Dr Oliver Mathematics
Mathematics: Advanced Higher
2017 Paper
3 hours

The total number of marks available is 100.
You must write down all the stages in your working.

1. Write down the binomial expansion (4)

$$\left(\frac{2}{y^2} - 5y\right)^3,$$

and simplify your answer.

Solution

$$\begin{aligned} & \left(\frac{2}{y^2} - 5y\right)^3 \\ = \quad & \left(\frac{2}{y^2}\right)^3 + 3\left(\frac{2}{y^2}\right)^2(-5y) + 3\left(\frac{2}{y^2}\right)(-5y)^2 + (-5y)^3 \\ & = \underline{\underline{\frac{8}{y^6} - \frac{60}{y^3} + 150 - 125y^3}}. \end{aligned}$$

2. Express (4)

$$\frac{x^2 - 6x + 20}{(x+1)(x-2)^2}$$

in partial fractions.

Solution

$$\begin{aligned} \frac{x^2 - 6x + 20}{(x+1)(x-2)^2} &\equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} \\ &= \frac{A(x-2)^2 + B(x+1)(x-2) + C(x+1)}{(x+1)(x-2)^2} \end{aligned}$$

and so

$$x^2 - 6x + 20 \equiv A(x-2)^2 + B(x+1)(x-2) + C(x+1).$$

$\underline{x = 2}$: $12 = 3C \Rightarrow C = 4$.
$\underline{x = -1}$: $27 = 9A \Rightarrow A = 3$.
$\underline{x = 0}$: $20 = 3 \cdot 4 - 2B + 4 \Rightarrow 2B = -4 \Rightarrow B = -2$.
Hence,

$$\frac{x^2 - 6x + 20}{(x+1)(x-2)^2} \equiv \underline{\underline{\frac{3}{(x+1)} - \frac{2}{(x-2)} + \frac{4}{(x-2)^2}}}.$$

3. On a suitable domain, a function is defined by (3)

$$\mathrm{f}(x) = \frac{\mathrm{e}^{x^2-1}}{x^2-1}.$$

Find $\mathrm{f}'(x)$, simplifying your answer.

Solution

$$u = \mathrm{e}^{x^2-1} \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x\,\mathrm{e}^{x^2-1}$$
$$v = x^2 - 1 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = 2x.$$

Finally,

$$\mathrm{f}(x) = \frac{\mathrm{e}^{x^2-1}}{x^2-1} \Rightarrow \mathrm{f}'(x) = \frac{(x^2-1)\,2x\,\mathrm{e}^{x^2-1} - 2x\,\mathrm{e}^{x^2-1}}{(x^2-1)^2}$$
$$\Rightarrow \mathrm{f}'(x) = \frac{2x\,\mathrm{e}^{x^2-1}[(x^2-1)-1]}{(x^2-1)^2}$$
$$\Rightarrow \mathrm{f}'(x) = \underline{\underline{\frac{2x\,\mathrm{e}^{x^2-1}(x^2-2)}{(x^2-1)^2}}}.$$

4. The fifth term of an arithmetic sequence is -6 and the twelfth term is -34.

(a) Determine the values of the first term and the common difference. (2)

Solution

$$\begin{aligned} \text{5th term}: &\quad a + 4d = -6 \quad (1) \\ \text{12th term}: &\quad a + 11d = -34 \quad (2). \end{aligned}$$

Do (2) − (1):

$$7d = -28 \Rightarrow \underline{\underline{d = -4}}$$

$$\begin{aligned} &\Rightarrow a - 16 = -6 \\ &\Rightarrow \underline{\underline{a = 10}}. \end{aligned}$$

(b) Obtain algebraically the value of n for which $S_n = -144$. (3)

