Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2010 June Paper 1 Variant 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Differentiate with respect to x:

   (a) $\sqrt{1+x^3}$, (2)

   Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{1+x^3}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(1+x^3)^{\frac{1}{2}}\right] \\ &= \tfrac{1}{2}(1+x^3)^{-\frac{1}{2}} \times 3x^2 \\ &= \underline{\underline{\tfrac{3}{2}x^2(1+x^3)^{-\frac{1}{2}}}}.\end{aligned}$$

   (b) $x^2 \cos 2x$. (3)

   Solution
   Product rule:

$$\begin{aligned}u = x^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \\ v = \cos 2x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -2\sin 2x\end{aligned}$$

   so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\cos 2x\right] &= (x^2)(-2\sin 2x) + (2x)(\cos 2x) \\ &= \underline{\underline{-2x^2\sin 2x + 2x\cos 2x}}.\end{aligned}$$

2. (a) Find the first 3 terms of the expansion, in ascending powers of x, of (2)

$$(1+3x)^6.$$

Solution

$$\begin{aligned}(1+3x)^6 &= 1^6 + \binom{6}{1}(1)^5(3x) + \binom{6}{2}(1)^4(3x)^2 + \dots \\ &= \underline{\underline{1 + 18x + 135x^2 + \dots}}.\end{aligned}$$

(b) Hence find the coefficient of x^2 in the expansion of (3)

$$(1+3x)^6(1-3x-5x^2).$$

Solution

Well,

$\times$	1	$+18x$	$+135x^2$
1	$\dots$	$\dots$	$+135x^2$
$-3x$	$\dots$	$-54x^2$	$\dots$
$-5x^2$	$-5x^2$	$\dots$	$\dots$

so the coefficient of x^2 is

$$135 - 54 - 5 = \underline{\underline{76}}.$$

3. Find the set of values of k for which the equation (5)

$$x^2 + (k-2)x + (2k-4) = 0$$

has real roots.

Solution

Now,

$$\begin{aligned}b^2 - 4ac \geqslant 0 &\Rightarrow (k-2)^2 - 4(1)(2k-4) \geqslant 0 \\ &\Rightarrow (k-2)^2 - 8(k-2) \geqslant 0 \\ &\Rightarrow (k-2)[(k-2)-8] \geqslant 0 \\ &\Rightarrow (k-2)(k-10) \geqslant 0.\end{aligned}$$

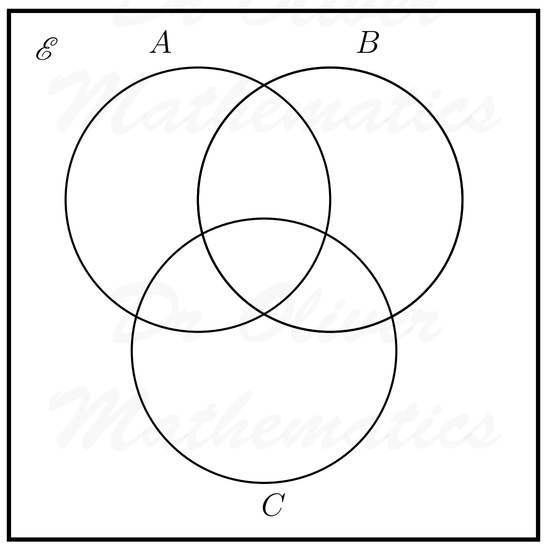
We need a 'table of signs':

	$k < 2$	$k = 2$	$2 < k < 10$	$k = 10$	$k > 10$
$k - 2$	$-$	0	$+$	$+$	$+$
$k - 10$	$-$	$-$	$-$	0	$+$
$(k-2)(k-10)$	$+$	0	$-$	0	$+$

