Dr Oliver Mathematics
Advance Level Further Mathematics
Further Mechanics 1: Calculator
1 hour 30 minutes

The total number of marks available is 75.
You must write down all the stages in your working.

1. Figure 1 represents the plan of part of a smooth horizontal floor, where W_1 and W_2 are two fixed parallel vertical walls. The walls are 3 metres apart.

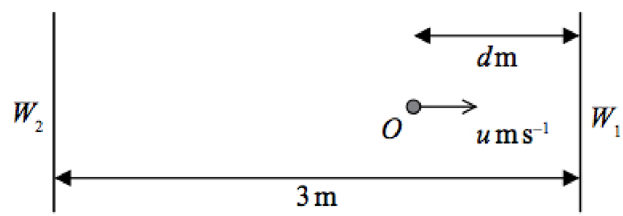


Figure 1: a smooth horizontal floor

A particle lies at rest at a point O on the floor between the two walls, where the point O is d metres, $0 < d \leqslant 3$, from W_1.

At time $t = 0$, the particle is projected from O towards W_1 with speed u ms^{-1} in a direction perpendicular to the walls.

The coefficient of restitution between the particle and each wall is $\frac{2}{3}$. The particle returns to O at time $t = T$ seconds, having bounced off each wall once.

(a) Show that (6)

$$T = \frac{45 - 5d}{4u}.$$

Solution

$\underline{O \text{ to } W_1}$: The time for to strike W_1 is

$$t_1 = \frac{d}{u}.$$

$\underline{W_1 \text{ to } W_2}$: the ball is moving at $\frac{2}{3}u$ and so the time taken is

$$t_2 = \frac{3}{\frac{2}{3}u} = \frac{9}{2u}.$$

$\underline{W_2 \text{ to } O}$: the ball is moving at $(\frac{2}{3})^2 u$ and so the time taken is

$$t_3 = \frac{3-d}{(\frac{2}{3})^2 u} = \frac{9(3-d)}{4u}.$$

Finally,

$$\begin{aligned} T &= \frac{d}{u} + \frac{9}{2u} + \frac{9(3-d)}{4u} \\ &= \frac{4d + 18 + 9(3-d)}{4u} \\ &= \underline{\underline{\frac{45-5d}{4u}}}, \end{aligned}$$

as required.

The value of u is fixed, the particle still hits each wall once but the value of d can now vary.

(b) Find the least possible value of T, giving your answer in terms of u. (2)
You must give a reason for your answer.

Solution
When $d = 3$, T is minimal. Therefore, the least possible value of T is

$$\begin{aligned} T &= \frac{45 - (5 \times 3)}{4u} \\ &= \frac{30}{4u} \\ &= \underline{\underline{\frac{15}{2u}}}. \end{aligned}$$

2. Figure 2 represents the plan view of part of a horizontal floor, where AB and BC are fixed vertical walls with AB perpendicular to BC.

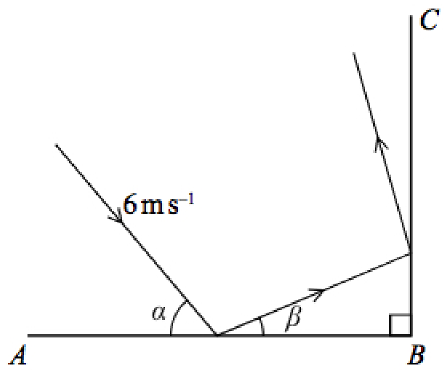


Figure 2: a horizontal floor

A small ball is projected along the floor towards AB with speed 6 ms^{-1} on a path that makes an angle α with AB, where $\tan\alpha = \frac{4}{3}$. The ball hits AB and then hits BC.

Immediately after hitting AB, the ball is moving at an angle β to AB, where $\tan\beta = \frac{1}{3}$.

The coefficient of restitution between the ball and AB is e.

