Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2011 November Paper 1 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. (a) Sets A and B are such that

$$\mathrm{n}(A) = 15 \text{ and } \mathrm{n}(B) = 7.$$

Find the greatest and least possible values of

(i) $\mathrm{n}(A \cap B)$, (2)

Solution
The greatest value is $\underline{\underline{7}}$ and the least value is $\underline{\underline{0}}$.

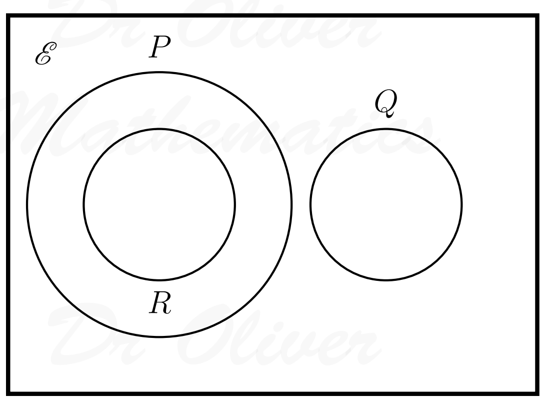
(ii) $\mathrm{n}(A \cup B)$ (2)

Solution
The greatest value is $15 + 7 = \underline{\underline{22}}$ and the least value is $\underline{\underline{15}}$.

(b) On a Venn diagram draw 3 sets P, Q, and R such that (2)

$$P \cup Q = \varnothing \text{ and } P \cup R = P.$$

Solution



2. The function f is such that (6)

$$\mathrm{f}(x) = 4x^3 - 8x^2 + ax + b,$$

where a and b are constants.

It is given that $(2x - 1)$ is a factor of $\mathrm{f}(x)$ and that when $\mathrm{f}(x)$ is divided by $(x + 2)$ the remainder is 20.

Find the remainder when $\mathrm{f}(x)$ is divided by $(x - 1)$.

Solution

We use synthetic division (twice):

$$\begin{array}{c|cccc}
\frac{1}{2} & 4 & -8 & a & b \\
 & \downarrow & 2 & -3 & \frac{1}{2}a - \frac{3}{2} \\
\hline
 & 4 & -6 & a - 3 & \frac{1}{2}a + b - \frac{3}{2}
\end{array}$$

so

$$\tfrac{1}{2}a + b - \tfrac{3}{2} = 0 \Rightarrow \boxed{a + 2b = 3 \quad (1).}$$

Now,

$$\begin{array}{c|cccc}
-2 & 4 & -8 & a & b \\
 & \downarrow & -8 & 32 & -2a - 64 \\
\hline
 & 4 & -16 & a + 32 & -2a + b - 64
\end{array}$$

so

$$-2a + b - 64 = 20 \Rightarrow -2a + b = 84 \quad (2).$$

Do $2 \times (2)$:

$$\boxed{-4a + 2b = 168 \quad (3)}$$

and do $(1) - (3)$:

$$\begin{aligned}
5a = -165 &\Rightarrow a = -33 \\
&\Rightarrow 2b = 3 - (-33) \\
&\Rightarrow 2b = 36 \\
&\Rightarrow b = 18.
\end{aligned}$$

[Check: $(-2) \times (-33) + 18 = 84$ ✓]

Next, the polynomial is

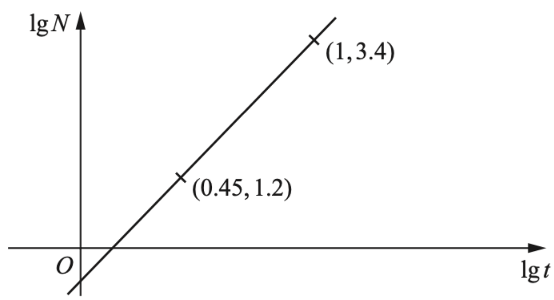
$$f(x) = 4x^3 - 8x^2 - 33x + 18$$

and

1	4	-8	-33	18
	↓	4	-4	-37
	4	-4	-37	-19

The remainder is $\underline{\underline{-19}}$.

3. Variables t and N are such that when $\log_{10} N$ is plotted against $\log_{10} t$, a straight line graph passing through the points $(0.45, 1.2)$ and $(1, 3.4)$ is obtained.