Hence,

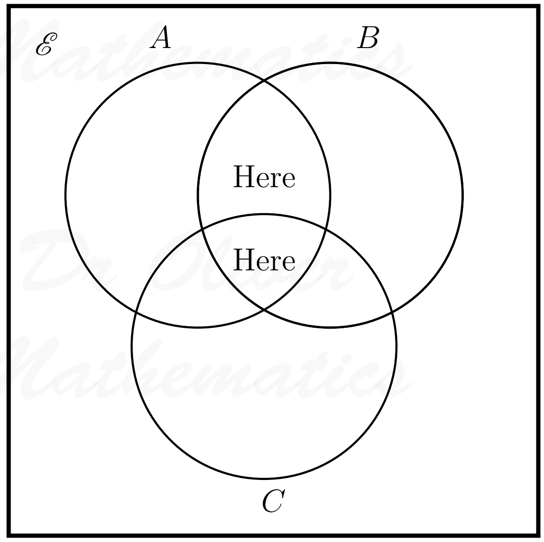
$$\underline{\underline{k \leqslant 2 \text{ or } k \geqslant 10}}.$$

4. We have a Venn diagram.



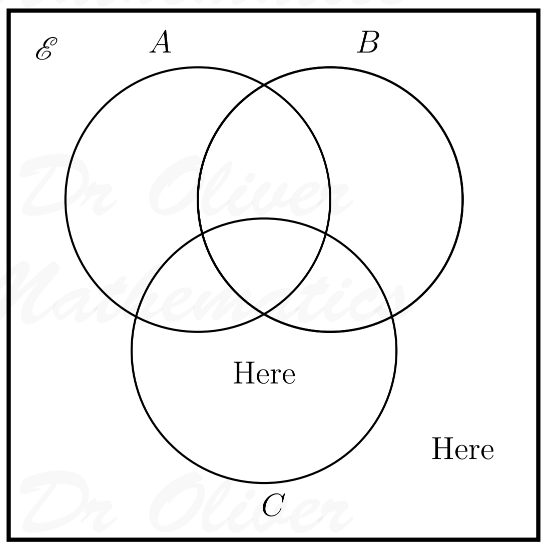
(a) Copy the Venn diagram above and shade the region that represents $(A \cap B) \cup C$. (1)

Solution



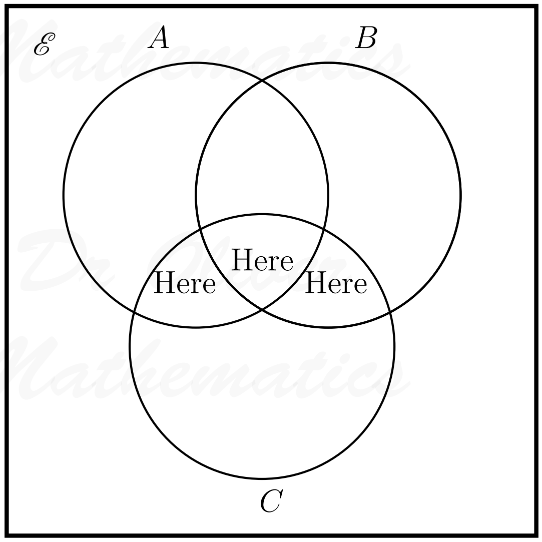
(ii) Copy the Venn diagram above and shade the region that represents $A' \cap B'$. (1)

Solution



(iii) Copy the Venn diagram above and shade the region that represents $(A \cup B) \cap C$. (1)

Solution



(b) It is given that the

- universal set $\mathscr{E} = \{x : 2 \leqslant x \leqslant 20,\ x \text{ is an integer}\}$,
- $X = \{x : 4 < x < 15,\ x \text{ is an integer}\}$,
- $Y = \{x : x \geqslant 9,\ x \text{ is an integer}\}$, and
- $Z = \{x : x \text{ is a multiple of } 5\}$.

(i) List the elements of $X \cap Y$. (1)

Solution
$\underline{\underline{\{9, 10, 11, 12, 13, 14\}}}$.

(ii) List the elements of $X \cup Y$. (1)

Solution
$\underline{\underline{\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}}}$.