The coefficient of restitution between the ball and BC is $\frac{1}{2}$.

By modelling the ball as a particle and the floor and walls as being smooth,

(a) show that the value of $e = \frac{1}{4}$, (5)

Solution

Let the speed be v ms^{-1} after the first collision. Then, horizontally,

$$6\cos\alpha = v\cos\beta \Rightarrow 3.6 = v\cos\beta \quad (1)$$

and, vertically,

$$\frac{v\sin\beta}{6\sin\alpha} = e \Rightarrow 4.8e = v\sin\beta \quad (2).$$

Divide (2) by (1):

$$\tan\beta = \frac{v\sin\beta}{v\cos\beta} \Rightarrow \tfrac{1}{3} = \frac{4.8e}{3.6}$$
$$\Rightarrow \tfrac{1}{3} = \frac{4e}{3}$$
$$\Rightarrow \underline{\underline{e = \tfrac{1}{4}}},$$

as required.

(b) find the speed of the ball immediately after it hits BC. (4)

Solution
First,

$$\tan\alpha = \tfrac{4}{3} \Rightarrow \sin\alpha = \tfrac{4}{5}, \cos\alpha = \tfrac{3}{5}.$$

Second,

$$\begin{aligned} \text{Horizontally}: &\quad 6\cos\alpha = 3.6, \\ \text{Vertically}: &\quad 6\sin\alpha = 4.8. \end{aligned}$$

Third,

AB	Before	After	BC	Before	After
Horizontally	3.6	3.6	Horizontally	1.2	1.2
Vertically	4.8	1.2	Vertically	3.6	1.8

Fourth (and finally!),

$$\begin{aligned} \text{speed} &= \sqrt{1.2^2 + 1.8^2} \\ &= \underline{\underline{\tfrac{3}{5}\sqrt{13} \text{ ms}^{-1}}}. \end{aligned}$$

(c) Suggest two ways in which the model could be refined to make it more realistic. (2)

Solution
E.g., include friction between the floor and the ball, include friction between the ball and the walls, give the ball dimensions, consider air resistance, spin, rotation.

3. A particle P, of mass 0.5 kg, is moving with velocity $(4\mathbf{i} + 4\mathbf{j})$ ms^{-1} when it receives an impulse $\mathbf{I}$ of magnitude 2.5 Ns. (9)

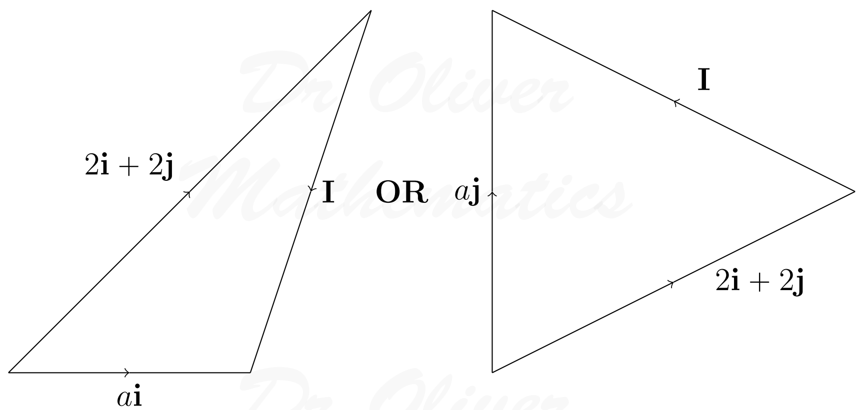
As a result of the impulse, the direction of motion of P is deflected through an angle of 45°.

Given that $\mathbf{I} = (\lambda\mathbf{i} + \mu\mathbf{j})$ Ns, find all the possible pairs of values of λ and μ.