(a) Express the equation of the straight line graph in the form (4)

$$\log_{10} N = m \log_{10} t + \log_{10} c,$$

where m and c are constants to be found.

Solution

Well,

$$\begin{aligned} m &= \frac{3.4 - 1.2}{1 - 0.45} \\ &= \frac{2.2}{0.55} \\ &= 4 \end{aligned}$$

and, using $(1, 3.4)$, the equation of the straight line is

$$\log_{10} N - 3.4 = 4(\log_{10} t - 1) \Rightarrow \log_{10} N - 3.4 = 4\log_{10} t - 4$$
$$\Rightarrow \underline{\underline{\log_{10} N = \log_{10} t^4 - 0.6}};$$

hence, $\underline{\underline{m = 4}}$ and $\underline{\underline{c = -0.6}}$.

(b) Hence express N in terms of t. (1)

Solution
Now,

$$\log_{10} N = 4\log_{10} t - 0.6 \Rightarrow \log_{10} N - \log_{10} t^4 = -0.6$$
$$\Rightarrow \log_{10}\left(\frac{N}{t^4}\right) = -0.6$$
$$\Rightarrow \frac{N}{t^4} = 10^{-0.6}$$
$$\Rightarrow \underline{\underline{N = (10^{-0.6})t^4}}.$$

4. Six-digit numbers are to be formed using the digits 3, 4, 5, 6, 7, and 9. Each digit may only be used once in any number.

(a) Find how many different six-digit numbers can be formed (1)

Solution

$$6! = \underline{\underline{720}}.$$

Find how many of these six-digit numbers are

(b) even, (1)

Solution

$$2 \times 5! = \underline{\underline{240}}.$$

(c) greater than 500 000, (1)

Solution

$$4 \times 5! = \underline{\underline{480}}.$$

(d) even and greater than 500 000. (3)

Solution

$$\begin{aligned} \text{even and greater than } 500\,000 &= \text{even first and last} + \text{odd first and even last} \\ &= (1 \times 4! \times 1) + (3 \times 4! \times 2) \\ &= 24 + 144 \\ &= \underline{\underline{168}}. \end{aligned}$$

5. A particle moves in a straight line such that its displacement, x m, from a fixed point O at time t s, is given by

$$x = 3 + \sin 2t, \text{ where } t \geqslant 0.$$

(a) Find the velocity of the particle when $t = 0$. (2)

Solution

Well,

$$x = 3 + \sin 2t \Rightarrow v = 2\cos 2t$$

and

$$t = 0 \Rightarrow \underline{\underline{v = 2 \text{ ms}^{-1}}}.$$

(b) Find the value of t when the particle is first at rest. (2)

Solution

Now,

$$\begin{aligned} v = 0 &\Rightarrow 2\cos 2t = 0 \\ &\Rightarrow 2t = \tfrac{1}{2}\pi \\ &\Rightarrow \underline{\underline{t = \tfrac{1}{4}\pi \text{ s}}}. \end{aligned}$$

(c) Find the distance travelled by the particle before it first comes to rest. (2)

Solution

Well,

$$\begin{aligned} \text{distance} &= [3 + \sin 2t]_{x=0}^{\frac{1}{4}\pi} \\ &= (3+1) - (3+0) \\ &= \underline{\underline{1 \text{ m}}}. \end{aligned}$$

(d) Find the acceleration of the particle when $t = \frac{3}{4}\pi$. (2)

