow x = -1 \text{ and } x = \tfrac{1}{3}. \end{aligned}$$

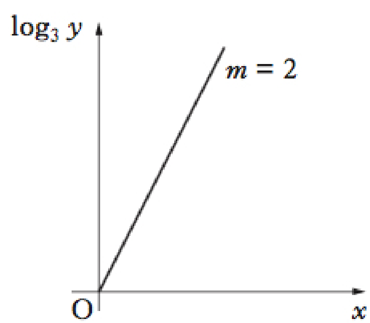
Now,

$$x = 0 \Rightarrow y = 1$$

and so we can see that

$$-1 < x < \tfrac{1}{3}.$$

20. The graph of $\log_3 y$ plotted against $x$ is a line through the origin with gradient 2, as shown. (2)




Express $y$ in terms of $x$.

A. $y = 2x$

B. $y = 9x$

C. $y = 6^x$

D. $y = 9^x$

**Solution**

**D**

$$\begin{aligned}\log_3 y = 2x &\Rightarrow y = 3^{2x} \\ &\Rightarrow y = 3^{2^x} \\ &\Rightarrow y = 9^x.\end{aligned}$$

# Section B

21. Express (3)

$$2x^2 + 12x + 1$$

in the form

$$a(x+b)^2 + c.$$

**Solution**

$$\begin{aligned}2x^2 + 12x + 1 &= 2(x^2 + 6x) + 1 \\ &= 2[(x^2 + 6x + 9) - 9] + 1 \\ &= 2[(x+3)^2 - 9] + 1 \\ &= 2(x+3)^2 - 18 + 1 \\ &= 2(x+3)^2 - 17;\end{aligned}$$

hence, $a = 2$, $b = 3$, and $c = -17$.

22. A circle $C_1$ has equation

$$x^2 + y^2 + 2x + 4y - 27 = 0.$$

(a) Write down the centre and calculate the radius of $C_1$. (2)

**Solution**

$$\begin{aligned}x^2 + y^2 + 2x + 4y - 27 = 0 &\Rightarrow (x^2 + 2x + 1) + (y^2 + 4y + 4) = 27 + 1 + 4 \\ &\Rightarrow (x+1)^2 + (y+2)^2 = 32 \\ &\Rightarrow (x+1)^2 + (y+2)^2 = (4\sqrt{2})^2;\end{aligned}$$

hence, the centre has coordinates $\underline{\underline{(-1, -2)}}$ and radius $\underline{\underline{4\sqrt{2}}}$.

The point $P(3, 2)$ lies on the circle $C_1$.

(b) Find the equation of the tangent at $P$. (3)

**Solution**

The gradient of $(-1, -2)$ and $P(3, 2)$ is

$$\frac{2+2}{3+1} = 1$$

and the gradient the normal is

$$-\frac{1}{m} = -1.$$

Finally, the equation of the tangent at $P$ is

$$\begin{aligned} y - 2 = -(x - 3) &\Rightarrow y - 2 = -x + 3 \\ &\Rightarrow \underline{\underline{y = -x + 5}}. \end{aligned}$$

A second circle $C_2$ has centre $(10, -1)$. The radius of $C_2$ is half of the radius of $C_1$.

(c) Show that the equation of $C_2$ is (3)

$$x^2 + y^2 - 20x + 2y + 93 = 0.$$

**Solution**

The radius of $C_2$ is $2\sqrt{2}$ and the equation is

$$\begin{aligned} (x - 10)^2 + (y + 1)^2 = (2\sqrt{2})^2 &\Rightarrow (x^2 - 20x + 100) + (y^2 + 2y + 1) = 8 \\ &\Rightarrow \underline{\underline{x^2 + y^2 - 20x + 2y + 93 = 0}}, \end{aligned}$$

as required.