Solution

$$\begin{aligned} S_n = -144 &\Rightarrow \tfrac{1}{2}n[20 - 4(n-1)] = -144 \\ &\Rightarrow n[20 - 4n + 4] = -288 \\ &\Rightarrow n(24 - 4n) = -288 \\ &\Rightarrow 24n - 4n^2 = -288 \\ &\Rightarrow 4n^2 - 24n - 288 = 0 \\ &\Rightarrow n^2 - 6n - 72 = 0 \end{aligned}$$

$$\left.\begin{array}{ll} \text{add to:} & -6 \\ \text{multiply to:} & -72 \end{array}\right\} -12,\ +6$$

$$\begin{aligned} &\Rightarrow (n-12)(n+6) = 0 \\ &\Rightarrow n = 12 \text{ or } n = -6; \end{aligned}$$

clearly, $n \neq -6$ and so $\underline{\underline{n = 12}}$.

5. (a) (i) Use Gaussian elimination on the system of equations below to give an expression for z in terms of λ. (4)

$$\begin{aligned} x + 2y - z &= -3 \\ 4x - 2y + 3z &= 11 \\ 3x + y + 2\lambda z &= 8. \end{aligned}$$

Solution

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 4 & -2 & 3 & 11 \\ 3 & 1 & 2\lambda & 8 \end{array}\right)$$

Do $R_2 - 4R_1$ and $R_3 - 3R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -10 & 7 & 23 \\ 0 & -5 & 2\lambda + 3 & 17 \end{array}\right)$$

Do $2R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -3 \\ 0 & -10 & 7 & 23 \\ 0 & 0 & 4\lambda - 1 & 11 \end{array}\right)$$

Hence,

$$(4\lambda - 1)z = 11 \Rightarrow \underline{\underline{z = \frac{11}{4\lambda - 1}}}.$$

(ii) For what value of λ is this system of equations inconsistent? (1)

Solution

$$\begin{aligned} 4\lambda - 1 = 0 &\Rightarrow 4\lambda = 1 \\ &\Rightarrow \underline{\underline{\lambda = \tfrac{1}{4}}}. \end{aligned}$$

(b) Determine the solution of this system when $\lambda = -2.5$. (1)

Solution

$$\begin{aligned} \lambda = -2.5 \Rightarrow z &= -1 \\ &\Rightarrow -10y - 7 = 23 \\ &\Rightarrow -10y = 30 \\ &\Rightarrow y = -3 \\ &\Rightarrow x - 6 + 1 = -3 \\ &\Rightarrow x = 2; \end{aligned}$$

hence,

$$\underline{\underline{x = 2,\ y = -3,\ z = -1}}.$$

6. Use the substitution $u = 5x^2$ to find the exact value of (6)

$$\int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1 - 25x^4}}\,\mathrm{d}x.$$

Solution

$$\begin{aligned} u = 5x^2 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 10x \\ &\Rightarrow \mathrm{d}u = 10x\,\mathrm{d}x \\ &\Rightarrow \tfrac{1}{10}\,\mathrm{d}u = x\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} x = 0 &\Rightarrow u = 0 \\ x = \tfrac{1}{\sqrt{10}} &\Rightarrow u = \tfrac{1}{2}. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1 - 25x^4}}\,\mathrm{d}x &= \int_0^{\frac{1}{\sqrt{10}}} \frac{x}{\sqrt{1 - (5x^2)^2}}\,\mathrm{d}x \\ &= \int_0^{\frac{1}{2}} \frac{\frac{1}{10}}{\sqrt{1 - u^2}}\,\mathrm{d}u \\ &= \tfrac{1}{10}\left[\arcsin u\right]_{u=0}^{\frac{1}{2}} \\ &= \tfrac{1}{10}\left(\arcsin \tfrac{1}{2} - 0\right) \\ &= \underline{\underline{\tfrac{1}{60}\pi}}. \end{aligned}$$

7. Matrices $\mathbf{P}$ and $\mathbf{Q}$ are defined by

$$\mathbf{P} = \begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix} \text{ and } \mathbf{Q} = \begin{pmatrix} 2 & -3 \\ 4 & y \end{pmatrix},$$

where x, $y \in \mathbb{R}$.