Solution

Now,

$$v = 2\cos 2t \Rightarrow a = -4\sin 2t$$

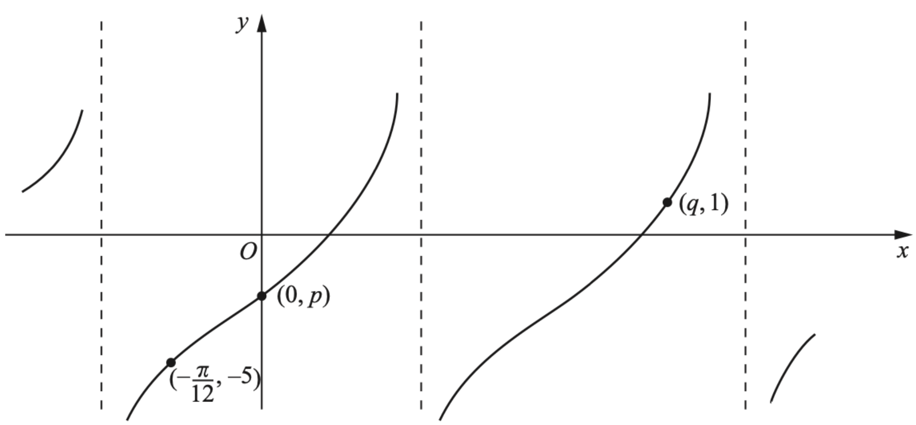
and

$$t = \tfrac{3}{4}\pi \Rightarrow \underline{\underline{a = 4 \text{ ms}^{-2}}}.$$

6. (a) The diagram shows part of the graph (4)

$$y = p + 3\tan 3x$$

passing through the points $(-\frac{1}{12}\pi, -5)$, $(0, p)$, and $(q, 1)$.



Find the value of p and of q.

Solution

Well,

$$\begin{aligned} x = -\tfrac{1}{12}\pi, y = -5 &\Rightarrow -5 = p + 3\tan(-\tfrac{1}{4}\pi) \\ &\Rightarrow -5 = p - 3 \\ &\Rightarrow \underline{\underline{p = -2}} \end{aligned}$$

and

$$\begin{aligned} x = q, y = 1 &\Rightarrow 1 = -2 + 3\tan 3q \\ &\Rightarrow 3\tan 3q = 3 \\ &\Rightarrow \tan 3q = 1 \\ &\Rightarrow 3q = \tfrac{1}{4}\pi \\ &\Rightarrow \underline{\underline{q = \tfrac{1}{12}\pi}}. \end{aligned}$$

(b) It is given that (4)

$$f(x) = a\cos(bx) + c,$$

where a, b, and c are integers.

The maximum value of f is 11, the minimum value of f is 3, and the period of f is 72°.

Find the value of a, of b, and of c.

Solution

Well,

$$b = \frac{360}{72} = \underline{\underline{5}}.$$

Now,

$$\begin{aligned} a + c &= 11 \quad (1) \\ -a + c &= 3 \quad (2). \end{aligned}$$

Do (1) + (2):

$$\begin{aligned} 2c = 14 &\Rightarrow \underline{\underline{c = 7}} \\ &\Rightarrow \underline{\underline{a = 4}}. \end{aligned}$$

7. The coefficient of x^2 in the expansion of

$$(1 + \tfrac{1}{5}x)^n,$$

where n is a positive integer, is $\frac{3}{5}$.

(a) Find the value of n. (4)

Solution

Well,

$$\begin{aligned}(1 + \tfrac{1}{5}x)^n &= 1 + \binom{n}{1}(\tfrac{1}{5}x) + \binom{n}{2}(\tfrac{1}{5}x)^2 + \dots \\ &= 1 + \tfrac{n}{5}x + \tfrac{n(n-1)}{50}x^2 + \dots\end{aligned}$$

and

$$\begin{aligned}\tfrac{n(n-1)}{50} = \tfrac{3}{5} &\Rightarrow n(n-1) = 30 \\ &\Rightarrow n^2 - n = 30 \\ &\Rightarrow n^2 - n - 30 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & -1 \\ \text{multiply to:} & -30\end{array}\right\} -6,\ +5$$

$$\begin{aligned}&\Rightarrow (n-6)(n+5) = 0 \\ &\Rightarrow n = 6 \text{ or } n = -5.\end{aligned}$$

But $n \neq -5$ (why?) so $\underline{\underline{n = 6}}$.

