Dr Oliver Mathematics
Cambridge O Level Additional Mathematics
2009 November Paper 1: Calculator
2 hours

The total number of marks available is 80.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You must write down all the stages in your working.

1. Given that
$$f(x) = 2x^3 - 7x^2 + 7ax + 16$$
is divisible by $(x - a)$, find

(a) the value of the constant a, (2)

Solution
Well,
$$\begin{aligned} f(a) = 0 &\Rightarrow 2a^3 - 7a^2 + 7a^2 + 16 = 0 \\ &\Rightarrow 2a^3 = -16 \\ &\Rightarrow a^3 = -8 \\ &\Rightarrow \underline{\underline{a = -2}}. \end{aligned}$$

(b) the remainder when $f(x)$ is divided by $(2x + 1)$. (2)

Solution
We use synthetic division:

$-\frac{1}{2}$	2	−7	−14	16
	↓	−1	4	5
	2	−8	−10	21

Hence, the remainder is $\underline{\underline{21}}$.

2. The table shows the results achieved by four teams in twelve events of an athletics match.

	1st	2nd	3rd	4th
Harriers	6	3	1	2
Strollers	3	2	4	3
Road Runners	2	5	5	0
Olympians	1	2	1	7

In each event,

- 1st place scores 5 points,
- 2nd place scores 3 points,
- 3rd place scores 2 points, and
- 4th place scores 1 point.

(a) Write down two matrices whose product shows the total number of points scored by each team. (2)

Solution

$$\begin{pmatrix} H \\ S \\ RR \\ O \end{pmatrix} = \underline{\underline{\begin{pmatrix} 6 & 3 & 1 & 2 \\ 3 & 2 & 4 & 3 \\ 2 & 5 & 5 & 0 \\ 1 & 2 & 2 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}}}.$$

(b) Evaluate this product of matrices. (2)

Solution

$$\begin{pmatrix} H \\ S \\ RR \\ O \end{pmatrix} = \underline{\underline{\begin{pmatrix} 43 \\ 32 \\ 35 \\ 22 \end{pmatrix}}}.$$

3. Find the values of k for which the equation (4)

$$x^2 - 2(2k+1)x + (k+2) = 0$$

has two equal roots.

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Solution

Well, $a = 1$, $b = -2(2k+1)$ and $c = k+2$. Then

$$b^2 - 4ac = 0 \Rightarrow [-2(2k+1)]^2 - 4(1)(k+2) = 0$$

×	$2k$	$+1$
$2k$	$4k^2$	$+2k$
$+1$	$+2k$	$+1$

$$\begin{aligned} &\Rightarrow 4(4k^2 + 4k + 1) - 4(k+2) = 0 \\ &\Rightarrow 4[(4k^2 + 4k + 1) - (k+2)] = 0 \\ &\Rightarrow 4(4k^2 + 3k - 1) = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & +3 \\ \text{multiply to:} & (+4) \times (-1) = -4 \end{array}\right\} +4, -1$$

e.g.,

$$\begin{aligned} &\Rightarrow 4[4k^2 + 4k - k - 1] = 0 \\ &\Rightarrow 4[4k(k+1) - 1(k+1)] = 0 \\ &\Rightarrow 4(4k-1)(k+1) = 0 \\ &\Rightarrow \underline{\underline{k = \tfrac{1}{4} \text{ or } k = -1.}} \end{aligned}$$

4. Solve the simultaneous equations (5)

$$\begin{aligned} x + 3y &= 13, \\ x^2 + 3y^2 &= 43. \end{aligned}$$

Solution

Insert the linear equation into the non-linear equation:

$$x + 3y = 13 \Rightarrow x = 13 - 3y$$

and

$$x^2 + 3y^2 = 43 \Rightarrow (13 - 3y)^2 + 3y^2 = 43$$

$\times$	13	$-3y$
13	169	$-39y$
$-3y$	$-39y$	$+9y^2$

$$\Rightarrow (169 - 78y + 9y^2) + 3y^2 = 43$$
$$\Rightarrow 12y^2 - 78y + 126 = 0$$
$$\Rightarrow 6(2y^2 - 13y + 21) = 0$$

$$\left.\begin{array}{ll}\text{add to:} & -13 \\ \text{multiply to:} & (+2) \times (+21) = +42\end{array}\right\} -7, -6$$