(d) Show that the tangent found in part (b) is also a tangent to circle $C_2$. (4)

**Solution**

$$\begin{aligned} y = -x + 5 &\Rightarrow x^2 + (-x+5)^2 - 20x + 2(-x+5) + 93 = 0 \\ &\Rightarrow x^2 + (x^2 - 10x + 25) - 20x - 2x + 10 + 93 = 0 \\ &\Rightarrow 2x^2 - 32x + 128 = 0 \\ &\Rightarrow x^2 - 16x + 64 = 0 \\ &\Rightarrow (x-8)^2 = 0 \\ &\Rightarrow x = 8 \text{ (repeated)}; \end{aligned}$$

hence, the tangent found in part (b) is also a tangent to circle $C_2$.

23. (a) The expression (4)

$$\sqrt{3}\sin x^\circ - \cos x^\circ$$

can be written

$$k\sin(x-a)^\circ,$$

where $k > 0$ and $0 \leqslant a < 360$.

Calculate the values of $k$ and $a$.

**Solution**

$$\begin{aligned} \sqrt{3}\sin x^\circ - \cos x^\circ &\equiv k\sin(x-a)^\circ \\ &\equiv k\sin x^\circ\cos a^\circ - k\cos x^\circ\sin a^\circ \end{aligned}$$

and, hence,

$$k\sin a^\circ = 1,\ k\cos a^\circ = \sqrt{3}.$$

Now,

$$\begin{aligned} k &= \sqrt{(k\sin a^\circ)^2 + (k\cos a^\circ)^2} \\ &= \sqrt{(\sqrt{3})^2 + (1)^2} \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} \tan a^\circ = \frac{k\sin a^\circ}{k\cos a^\circ} &\Rightarrow \tan a^\circ = \frac{1}{\sqrt{3}} \\ &\Rightarrow a = 30. \end{aligned}$$

(b) Determine the maximum value of (2)

$$4 + 5\cos x^\circ - 5\sqrt{3}\sin x^\circ,$$

where $0 \leqslant x < 360$.

**Solution**

$$\begin{aligned} 4 + 5\cos x^\circ - 5\sqrt{3}\sin x^\circ &= 4 - 5(\sqrt{3}\sin x^\circ - \cos x^\circ) \\ &= 4 - 5[2\sin(x-30)^\circ] \\ &= 4 - 10\sin(x-30)^\circ; \end{aligned}$$

so, the maximum value is

$$4 + 10 = \underline{\underline{14}}.$$

24. (a) (i) Show that the points $A(-7,-8,1)$, $T(3,2,5)$, and $B(18,17,11)$ are collinear. (4)

**Solution**

$$\overrightarrow{AT} = \begin{pmatrix} 10 \\ 10 \\ 4 \end{pmatrix} \text{ and } \overrightarrow{TB} = \begin{pmatrix} 15 \\ 15 \\ 6 \end{pmatrix}$$

and

$$\overrightarrow{TB} = \tfrac{3}{2}\overrightarrow{AT}.$$

Hence, $A$, $T$, and $B$ are collinear.

(ii) Find the ratio in which $T$ divides $AB$.

**Solution**

$$\underline{\underline{1 : \tfrac{3}{2}}} \text{ or } \underline{\underline{2 : 3}}.$$

The point $C$ lies on the $x$-axis.

(b) If $TB$ and $TC$ are perpendicular, find the coordinates of $C$. (5)

**Solution**

Let $C(x, 0, 0)$. Then

$$\overrightarrow{TB}.\overrightarrow{TC} = 0 \Rightarrow \begin{pmatrix} 15 \\ 15 \\ 6 \end{pmatrix} . \begin{pmatrix} 3 - x \\ 2 \\ 5 \end{pmatrix} = 0$$
$$\Rightarrow 15(3 - x) + 30 + 30 = 0$$
$$\Rightarrow 15(3 - x) = -60$$
$$\Rightarrow 3 - x = -4$$
$$\Rightarrow x = 7;$$

hence, $\underline{\underline{C(7, 0, 0)}}$.