(b) Using this value of n, find the term independent of x in the expansion of (4)

$$(1 + \tfrac{1}{5}x)^n \left(2 - \frac{3}{x}\right)^2.$$

Solution

Well,

$\times$	2	$-\frac{3}{x}$
2	4	$-\frac{6}{x}$
$-\frac{3}{x}$	$-\frac{6}{x}$	$+\frac{9}{x^2}$

and

$$(1+\tfrac{1}{5}x)^6\left(2-\frac{3}{x}\right)^2 = (1+\tfrac{6}{5}x+\tfrac{3}{5}x^2+\ldots)\left(4-\frac{12}{x}+\frac{9}{x^2}\right).$$

Now,

$\times$	1	$+\frac{6}{5}x$	$+\frac{3}{5}x^5$
4	4	$\ldots$	$\ldots$
$-\frac{12}{x}$	$\ldots$	$-\frac{72}{5}$	$\ldots$
$+\frac{9}{x^2}$	$\ldots$	$\ldots$	$+\frac{27}{5}$

and, hence, the term independent of x is

$$4-\tfrac{72}{5}+\tfrac{27}{5} = \underline{\underline{-5}}.$$

8. (a) Find (6)

$$\int (\mathrm{e}^x+1)^2\,\mathrm{d}x$$

and hence evaluate

$$\int_0^2 (\mathrm{e}^x+1)^2\,\mathrm{d}x.$$

Solution

Well,

$\times$	e^x	$+1$
e^x	e^{2x}	$+\mathrm{e}^x$
-9	$+\mathrm{e}^x$	$+1$

and

$$\int (\mathrm{e}^x+1)^2\,\mathrm{d}x = \int (\mathrm{e}^{2x}+2\mathrm{e}^x+1)\,\mathrm{d}x$$
$$= \underline{\underline{\tfrac{1}{2}\mathrm{e}^{2x}+2\mathrm{e}^x+x+c}}.$$

Now,

$$\int_0^2 (\mathrm{e}^x + 1)^2 \,\mathrm{d}x = \left[\tfrac{1}{2}\mathrm{e}^{2x} + 2\mathrm{e}^x + x\right]_{x=0}^2$$
$$= \left(\tfrac{1}{2}\mathrm{e}^4 + 2\mathrm{e}^2 + 2\right) - \left(\tfrac{1}{2} + 2 + 0\right)$$
$$= \underline{\underline{\tfrac{1}{2}\mathrm{e}^4 + 2\mathrm{e}^2 - \tfrac{1}{2}}}.$$

(b) A curve is such that (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4x+1)^{-\frac{1}{2}}.$$

Given that the curve passes through the point with coordinates $(2, 4.5)$, find the equation of the curve.

Solution

Well,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (4x+1)^{-\frac{1}{2}} \Rightarrow y = \tfrac{1}{2}(4x+1)^{\frac{1}{2}} + c.$$

Now,

$$x = 2, y = 4.5 \Rightarrow 4.5 = \tfrac{1}{2}(4 \times 2 + 1)^{\frac{1}{2}} + c$$
$$\Rightarrow 9 = (9)^{\frac{1}{2}} + c$$
$$\Rightarrow 9 = 2(3) + c$$
$$\Rightarrow c = 3;$$

hence,

$$\underline{\underline{y = 2(4x+1)^{\frac{1}{2}} + 3}}.$$

9. (a) Solve (5)

$$1 + \cot^2 x = 8 \sin x \text{ for } 0^\circ \leqslant x \leqslant 360^\circ.$$

Solution

Well,

$$\begin{aligned}1+\cot^2 x = 8\sin x &\Rightarrow \operatorname{cosec}^2 x = 8\sin x\\ &\Rightarrow \frac{1}{\sin^2 x} = 8\sin x\\ &\Rightarrow \sin^3 x = \tfrac{1}{8}\\ &\Rightarrow \sin x = \tfrac{1}{2}\\ &\Rightarrow \underline{\underline{x = 30^\circ,\ 150^\circ}}.\end{aligned}$$

(b) Solve (5)

$$4\sin(2y-0.3)+5\cos(2y-0.3)=0 \text{ for } 0 \leqslant y \leqslant \pi \text{ radians.}$$