e.g.,

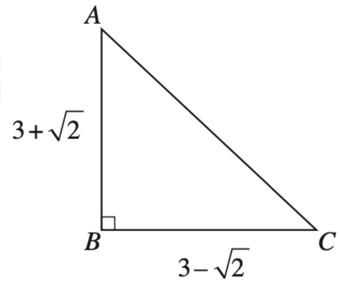
$$\Rightarrow 6[2y^2 - 7y - 6y + 21] = 0$$
$$\Rightarrow 6[y(2y - 7) - 3(2y - 7)] = 0$$
$$\Rightarrow 6(y - 3)(2y - 7) = 0$$
$$\Rightarrow y = 3 \text{ or } y = 3\tfrac{1}{2}$$
$$\Rightarrow x = 4 \text{ or } x = 2\tfrac{1}{2};$$

hence,

$$\underline{\underline{x = 4, y = 3 \text{ or } x = 2\tfrac{1}{2}, y = 3\tfrac{1}{2}}}.$$

5. The diagram shows a triangle ABC, where

- angle ABC is a right angle,
- the length of $AB = (3 + \sqrt{2})$, and
- the length of $BC = (3 - \sqrt{2})$.



(a) Find the length of AC in the form $\sqrt{k}$, where k is an integer. (2)

Solution

Well,

$\times$	3	$\pm\sqrt{2}$
3	9	$\pm 3\sqrt{2}$
$\pm\sqrt{2}$	$\pm 3\sqrt{2}$	$+2$

and

$$\begin{aligned} AC^2 = AB^2 + BC^2 &\Rightarrow AC^2 = (9 + 6\sqrt{2} + 2) + (9 - 6\sqrt{2} + 2) \\ &\Rightarrow AC^2 = 22 \\ &\Rightarrow \underline{\underline{AC = \sqrt{22}}}. \end{aligned}$$

(b) Find $\tan BAC$ in the form (3)

$$\frac{a + b\sqrt{2}}{c},$$

where a, b, and c are integers.

Solution

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Now,

$$\begin{aligned}
\tan = \frac{\text{opp}}{\text{adj}} &\Rightarrow \tan BAC = \frac{3-\sqrt{2}}{3+\sqrt{2}} \\
&\Rightarrow \tan BAC = \left(\frac{3-\sqrt{2}}{3+\sqrt{2}}\right) \times \left(\frac{3-\sqrt{2}}{3-\sqrt{2}}\right) \\
&\Rightarrow \tan BAC = \frac{11-6\sqrt{2}}{9-2} \\
&\Rightarrow \underline{\underline{\tan BAC = \frac{11-6\sqrt{2}}{7}}};
\end{aligned}$$

hence, $\underline{\underline{a = 11}}$, $\underline{\underline{b = -6}}$, and $\underline{\underline{c = 7}}$.

6. Set A is such that

$$A = \left\{x : 3x^2 - 10x - 8 \leqslant 0\right\}.$$

(a) Find the set of values of x which define the set A. (3)

Solution

Well,

$$\left.\begin{array}{ll} \text{add to:} & -10 \\ \text{multiply to:} & (+3) \times (-8) = -24 \end{array}\right\} -12,\ +2$$

e.g.,

$$\begin{aligned}
3x^2 - 10x - 8 \leqslant 0 &\Rightarrow 3x^2 - 12x + 2x - 8 \leqslant 0 \\
&\Rightarrow 3x(x-4) + 2(x-4) \leqslant 0 \\
&\Rightarrow (3x+2)(x-4) \leqslant 0
\end{aligned}$$

and we need a 'table of signs':

	$x < -\frac{2}{3}$	$x = -\frac{2}{3}$	$-\frac{2}{3} < x < 4$	$x = 4$	$x > 4$
$3x + 2$	$-$	0	$+$	$+$	$+$
$x - 4$	$-$	$-$	$-$	0	$+$
$(3x + 2)(x - 4)$	$+$	0	$-$	0	$+$

Hence,

$$\underline{\underline{A = \left\{x : -\tfrac{2}{3} \leqslant x \leqslant 4\right\}}}.$$

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Set B is such that

$$B = \{x : 7 - 2x \leqslant 1\}.$$

(b) Find the set of values of x which define the set $A \cap B$. (2)

Solution

Now,

$$\begin{aligned} 7 - 2x \leqslant 1 &\Rightarrow 6 \leqslant 2x \\ &\Rightarrow x \geqslant 3. \end{aligned}$$

Hence,

$$\underline{\underline{A \cap B = \{x : 3 \leqslant x \leqslant 4\}}}.$$

7. A committee of 8 people is to be selected from 7 teachers and 6 students.

Find the number of different ways in which the committee can be selected if

(a) there are no restrictions, (2)

Solution

$$\binom{13}{8} = \underline{\underline{1\,287}}.$$

(b) there are to be more teachers than students on the committee. (4)