Solution

$$0.5(4\mathbf{i} + 4\mathbf{j}) = (2\mathbf{i} + 2\mathbf{j}) \text{ Ns}$$

and we have a picture:



Now, the left-hand picture:

$$\begin{aligned}
\mathbf{I} &= -(2\mathbf{i} + 2\mathbf{j}) + a\mathbf{i} \Rightarrow \mathbf{I} = -2\mathbf{i} - 2\mathbf{j} + a\mathbf{i} \\
&\Rightarrow \mathbf{I} = (a-2)\mathbf{i} - 2\mathbf{j} \\
&\Rightarrow 2.5^2 = (a-2)^2 + (-2)^2 \\
&\Rightarrow 6.25 = (a-2)^2 + 4 \\
&\Rightarrow (a-2)^2 = 2.25 \\
&\Rightarrow a - 2 = \pm 1.5 \\
&\Rightarrow a = 2 \pm 1.5 \\
&\Rightarrow a = 0.5 \text{ or } a = 3.5.
\end{aligned}$$

On the other hand, the right-hand picture:

$$\begin{aligned}
\mathbf{I} &= -(2\mathbf{i} + 2\mathbf{j}) + a\mathbf{j} \Rightarrow \mathbf{I} = -2\mathbf{i} - 2\mathbf{j} + a\mathbf{j} \\
&\Rightarrow \mathbf{I} = -2\mathbf{i} + (a-2)\mathbf{j} \\
&\Rightarrow 2.5^2 = (-2)^2 + (a-2)^2,
\end{aligned}$$

and so we end up with

$$a = 0.5 \text{ or } a = 3.5.$$

Hence,

$$\mathbf{I} = \underline{\underline{(-1.5\mathbf{i} - 2\mathbf{j})\ \text{Ns}}},$$

$$\mathbf{I} = \underline{\underline{(1.5\mathbf{i} - 2\mathbf{j})\ \text{Ns}}},$$

$$\mathbf{I} = \underline{\underline{(-2\mathbf{i} - 1.5\mathbf{j})\ \text{Ns}}}, \text{ and}$$

$$\mathbf{I} = \underline{\underline{(-2\mathbf{i} + 1.5\mathbf{j})\ \text{Ns}}}.$$

4. A car of mass 600 kg pulls a trailer of mass 150 kg along a straight horizontal road. The trailer is connected to the car by a light inextensible towbar, which is parallel to the direction of motion of the car. The resistance to the motion of the trailer is modelled as a constant force of magnitude 200 N. At the instant when the speed of the car is v ms^{-1}, the resistance to the motion of the car is modelled as a force of magnitude $(200 + \lambda v)$ N, where λ is a constant.

When the engine of the car is working at a constant rate of 15 kW, the car is moving at a constant speed of 25 ms^{-1}.

(a) Show that $\lambda = 8$. (4)

Solution

$$\begin{aligned} P = Fv &\Rightarrow 15\,000 = F \times 25 \\ &\Rightarrow F = 600 \text{ N}. \end{aligned}$$

The equation of motion:

$$\begin{aligned} F - 200 - [200 + (\lambda \times 25)] = 0 &\Rightarrow 25\lambda = 200 \\ &\Rightarrow \underline{\underline{\lambda = 8}}, \end{aligned}$$

as required.

Later on, the car is pulling the trailer up a straight road inclined at an angle θ to the horizontal, where $\sin\theta = \frac{1}{15}$.

The resistance to the motion of the trailer from non-gravitational forces is modelled as a constant force of magnitude 200 N at all times. At the instant when the speed of the car is v ms^{-1}, the resistance to the motion of the car from non-gravitational forces is modelled as a force of magnitude $(200 + 8v)$ N.

The engine of the car is again working at a constant rate of 15 kW.

When $v = 10$, the towbar breaks. The trailer comes to instantaneous rest after moving a distance d metres up the road from the point where the towbar broke.