Solution

Now,

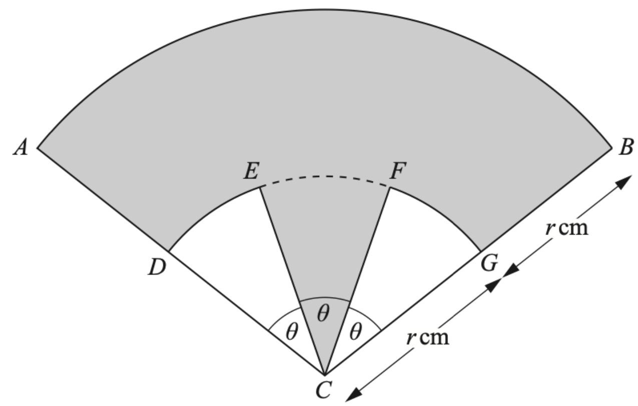
$$\begin{aligned}0 \leqslant y \leqslant \pi &\Rightarrow 0 \leqslant 2y \leqslant 2\pi\\ &\Rightarrow 0 \leqslant (2y-0.3) \leqslant 5.983\ldots.\end{aligned}$$

Next,

$$\begin{aligned}& 4\sin(2y-0.3)+5\cos(2y-0.3)=0\\ \Rightarrow\ & 4\sin(2y-0.3) = -5\cos(2y-0.3)\\ \Rightarrow\ & \tan(2y-0.3) = -\tfrac{5}{4}\\ \Rightarrow\ & 2y-0.3 = 2.245\,537\,269,\ 5.387\,129\,923 \text{ (FCD)}\\ \Rightarrow\ & 2y = 2.545\,537\,269,\ 5.687\,129\,923 \text{ (FCD)}\\ \Rightarrow\ & y = 1.272\,768\,635,\ 2.843\,564\,961 \text{ (FCD)}\\ \Rightarrow\ & \underline{\underline{y = 1.27,\ 2.84 \text{ (3 sf)}}}.\end{aligned}$$

EITHER

10. The figure shows a sector ABC of a circle centre C, radius $2r$ cm, where angle ACB is 3θ radians.



- The points D, E, F, and G lie on an arc of a circle centre C, radius r cm.
- The points D and G are the midpoints of CA and CB respectively.
- Angles DCE and FCG are each θ radians.
- The area of the shaded region is 5 cm^2.

(a) By first expressing θ in terms of r, show that the perimeter, P cm, of the shaded region is given by (6)

$$P = 4r + \frac{8}{r}.$$

Solution

Well, the area of the shaded region is 5 cm^2 so

$$\begin{aligned} & \left[\tfrac{1}{2} \times (2r)^2 \times 3\theta\right] - \left[\tfrac{1}{2} \times r^2 \times 3\theta\right] + \left(\tfrac{1}{2}r^2\theta\right) = 5 \\ \Rightarrow \quad & 6r^2\theta - \tfrac{3}{2}r^2\theta + \tfrac{1}{2}r^2\theta = 5 \\ \Rightarrow \quad & 5r^2\theta = 5 \\ \Rightarrow \quad & \theta = \frac{1}{r^2} \end{aligned}$$

and

$$\begin{aligned} P &= (2r)(3r\theta) + r + (r\theta) + r + r + (r\theta) + r \\ &= 4r + 8r\theta \\ &= 4r + 8r\left(\frac{1}{r^2}\right) \\ &= \underline{\underline{4r + \frac{8}{r}}}, \end{aligned}$$

as required.

(b) Given that r can vary, show that the stationary value of P can be written in the form $k\sqrt{2}$, where k is a constant to be found. (4)

Solution

Now,

$$P = 4r + \frac{8}{r} \Rightarrow P = 4r + 8r^{-1}$$
$$\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}r} = 4 - 8r^{-2}$$

and

$$\frac{\mathrm{d}P}{\mathrm{d}r} = 0 \Rightarrow 4 - 8r^{-2} = 0$$
$$\Rightarrow 8r^{-2} = 4$$
$$\Rightarrow r^{-2} = \tfrac{1}{2}$$
$$\Rightarrow r^2 = 2$$
$$\Rightarrow r = \sqrt{2}.$$

Finally,

$$r = \sqrt{2} \Rightarrow \underline{\underline{P = 8\sqrt{2}\text{ cm}}};$$

hence, $\underline{\underline{k = 8}}$.