(iii) Find $(X \cup Y)' \cap Z$. (1)

Solution
$\underline{\underline{\varnothing}}$.

5. Solve the equation (6)

$$3x(x^2 + 6) = 8 - 17x^2.$$

Solution

Well,

$$\begin{aligned} 3x(x^2+6) = 8 - 17x^2 &\Rightarrow 3x^3 + 18x = 8 - 17x^2 \\ &\Rightarrow 3x^3 + 17x^2 + 18x - 8 = 0. \end{aligned}$$

Let

$$\mathrm{f}(x) = 3x^3 + 17x^2 + 18x - 8 = 0.$$

Now,

$$\begin{aligned} \mathrm{f}(1) &= 30 \\ \mathrm{f}(-1) &= -12 \\ \mathrm{f}(2) &= 120 \\ \mathrm{f}(-2) &= 0, \end{aligned}$$

so $(x+2)$ is a factor of $\mathrm{f}(x)$.

We use synthetic division:

-2	3	17	18	-8
	$\downarrow$	-6	-22	8
	3	11	-4	0

and

$$\mathrm{f}(x) = (x+2)(3x^2 - 11x - 4).$$

$$\left.\begin{array}{ll} \text{add to:} & +11 \\ \text{multiply to:} & (+3) \times (-4) = -12 \end{array}\right\} -12, +1$$

E.g.,

$$\begin{aligned} 3x^2 + 11x - 4 = 0 &\Rightarrow 3x^2 + 12x - x - 4 = 0 \\ &\Rightarrow 3x(x+4) - 1(x+4) = 0 \\ &\Rightarrow (3x-1)(x+4) = 0 \\ &\Rightarrow x = \tfrac{1}{3} \text{ or } x = -4. \end{aligned}$$

Hence, the solutions are

$$\underline{\underline{x = -4,\ x = -2,\ \text{ or } x = -\tfrac{1}{3}}}.$$

6. Given that

$$\log_8 p = x \text{ and } \log_8 q = y,$$

express in terms of x and/or y:

(a) $\log_8 \sqrt{p} + \log_8 q^2$, (2)

Solution

$$\begin{aligned}\log_8 \sqrt{p} + \log_8 q^2 &= \log_8 p^{\frac{1}{2}} + 2\log_8 q \\ &= \tfrac{1}{2}\log_8 p + 2\log_8 q \\ &= \underline{\underline{\tfrac{1}{2}x + 2y}}.\end{aligned}$$

(b) $\log_8 \left(\frac{q}{8}\right)$, (2)

Solution

$$\begin{aligned}\log_8 \left(\frac{q}{8}\right) &= \log_8 q - \log_8 8 \\ &= \underline{\underline{y - 1}}.\end{aligned}$$

(c) $\log_2(64p)$. (3)

Solution

$$\begin{aligned}\log_2(64p) &= \log_2 64 + \log_2 p \\ &= \log_2 2^6 + \frac{\log_8 p}{\log_8 2} \\ &= 6 + \frac{\log_8 p}{\log_8 8^{\frac{1}{3}}} \\ &= 6 + \frac{x}{\frac{1}{3}} \\ &= \underline{\underline{6 + 3x}}.\end{aligned}$$

7. The function f is defined by

$$\mathrm{f}(x) = (2x+1)^2 - 3, \text{ for } x \geqslant -\tfrac{1}{2}.$$

Find

(a) the range of f. (1)

Solution
Well,

$$\begin{aligned} f(-\tfrac{1}{2}) &= (-1+1)^2 - 3 \\ &= -3, \end{aligned}$$

and so

$$\underline{\underline{f(x) \geqslant -3}}.$$

(b) an expression for $f^{-1}(x)$. (3)