(b) Find the acceleration of the car immediately after the towbar breaks. (4)

Solution

Let a ms^{-2} be the acceleration. Then,

$$\begin{aligned} v = 10 &\Rightarrow 15\,000 = 10F \\ &\Rightarrow F = 1\,500 \text{ N} \end{aligned}$$

and we have the equation of motion:

$$\begin{aligned} 1\,500 - [200 + (6 \times 10)] - 600g\sin\theta = 600a &\Rightarrow 1\,500 - 280 - 40g = 600a \\ &\Rightarrow a = \frac{1\,220 - 40g}{600} \\ &\Rightarrow a = 1.38 \\ &\Rightarrow \underline{\underline{a = 1.4 \text{ ms}^{-2} \text{ (2 sf)}}}. \end{aligned}$$

(c) Use the work-energy principle to find the value of d. (4)

Solution

Well, the work-energy equation is

$$\begin{aligned} \tfrac{1}{2} \times 150 \times 10^2 = 200d + 150dg\sin\theta &\Rightarrow 7\,500 = d(200 + 10g) \\ &\Rightarrow d = \frac{7\,500}{200 + 10g} \\ &\Rightarrow d = 25.167\,785\,23 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{d = 25 \text{ m (2 sf)}}}. \end{aligned}$$

5. A particle P of mass $3m$ and a particle Q of mass $2m$ are moving along the same straight line on a smooth horizontal plane. The particles are moving in opposite directions towards each other and collide directly.

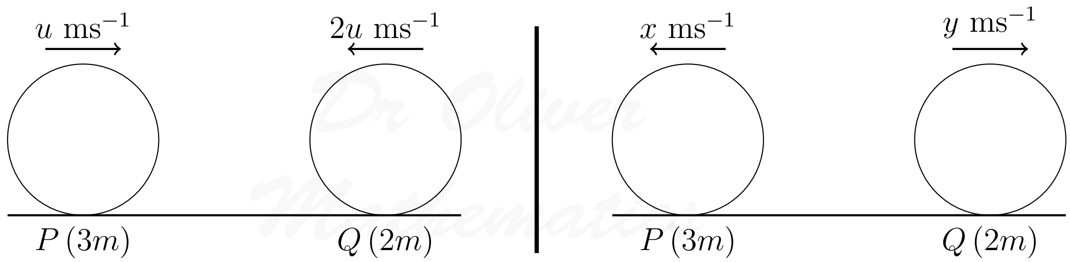
Immediately before the collision the speed of P is u and the speed of Q is $2u$.

Immediately after the collision P and Q are moving in opposite directions.

The coefficient of restitution between P and Q is e.

(a) Find the range of possible values of e, justifying your answer. (8)

Solution



$$\text{Conservation of momentum:} \quad (3m)(u) - (2m)(2u) = -(3m)x + (2m)y$$

$$\text{Newton's Law of Restitution:} \quad \frac{y+x}{3u} = e.$$

Now,

$$\frac{y+x}{3u} = e \Rightarrow x + y = 3eu \quad (1)$$

and

$$\begin{aligned}(3m)(u) - (2m)(2u) = -(3m)x + (2m)y &\Rightarrow -u = -3x + 2y \\ &\Rightarrow 3x - 2y = u \quad (2).\end{aligned}$$

Do $2 \times (1) + (2)$:

$$\begin{aligned}5x = u(1+6e) &\Rightarrow x = \tfrac{1}{5}u(1+6e) \\ &\Rightarrow \tfrac{1}{5}u(1+6e) + y = 3eu \\ &\Rightarrow \tfrac{1}{5}u + \tfrac{6}{5}eu + y = 3eu \\ &\Rightarrow y = \tfrac{9}{5}eu - \tfrac{1}{5}u \\ &\Rightarrow y = \tfrac{1}{5}u(9e-1).\end{aligned}$$