(c) Determine the nature of this stationary value and find the value of θ for which it occurs. (2)

Solution

Well,

$$\frac{\mathrm{d}^2P}{\mathrm{d}r^2} = 16r^{-3}$$

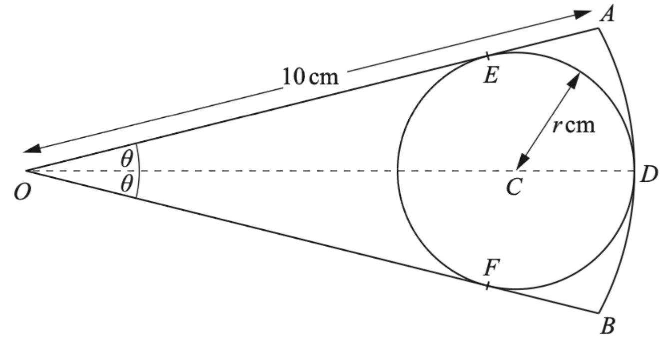
and, for any $r > 0$, $\frac{\mathrm{d}^2P}{\mathrm{d}r^2} > 0$. Hence, it is a $\underline{\underline{\text{minimum}}}$.

Finally, the value of θ is

$$\theta = \frac{1}{(\sqrt{2})^2} = \underline{\underline{\tfrac{1}{2}}}.$$

OR

11. The figure shows a sector OAB of a circle, centre O, radius 10 cm.



- Angle $AOB = 2\theta$ radians where $0 < \theta < \frac{1}{2}\pi$.
- A circle centre C, radius r cm, touches the arc AB at the point D.
- The lines OA and OB are tangents to the circle at the points E and F respectively.

(a) Write down, in terms of r, the length of OC. (1)

Solution

$$\begin{aligned} OE &= OA - EA \\ &= \underline{\underline{10 - r}}. \end{aligned}$$

(b) Hence show that (2)

$$r = \frac{10 \sin\theta}{1 + \sin\theta}.$$

Solution

Well,

$$\begin{aligned} \sin = \frac{\text{opp}}{\text{hyp}} &\Rightarrow \sin\theta = \frac{r}{10 - r} \\ &\Rightarrow \sin\theta(10 - r) = r \\ &\Rightarrow 10\sin\theta - r\sin\theta = r \\ &\Rightarrow 10\sin\theta = r + r\sin\theta \\ &\Rightarrow 10\sin\theta = r(1 + \sin\theta) \\ &\Rightarrow \underline{\underline{r = \frac{10\sin\theta}{1 + \sin\theta}}}, \end{aligned}$$

as required.

(c) Given that θ can vary, find $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$ when $r = \frac{10}{3}$. (6)

Solution

Quotient rule:

$$u = 10\sin\theta \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}\theta} = 10\cos\theta$$
$$v = 1 + \sin\theta \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}\theta} = \cos\theta$$

and

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{(1+\sin\theta)(10\cos\theta) - (\cos\theta)(10\sin\theta)}{(1+\sin\theta)^2} \\ &= \frac{10\cos\theta + 10\sin\theta\cos\theta - 10\sin\theta\cos\theta}{(1+\sin\theta)^2} \\ &= \frac{10\cos\theta}{(1+\sin\theta)^2}.\end{aligned}$$

Cross-multipy:

$$\begin{aligned}\frac{10\sin\theta}{1+\sin\theta} = \tfrac{10}{3} &\Rightarrow 30\sin\theta = 10(1+\sin\theta) \\ &\Rightarrow 30\sin\theta = 10 + 10\sin\theta \\ &\Rightarrow 20\sin\theta = 10 \\ &\Rightarrow \sin\theta = \tfrac{1}{2} \\ &\Rightarrow \cos\theta = \tfrac{\sqrt{3}}{2}.\end{aligned}$$

Finally,

$$r = \tfrac{10}{3} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{10 \times \frac{\sqrt{3}}{2}}{(1+\frac{1}{2})^2} = \underline{\underline{\tfrac{20\sqrt{3}}{9}}}.$$

(d) Given that r is increasing at 2 $\mathrm{cm\,s^{-1}}$, find the rate at which θ is increasing when $\theta = \frac{1}{6}\pi$. (3)

Solution

Now,

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}\theta} &\Rightarrow \tfrac{20\sqrt{3}}{9} = \frac{2}{\frac{\mathrm{d}\theta}{\mathrm{d}t}} \\ &\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{\frac{20\sqrt{3}}{9}} \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}\theta}{\mathrm{d}t} = \tfrac{3\sqrt{3}}{10}}}.\end{aligned}$$