Solution

$$\begin{aligned} y = (2x+1)^2 - 3 \Rightarrow y + 3 &= (2x+1)^2 \\ \Rightarrow \sqrt{y+3} &= 2x + 1 \\ \Rightarrow \sqrt{y+3} - 1 &= 2x \\ \Rightarrow \frac{\sqrt{y+3} - 1}{2} &= x, \end{aligned}$$

and so

$$\underline{\underline{f^{-1}(x) = \frac{\sqrt{x+3} - 1}{2}}}.$$

The function g is defined by

$$g(x) = \frac{3}{1+x}, \text{ for } x > -1.$$

(c) Find the value of x for which (4)

$$f\,g(x) = 13.$$

Solution

Well,

$$\begin{aligned} \mathrm{f}\,\mathrm{g}(x) = 13 &\Rightarrow \mathrm{f}(\mathrm{g}(x)) = 13 \\ &\Rightarrow \mathrm{g}(x) = \mathrm{f}^{-1}(13) \\ &\Rightarrow \frac{3}{1+x} = \tfrac{3}{2} \end{aligned}$$

invert:

$$\begin{aligned} &\Rightarrow \frac{1+x}{3} = \tfrac{2}{3} \\ &\Rightarrow 1 + x = 2 \\ &\Rightarrow \underline{\underline{x = 1}}. \end{aligned}$$

8. (a) Solve the equation (3)

$$(2^{3-4x})(4^{x+4}) = 2.$$

Solution

$$\begin{aligned} (2^{3-4x})(4^{x+4}) = 2 &\Rightarrow (2^{3-4x})[(2^2)^{x+4}] = 2 \\ &\Rightarrow (2^{3-4x})((2^{2x+8}) = 2 \\ &\Rightarrow 2^{3-4x+2x+8} = 2 \\ &\Rightarrow 2^{11-2x} = 2^1 \\ &\Rightarrow 11 - 2x = 1 \\ &\Rightarrow 2x = 10 \\ &\Rightarrow \underline{\underline{x = 5}}. \end{aligned}$$

(b) (i) Simplify (2)

$$\sqrt{108} - \frac{12}{\sqrt{3}},$$

giving your answer in the form $k\sqrt{3}$, where k is an integer.

Solution

$$\begin{aligned}\sqrt{108} - \frac{12}{\sqrt{3}} &= \sqrt{36 \times 3} - \left(\frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\right) \\ &= \sqrt{36} \times \sqrt{3} - \frac{12\sqrt{3}}{3} \\ &= 6\sqrt{3} - 4\sqrt{3} \\ &= \underline{\underline{2\sqrt{3}}};\end{aligned}$$

so, $\underline{\underline{k = 2}}$.

(ii) Simplify (3)

$$\frac{\sqrt{5}+3}{\sqrt{5}-2},$$

giving your answer in the form

$$a\sqrt{5} + b,$$

where a and b are integers.

Solution

Well,

$\times$	$\sqrt{5}$	-2
$\sqrt{5}$	5	$-2\sqrt{5}$
$+2$	$+2\sqrt{5}$	-4

and

$\times$	$\sqrt{5}$	$+2$
$\sqrt{5}$	5	$+2\sqrt{5}$
$+3$	$+3\sqrt{5}$	$+6$

and so

$$\begin{aligned}\frac{\sqrt{5}+3}{\sqrt{5}-2} &= \frac{\sqrt{5}+3}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2} \\ &= \frac{11+5\sqrt{5}}{5-4} \\ &= \underline{\underline{5\sqrt{5}+11}};\end{aligned}$$

hence, $\underline{\underline{a = 5}}$ and $\underline{\underline{b = 11}}$.

9. (a) Variables x and y are related by the equation

$$y = 5x + 2 - 4e^{-x}.$$

(i) Find $\frac{dy}{dx}$. (2)

Solution

$$y = 5x + 2 - 4e^{-x} \Rightarrow \underline{\underline{\frac{dy}{dx} = 5 + 4e^{-x}}}.$$

(ii) Hence find the approximate change in y when x increases from 0 to p, where p is small. (2)

Solution

Well,

$$x = 0 \Rightarrow \frac{dy}{dx} = 9$$

and

$$\begin{aligned} \delta y &\approx \frac{dy}{dx} \times \delta x \\ &= \underline{\underline{9p}}. \end{aligned}$$

(b) A square of area A cm^2 has a side of length x cm. (4)

Given that the area is increasing at a constant rate of 0.5 cm^{2}s^{-1}, find the rate of increase of x when $A = 9$.