Solution

Well,

$$\begin{aligned} & \text{P(more teachers than students)} \\ = \; & \text{P}(5T, 3S) + \text{P}(6T, 2S) + \text{P}(7T, S) \\ = \; & \left(\binom{7}{5} \times \binom{6}{3}\right) + \left(\binom{7}{6} \times \binom{6}{2}\right) + \left(\binom{7}{7} \times \binom{6}{1}\right) \\ = \; & 420 + 105 + 6 \\ = \; & \underline{\underline{531}}. \end{aligned}$$

8. The number, N, of bacteria present in an experiment, t minutes after measurements begin, is given by

$$N = 1\,000\mathrm{e}^{-kt},$$

where k is a constant.

(a) State the number of bacteria when $t = 0$. (1)

Solution

$\underline{\underline{1\,000}}$.

When $t = 0$, the number of bacteria is decreasing at the rate of 20 per minute.

Find

(b) the value of k, (3)

Solution

Well,

$$N = 1\,000\mathrm{e}^{-kt} \Rightarrow \frac{\mathrm{d}N}{\mathrm{d}t} = -1\,000k\mathrm{e}^{-kt}$$

and

$$t = 0, \frac{\mathrm{d}N}{\mathrm{d}t} = -20 \Rightarrow -20 = -1\,000k$$
$$\Rightarrow \underline{\underline{k = \tfrac{1}{50}}}$$

(c) the time taken for the number of bacteria to decrease by 50%. (3)

Solution

Now,

$$500 = 1\,000\mathrm{e}^{-\frac{1}{50}t} \Rightarrow \tfrac{1}{2} = \mathrm{e}^{-\frac{1}{50}t}$$
$$\Rightarrow 2 = \mathrm{e}^{\frac{1}{50}t}$$
$$\Rightarrow \tfrac{1}{50}t = \ln 2$$
$$\Rightarrow \underline{\underline{t = 50\ln 2 \text{ or } 34.7 \text{ minutes (3 sf)}}}.$$

9. Differentiate, with respect to x,

(a) $(1 - 2x)^{20}$, (2)

Solution

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$$\frac{d}{dx}\left[(1-2x)^{20}\right] = 20(1-2x)^{19} \times (-2)$$
$$= \underline{\underline{-40(1-2x)^{19}}}.$$

(b) $x^2 \ln x$, (3)

Solution

Product rule:

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$
$$v = \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

and

$$\frac{d}{dx}\left[x^2 \ln x\right] = (x^2)\left(\frac{1}{x}\right) + (2x)(\ln x)$$
$$= \underline{\underline{x + 2x \ln x}}.$$

(c) $\dfrac{\tan(2x+1)}{x}$. (3)

Solution

Quotient rule:

$$u = \tan(2x+1) \Rightarrow \frac{du}{dx} = 2\sec^2(2x+1)$$
$$v = x \Rightarrow \frac{dv}{dx} = 1$$

and

$$\frac{d}{dx}\left[\frac{\tan(2x+1)}{x}\right] = \frac{(x)(2\sec^2(2x+1)) - (\tan(2x+1))(1)}{x^2}$$
$$= \underline{\underline{\frac{2x\sec^2(2x+1) - \tan(2x+1)}{x^2}}}.$$

10. A curve has equation

$$y = 3x^3 - 2x^2 + 2x.$$

(a) Show that the equation of the tangent to the curve at the point where $x = 1$ is (4)

$$y = 7x - 4.$$

Solution

Well,

$$y = 3x^3 - 2x^2 + 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 - 4x + 2$$

and

$$x = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 7.$$

Now,

$$x = 1 \Rightarrow y = 3$$

and the equation is

$$\begin{aligned} y - 3 = 7(x - 1) &\Rightarrow y - 3 = 7x - 7 \\ &\Rightarrow \underline{\underline{y = 7x - 4}}, \end{aligned}$$

as required.