(a) Given the determinant of $\mathbf{P}$ is 2, obtain:

(i) The value of x. (1)

Solution

$$\det \mathbf{P} = 2 \Rightarrow -x + 10 = 2$$
$$\Rightarrow \underline{\underline{x = 8}}.$$

(ii) $\mathbf{P}^{-1}$. (1)

Solution

$$\mathbf{P}^{-1} = \underline{\underline{\tfrac{1}{2}\begin{pmatrix} -1 & -2 \\ 5 & 8 \end{pmatrix}}}.$$

(iii) $\mathbf{P}^{-1}\mathbf{Q}^{\mathrm{T}}$, where $\mathbf{Q}^{\mathrm{T}}$ is the transpose of $\mathbf{Q}$. (2)

Solution

$$\mathbf{P}^{-1}\mathbf{Q}^{\mathrm{T}} = \tfrac{1}{2}\begin{pmatrix} -1 & -2 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & y \end{pmatrix}$$
$$= \tfrac{1}{2}\begin{pmatrix} 4 & -4-2y \\ -14 & 20+8y \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} 2 & -2-y \\ -7 & 10+4y \end{pmatrix}}}.$$

The matrix $\mathbf{R}$ is defined by

$$\mathbf{R} = \begin{pmatrix} 5 & -2 \\ z & -6 \end{pmatrix},$$

where $z \in \mathbb{R}$.

(b) Determine the value of z such that $\mathbf{R}$ is singular. (2)

Solution

$$\det \mathbf{R} = 0 \Rightarrow -30 + 2z = 0$$
$$\Rightarrow 2z = 30$$
$$\Rightarrow \underline{\underline{z = 15}}.$$

8. Use the Euclidean algorithm to find integers a and b such that (4)

$$1\,595a + 1\,218b = 29.$$

Solution

$$
\begin{aligned}
1\,595 &= 1 \times 1\,218 + 377 \\
1\,218 &= 3 \times 377 + 87 \\
377 &= 4 \times 87 + 29 \\
87 &= 3 \times 29 + 0
\end{aligned}
$$

and so

$$
\begin{aligned}
29 &= 377 - 4 \times 87 \\
&= 377 - 4(1\,218 - 3 \times 377) \\
&= 13 \times 377 - 4 \times 1\,218 \\
&= 13(1\,595 - 1\,218) - 4 \times 1\,218 \\
&= \underline{\underline{13 \times 1\,595 - 17 \times 1\,218}};
\end{aligned}
$$

hence,

$$\underline{\underline{a = 13 \text{ and } b = -17}}.$$

9. Solve (5)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x}(1 + y^2),$$

given that when $x = 0$, $y = 1$.

Express y in terms of x.

Solution

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{2x}(1 + y^2) &\Rightarrow \frac{1}{1 + y^2}\,\mathrm{d}y = \mathrm{e}^{2x}\,\mathrm{d}x \\
&\Rightarrow \int \frac{1}{1 + y^2}\,\mathrm{d}y = \int \mathrm{e}^{2x}\,\mathrm{d}x \\
&\Rightarrow \arctan y = \tfrac{1}{2}\mathrm{e}^{2x} + c.
\end{aligned}
$$

Now,

$$
\begin{aligned}
x = 0, y = 1 &\Rightarrow \tfrac{1}{4}\pi = \tfrac{1}{2} + c \\
&\Rightarrow c = \tfrac{1}{4}\pi - \tfrac{1}{2}
\end{aligned}
$$

and

$$\arctan y = \tfrac{1}{2}\mathrm{e}^{2x} + \tfrac{1}{4}\pi - \tfrac{1}{2} \Rightarrow \underline{\underline{y = \tan(\tfrac{1}{2}\mathrm{e}^{2x} + \tfrac{1}{4}\pi - \tfrac{1}{2})}}.$$

10. S_n is defined by

$$\sum_{r=1}^{n} \left(r^2 + \tfrac{1}{3}r\right).$$

(a) Find an expression for S_n, fully factorising your answer. (2)