Solution

Well,

$$A = x^2 \Rightarrow \frac{dA}{dx} = 2x$$

and

$$A = 9 \Rightarrow x = 3.$$

Now,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}A}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\frac{\mathrm{d}A}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\frac{\mathrm{d}A}{\mathrm{d}t}}{\frac{\mathrm{d}A}{\mathrm{d}x}}$$
$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{0.5}{2 \times 3}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{1}{12} \text{ cms}^{-1}}}.$$

10. Solve

(a) $4 \sin x = \cos x$ for $0^\circ < x < 360^\circ$, (3)

Solution

$$4 \sin x = \cos x \Rightarrow \tan x = \tfrac{1}{4}$$
$$\Rightarrow x = 14.036\,243\,47,\ 194.036\,243\,47 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 14.0,\ 194 \text{ (3 sf)}}}.$$

(b) $3 + \sin y = 3 \cos^2 y$ for $0^\circ < y < 360^\circ$, (5)

Solution

$$3 + \sin y = 3 \cos^2 y \Rightarrow 3 + \sin y = 3(1 - \sin^2 y)$$
$$\Rightarrow 3 + \sin y = 3 - 3 \sin^2 y$$
$$\Rightarrow 3 \sin^2 y + \sin y = 0$$
$$\Rightarrow \sin y(3 \sin y + 1) = 0$$
$$\Rightarrow \sin y = 0 \text{ or } \sin y = -\tfrac{1}{3}.$$

$\underline{\sin y = 0}$:

$$\sin y = 0 \Rightarrow \underline{\underline{y = 180}}.$$

$\underline{\sin y = -\tfrac{1}{3}}$:

$$\sin y = -\tfrac{1}{3} \Rightarrow y = 199.471\,220\,6,\ 340.528\,779\,4 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{y = 199,\ 341 \text{ (3 sf)}}}.$$

(c) $\sec\left(\frac{z}{3}\right) = 4$ for $0 < z < 5$ radians. (3)

Solution
Well,

$$0 < z < 5 \Rightarrow 0 < \frac{z}{3} < \frac{5}{3}$$

and

$$\begin{aligned}\sec\left(\frac{z}{3}\right) = 4 \Rightarrow \cos\left(\frac{z}{3}\right) &= \tfrac{1}{4} \\ \Rightarrow \frac{z}{3} &= 1.318\,116\,072 \text{ (FCD)} \\ \Rightarrow z &= 3.954\,348\,215 \text{ (FCD)} \\ \Rightarrow \underline{\underline{z = 3.96 \text{ (3 sf)}}}.\end{aligned}$$

EITHER

11. A curve has equation

$$y = \frac{\ln x}{x^2}, \text{ where } x > 0.$$

(a) Find the exact coordinates of the stationary point of the curve. (6)

Solution
Quotient rule:

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$$
$$v = x^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x$$

and so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x^2)(\frac{1}{x}) - (\ln x)(2x)}{(x^2)^2} \\ &= \frac{x - 2x\ln x}{x^4}.\end{aligned}$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow \frac{x - 2x\ln x}{x^4} = 0 \\ &\Rightarrow x(1 - 2\ln x) = 0 \\ &\Rightarrow x = 0 \text{ (no!) or } \ln x = \tfrac{1}{2} \\ &\Rightarrow x = \mathrm{e}^{\frac{1}{2}} \\ &\Rightarrow y = \tfrac{1}{2}\mathrm{e}^{-1};\end{aligned}$$

hence, the exact coordinates of the stationary point of the curve are

$$\underline{\underline{(\mathrm{e}^{\frac{1}{2}}, \tfrac{1}{2}\mathrm{e}^{-1})}}.$$

(b) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ can be written in the form (3)

$$\frac{a\ln x + b}{x^4},$$

where a and b are integers.