(b) Find the coordinates of the point where this tangent meets the curve again. (5)

Solution

Well,

$$3x^3 - 2x^2 + 2x = 7x - 4 \Rightarrow 3x^3 - 2x^2 - 5x + 4 = 0.$$

We use synthetic division:

1	3	−2	−5	4
	↓	3	1	−4
	3	1	−4	0

and

$$3x^3 - 2x^2 - 5x + 4 = 0 \Rightarrow (x - 1)(3x^2 + x - 4) = 0.$$

$$\left.\begin{array}{ll} \text{add to:} & +1 \\ \text{multiply to:} & (+3) \times (-4) = -12 \end{array}\right\} +4,\ -3$$

E.g.,

$$\begin{aligned}3x^2 + x - 4 = 0 &\Rightarrow 3x^2 + 4x - 3x - 4 = 0\\ &\Rightarrow x(3x+4) - 1(3x+4) = 0\\ &\Rightarrow (x-1)(3x+4) = 0\\ &\Rightarrow x = 1 \text{ or } x = -\tfrac{4}{3}.\end{aligned}$$

Finally

$$x = -\tfrac{4}{3} \Rightarrow y = -\tfrac{64}{9}$$

and, hence, the coordinates of the point where this tangent meets the curve again are

$$\underline{\underline{\left(-\tfrac{4}{3}, -\tfrac{40}{3}\right)}}.$$

11. (a) Show that (3)

$$\tan\theta + \cot\theta \equiv \operatorname{cosec}\theta\sec\theta.$$

Solution

$$\begin{aligned}\tan\theta + \cot\theta &\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &\equiv \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\\ &\equiv \frac{1}{\sin\theta\cos\theta}\\ &\equiv \underline{\underline{\operatorname{cosec}\theta\sec\theta}},\end{aligned}$$

as required.

(b) Solve the equation

(i) $\tan x = 3\sin x$ for $0^\circ \leqslant x \leqslant 360^\circ$, (4)

Solution

$$\begin{aligned}\tan x = 3\sin x &\Rightarrow \frac{\sin\theta}{\cos\theta} = 3\sin x\\ &\Rightarrow \sin\theta = 3\sin x\cos\theta\\ &\Rightarrow \sin\theta - 3\sin x\cos\theta = 0\\ &\Rightarrow \sin\theta(1-3\cos\theta) = 0\\ &\Rightarrow \sin\theta = 0 \text{ or } \cos\theta = \tfrac{1}{3}.\end{aligned}$$

$\underline{\sin\theta = 0}$:

$$\sin\theta = 0 \Rightarrow \underline{\underline{\theta = 180}}.$$

$\underline{\cos\theta = \frac{1}{3}}$:

$$\begin{aligned}\cos\theta = \tfrac{1}{3} &\Rightarrow \theta = 70.528\,779\,37,\ 289.471\,220\,6 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{\theta = 70.5,\ 289 \text{ (3 sf)}}}.\end{aligned}$$

(ii) $2\cot^2 y + 3\operatorname{cosec} y = 0$ for $0 < y < 2\pi$ radians. (5)

Solution

$$\begin{aligned}2\cot^2 y + 3\operatorname{cosec} y = 0 &\Rightarrow 2(\operatorname{cosec}^2 y - 1) + 3\operatorname{cosec} y = 0\\ &\Rightarrow 2\operatorname{cosec}^2 y + 3\operatorname{cosec} y - 2 = 0\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +3\\ \text{multiply to:} & (+2)\times(-2) = -4\end{array}\right\} +4,\ -1$$

e.g.,

$$\begin{aligned}&\Rightarrow 2\operatorname{cosec}^2 y + 4\operatorname{cosec} y - \operatorname{cosec} y - 2 = 0\\ &\Rightarrow 2\operatorname{cosec} y(\operatorname{cosec} y + 2) - 1(\operatorname{cosec} y + 2) = 0\\ &\Rightarrow (2\operatorname{cosec} y - 1)(\operatorname{cosec} y + 2) = 0\\ &\Rightarrow \operatorname{cosec} y = \tfrac{1}{2} \text{ or } \operatorname{cosec} y = -2\\ &\Rightarrow \sin y = 2 \text{ (no!) or } \sin y = -\tfrac{1}{2}\\ &\Rightarrow \underline{\underline{y = \tfrac{7}{6}\pi,\ \tfrac{11}{6}\pi}}.\end{aligned}$$

EITHER

12. A solid circular cylinder has radius r cm and height h cm.

The volume of the cylinder is $1\,000$ cm^3.

(a) Find an expression for h in terms of r. (2)

Solution

$$\pi r^2 h = 1\,000 \Rightarrow \underline{\underline{h = \frac{1\,000}{\pi r^2}}}.$$

(b) Hence show that the total surface area, A cm^2, of the cylinder is given by (2)

$$A = 2\pi r^2 + \frac{2\,000}{r}.$$

Solution

Well, the total surface area is given by

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{1\,000}{\pi r^2}\right) \\ &= \underline{\underline{2\pi r^2 + \frac{2\,000}{r}}}, \end{aligned}$$

as required.