Next,

$$y > 0 \Rightarrow 9e - 1 > 0$$
$$\Rightarrow e > \tfrac{1}{9}.$$

But

$$0 \leqslant e \leqslant 1$$

and so we have

$$\underline{\underline{\tfrac{1}{9} < e \leqslant 1}}.$$

Given that Q loses 75% of its kinetic energy as a result of the collision,

(b) find the value of e. (3)

Solution

Well,

$$\begin{aligned}\frac{\text{Final KE}}{\text{Initial KE}} = \tfrac{1}{4} &\Rightarrow \frac{\frac{1}{2} \times 2m \times [\frac{1}{5}u(9e-1)]^2}{\frac{1}{2} \times 2m \times (2u)^2} = \tfrac{1}{4} \\ &\Rightarrow \frac{u^2(9e-1)^2}{100u^2} = \tfrac{1}{4} \\ &\Rightarrow (9e-1)^2 = 25 \\ &\Rightarrow 9e - 1 = 5 \\ &\Rightarrow 9e = 6 \\ &\Rightarrow \underline{\underline{e = \tfrac{2}{3}}}.\end{aligned}$$

6. (In this question $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors in a horizontal plane.) A smooth uniform sphere A has mass 0.2 kg and another smooth uniform sphere B, with the same radius as A, has mass 0.4 kg.

The spheres are moving on a smooth horizontal surface when they collide obliquely. Immediately before the collision, the velocity of A is $(3\mathbf{i} + 2\mathbf{j})$ ms^{-1} and the velocity of B is $(-4\mathbf{i} - \mathbf{j})$ ms^{-1}.

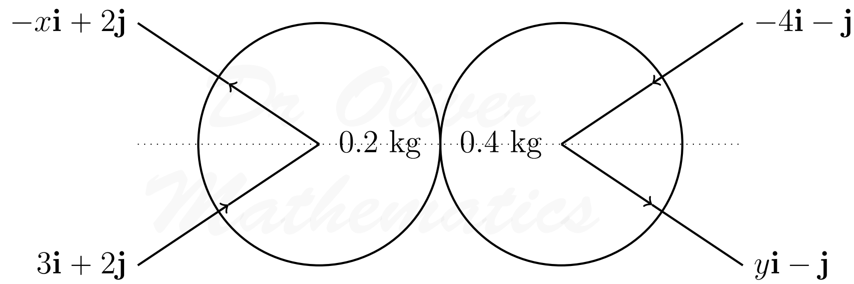
At the instant of collision, the line joining the centres of the spheres is parallel to $\mathbf{i}$.

The coefficient of restitution between the spheres is $\frac{3}{7}$.

(a) Find the velocity of A immediately after the collision. (7)

Solution

We begin with a picture:



$$\text{CLM (par. to line of centres):} \quad (0.2)(3) - (0.4)(-4) = -(0.2)x + (0.4)y$$
$$\text{Newton's Law of Restitution:} \quad \frac{y+x}{7} = \tfrac{3}{7}.$$

Now,

$$\frac{y+x}{7} = \tfrac{3}{7} \Rightarrow x + y = 3 \quad (1)$$

and

$$\begin{aligned}(0.2)(3) - (0.4)(-4) = -(0.2)x + (0.4)y &\Rightarrow -1 = -0.2x + 0.4y \\ &\Rightarrow -x + 2y = -5 \quad (2).\end{aligned}$$

Do (1) + (2):

$$\begin{aligned}3y = -2 &\Rightarrow y = -\tfrac{2}{3} \\ &\Rightarrow x = 3 - (-\tfrac{2}{3}) \\ &\Rightarrow x = \tfrac{11}{3};\end{aligned}$$

hence, the velocity of A immediately after the collision is

$$\underline{\underline{(-\tfrac{11}{3}\mathbf{i} + 2\mathbf{j}) \text{ ms}^{-1}}}.$$

(b) Find the magnitude of the impulse received by A in the collision. (2)