Solution

$$\begin{aligned} S_n &= \sum_{r=1}^{n} \left(r^2 + \tfrac{1}{3}r\right) \\ &= \sum_{r=1}^{n} r^2 + \tfrac{1}{3}\sum_{r=1}^{n} r \\ &= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{3}\cdot\tfrac{1}{2}n(n+1) \\ &= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{6}n(n+1) \\ &= \tfrac{1}{6}n(n+1)[(2n+1)+1] \\ &= \tfrac{1}{6}n(n+1)(2n+2) \\ &= \underline{\underline{\tfrac{1}{3}n(n+1)^2}}. \end{aligned}$$

(b) Hence find an expression for (2)

$$\sum_{r=10}^{2p} \left(r^2 + \tfrac{1}{3}r\right),$$

where $p > 5$.

Solution

$$\begin{aligned} \sum_{r=10}^{2p} \left(r^2 + \tfrac{1}{3}r\right) &= S_{2p} - S_9 \\ &= \tfrac{1}{3}(2p)[(2p)+1]^2 - \tfrac{1}{3}(9)(9+1)^2 \\ &= \underline{\underline{\tfrac{2}{3}p(2p+1)^2 - 300}}. \end{aligned}$$

11. Given (5)

$$y = x^{2x^3+1},$$

use logarithmic differentiation to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Write your answer in terms of x.

Solution

$$\begin{aligned} y = x^{2x^3+1} &\Rightarrow \ln y = \ln x^{2x^3+1} \\ &\Rightarrow \ln y = (2x^3+1)\ln x \end{aligned}$$

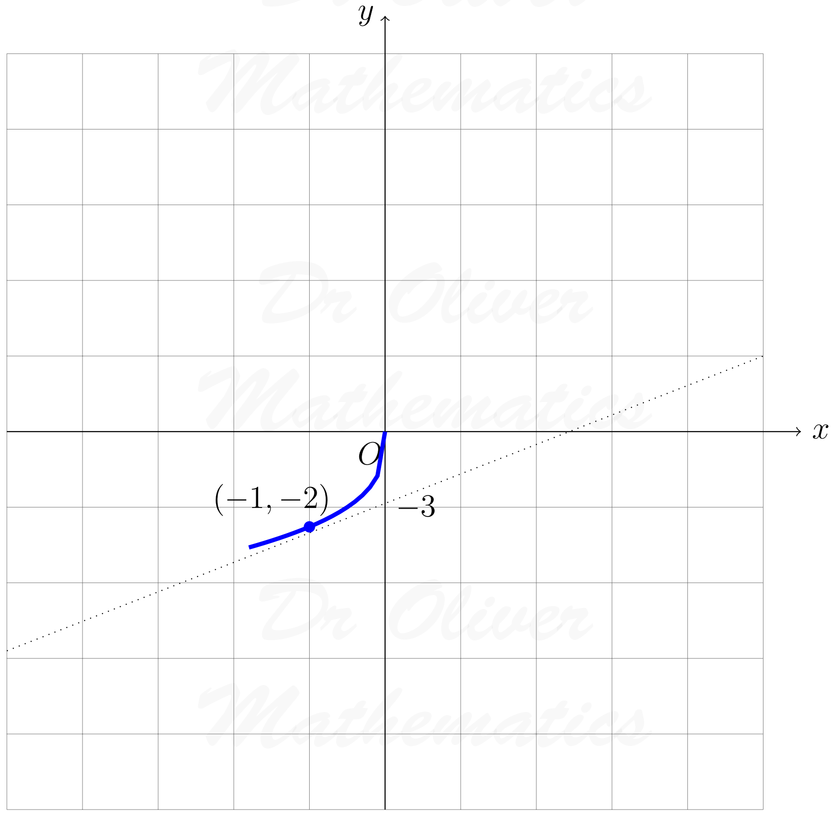
$$\begin{aligned} u = 2x^3 + 1 &\Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 6x^2 \\ u = \ln x &\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{x} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x^3+1}{x} + 6x^2 \ln x \\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y\left(\frac{2x^3+1}{x} + 6x^2\ln x\right) \\ &\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = x^{2x^3+1}\left(\frac{2x^3+1}{x} + 6x^2\ln x\right)}}. \end{aligned}$$