Solution
Quotient rule:

$$\begin{aligned}u = x - 2x\ln x &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 - (2 + 2\ln x) = -2\ln x - 1 \\ v = x^4 &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 4x^3\end{aligned}$$

and so

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{(x^4)(-2\ln x - 1) - (4x^3)(x - 2x\ln x)}{(x^4)^2} \\ &= \frac{-2x\ln x - x - 4x + 8x\ln x}{x^5} \\ &= \frac{6x\ln x - 5x}{x^5} \\ &= \underline{\underline{\frac{6\ln x - 5}{x^4}}};\end{aligned}$$

hence, $\underline{\underline{a = 6}}$ and $\underline{\underline{b = -5}}$.

(c) Hence, or otherwise, determine the nature of the stationary point of the curve. (2)

Solution
Well,

$$x = e^{\frac{1}{2}} \Rightarrow \frac{d^2y}{dx^2} = -0.270\ldots < 0$$

and so the nature of the stationary point of the curve is a maximum.

OR

12. A curve is such that

$$\frac{dy}{dx} = 6\cos(2x + \tfrac{1}{2}\pi) \text{ for } -\tfrac{1}{4}\pi \leqslant x \leqslant \tfrac{5}{4}\pi.$$

The curve passes through the point $(\frac{1}{4}\pi, 5)$.

Find

(a) the equation of the curve, (4)

Solution
Well,

$$\frac{dy}{dx} = 6\cos(2x + \tfrac{1}{2}\pi) \Rightarrow y = 3\sin(2x + \tfrac{1}{2}\pi) + c,$$

for some constant c. Now,

$$\begin{aligned} x = \tfrac{1}{4}\pi, y = 5 &\Rightarrow 3\sin(\pi) + c = 5 \\ &\Rightarrow c = 5; \end{aligned}$$

hence,

$$y = 3\sin(2x + \tfrac{1}{2}\pi) + 5.$$

(b) the x-coordinates of the stationary points of the curve, (3)

Solution
Now,

$$\begin{aligned} -\tfrac{1}{4}\pi \leqslant x \leqslant \tfrac{5}{4}\pi &\Rightarrow -\tfrac{1}{2}\pi \leqslant 2x \leqslant \tfrac{5}{2}\pi \\ &\Rightarrow 0 \leqslant (2x + \tfrac{1}{2}\pi) \leqslant 3\pi \end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 6\cos(2x + \tfrac{1}{2}\pi) = 0 \\ &\Rightarrow \cos(2x + \tfrac{1}{2}\pi) = 0 \\ &\Rightarrow 2x + \tfrac{1}{2}\pi = \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi,\ \tfrac{5}{2}\pi \\ &\Rightarrow 2x = 0,\ \pi,\ 2\pi \\ &\Rightarrow \underline{\underline{x = 0,\ \tfrac{1}{2}\pi,\ \pi.}}\end{aligned}$$

(c) the equation of the normal to the curve at the point on the curve where $x = \frac{3}{4}\pi$. (4)

Solution

Well,

$$\begin{aligned}x = \tfrac{3}{4}\pi &\Rightarrow y = 5 \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 6 \\ &\Rightarrow m_{\text{normal}} = -\tfrac{1}{6}.\end{aligned}$$

Finally, the equation of the normal is

$$\begin{aligned}y - 5 = -\tfrac{1}{6}(x - \tfrac{3}{4}\pi) &\Rightarrow y - 5 = -\tfrac{1}{6}x + \tfrac{1}{8}\pi \\ &\Rightarrow \underline{\underline{y = -\tfrac{1}{6}x + \tfrac{1}{8}\pi + 5.}}\end{aligned}$$