Solution

$$\begin{aligned}\text{Magnitude of impulse on } A &= 0.2\left(\tfrac{11}{3} - (-3)\right) \\ &= \underline{\underline{1\tfrac{1}{3}\text{ Ns}}}.\end{aligned}$$

(c) Find, to the nearest degree, the size of the angle through which the direction of motion of A is deflected as a result of the collision. (3)

Solution

$$|3\mathbf{i} + 2\mathbf{j}| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

and

$$|\tfrac{11}{3}\mathbf{i} + 2\mathbf{j}| = \sqrt{(\tfrac{11}{3})^2 + 2^2} = \frac{\sqrt{157}}{3}.$$

Finally,

$$\begin{aligned}\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta &\Rightarrow -11 + 4 = \sqrt{13} \cdot \frac{\sqrt{157}}{3} \cdot \cos\theta \\ &\Rightarrow \cos\theta = \frac{-7}{\sqrt{13} \cdot \frac{\sqrt{157}}{3}} \\ &\Rightarrow \theta = 117.699\,472\,8 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{\theta = 118^\circ \text{ (3 sf)}}}.\end{aligned}$$

7. A particle P, of mass m, is attached to one end of a light elastic spring of natural length a and modulus of elasticity kmg.

The other end of the spring is attached to a fixed point O on a ceiling.

The point A is vertically below O such that $OA = 3a$.

The point B is vertically below O such that $OB = \frac{1}{2}a$.

The particle is held at rest at A, then released and first comes to instantaneous rest at the point B.

(a) Show that $k = \frac{4}{3}$. (3)

Solution

First, $3a - a = 2a$ and $a - \frac{1}{2}a = \frac{1}{2}a$. Second,

$$\begin{aligned} & \text{elastic potential energy lost} = \text{gravitational potential energy gained} \\ \Rightarrow \quad & \frac{kmg}{2a}(2a)^2 - \frac{kmg}{2a}(\tfrac{1}{2}a)^2 = mg[\tfrac{1}{2}a - (-2a)] \\ \Rightarrow \quad & 2akmg - \tfrac{1}{8}akmg = \tfrac{5}{2}amg \\ \Rightarrow \quad & \tfrac{15}{8}k = \tfrac{5}{2} \\ \Rightarrow \quad & \underline{\underline{k = \tfrac{4}{3}}}. \end{aligned}$$

(b) Find, in terms of g, the acceleration of P immediately after it is released from rest at A. (3)

Solution

Let b ms^{-2} be the acceleration. Then, an equation of motion is

$$\begin{aligned} \frac{4mg}{3a}(2a) - mg = mb &\Rightarrow \tfrac{8}{3}g - g = b \\ &\Rightarrow \underline{\underline{b = \tfrac{5g}{3}}}. \end{aligned}$$

(c) Find, in terms of g and a, the maximum speed attained by P as it moves from A to B. (6)

Solution

Well, the maximum speed is at the equilibrium position:

$$\begin{aligned} \frac{4mge}{3a} = mg &\Rightarrow \frac{4e}{3a} = 1 \\ &\Rightarrow e = \tfrac{3}{4}a. \end{aligned}$$

Let the maximum speed be v ms^{-1}. Finally, we use the conservation of energy:

$$\begin{aligned} & \frac{4mg}{3 \cdot 2a}(2a)^2 = \frac{4mg}{3 \cdot 2a}(\tfrac{3}{4}a)^2 + \tfrac{1}{2}mv^2 + mg(\tfrac{5}{4}a) \\ \Rightarrow \quad & \frac{8ag}{3} = \frac{3ag}{8} + \tfrac{1}{2}v^2 + \tfrac{5ag}{4} \\ \Rightarrow \quad & \tfrac{1}{2}v^2 = \frac{25ag}{24} \\ \Rightarrow \quad & v^2 = \frac{25ag}{12} \\ \Rightarrow \quad & \underline{\underline{v = \sqrt{\frac{25ag}{12}}}}. \end{aligned}$$