(c) Given that r varies, find, correct to 2 decimal places, the value of r when A has a stationary value. (4)

Solution

Well,

$$\begin{aligned} A = 2\pi r^2 + \frac{2\,000}{r} &\Rightarrow A = 2\pi r^2 + 2\,000 r^{-1} \\ &\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - 2\,000 r^{-2} \end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}r} = 0 &\Rightarrow 4\pi r - 2\,000r^{-2} = 0\\ &\Rightarrow 4\pi r = 2\,000r^{-2}\\ &\Rightarrow r^3 = \frac{500}{\pi}\\ &\Rightarrow r = \sqrt[3]{\frac{500}{\pi}}\\ &\Rightarrow r = 5.419\,260\,701 \text{ (FCD)}\\ &\Rightarrow \underline{\underline{r = 5.42 \text{ cm (2 dp)}}}.\end{aligned}$$

(d) Find this stationary value of A and determine its nature. (3)

Solution

Well,

$$\begin{aligned}r = 5.419\ldots \Rightarrow A &= 2\pi(5.419\ldots^2) + \frac{2\,000}{5.419\ldots}\\ \Rightarrow A &= 553.581\,044\,6 \text{ (FCD)}\\ \Rightarrow \underline{\underline{A}} &\underline{\underline{= 554 \text{ cm}^2 \text{ (3 sf)}}}.\end{aligned}$$

Now,

$$\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - 2\,000r^{-2} \Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}r^2} = 4\pi + 4\,000r^{-3}$$

and

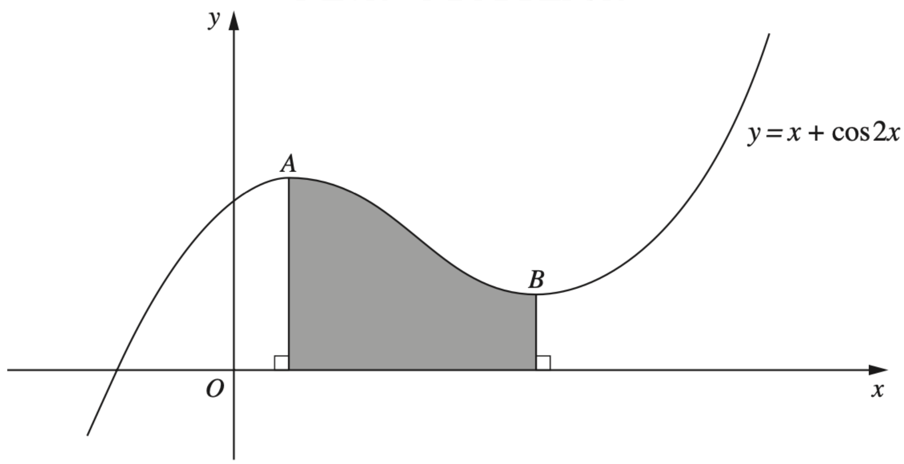
$$\begin{aligned}r = 5.419\ldots &\Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}r^2} = 12\pi\\ &\Rightarrow \frac{\mathrm{d}^2A}{\mathrm{d}r^2} > 0\end{aligned}$$

and this is a minumum.

OR

13. The diagram shows part of the curve

$$y = x + \cos 2x.$$



The curve has a maximum point at A and a minimum point at B.

(a) Find the x-coordinate of the point A and of the point B. (6)

Solution

Well,

$$y = x + \cos 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 1 - 2\sin 2x$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 &\Rightarrow 1 - 2\sin 2x = 0 \\ &\Rightarrow 2\sin 2x = 1 \\ &\Rightarrow \sin 2x = \tfrac{1}{2} \\ &\Rightarrow 2x = \tfrac{1}{6}\pi,\ \tfrac{5}{6}\pi \\ &\Rightarrow \underline{\underline{x = \tfrac{1}{12}\pi,\ \tfrac{5}{12}\pi}}.\end{aligned}$$

(b) Find, in terms of π, the area of the shaded region. (5)

Solution

$$
\begin{aligned}
\text{Area} &= \int_{\frac{1}{12}\pi}^{\frac{5}{12}\pi} (x + \cos 2x)\,\mathrm{d}x \\
&= \left[\tfrac{1}{2}x^2 + \tfrac{1}{2}\sin 2x\right]_{x=\frac{1}{12}\pi}^{\frac{5}{12}\pi} \\
&= \left(\tfrac{1}{2}(\tfrac{5}{12}\pi)^2 + \tfrac{1}{4}\right) - \left(\tfrac{1}{2}(\tfrac{1}{12}\pi)^2 + \tfrac{1}{4}\right) \\
&= \underline{\underline{\tfrac{1}{12}\pi^2}}.
\end{aligned}
$$