12. In the diagram below part of the graph of $y = \mathrm{f}(x)$ has been omitted. The point $(-1, -2)$ lies on the graph and the line

$$y = \tfrac{1}{2}x - 3$$

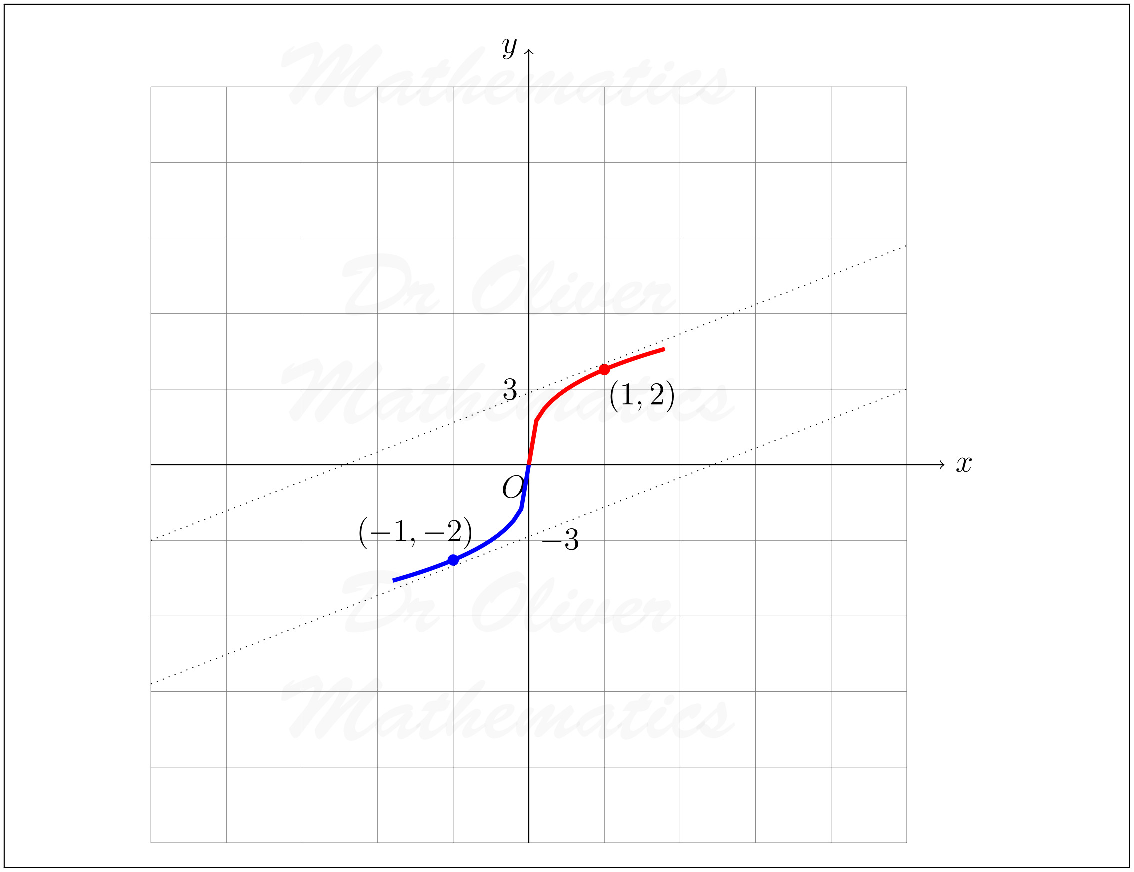
is an asymptote.



Given that $f(x)$ is an odd function:

(a) Copy and complete the diagram, including any asymptotes and any points you know to be on the graph. (2)

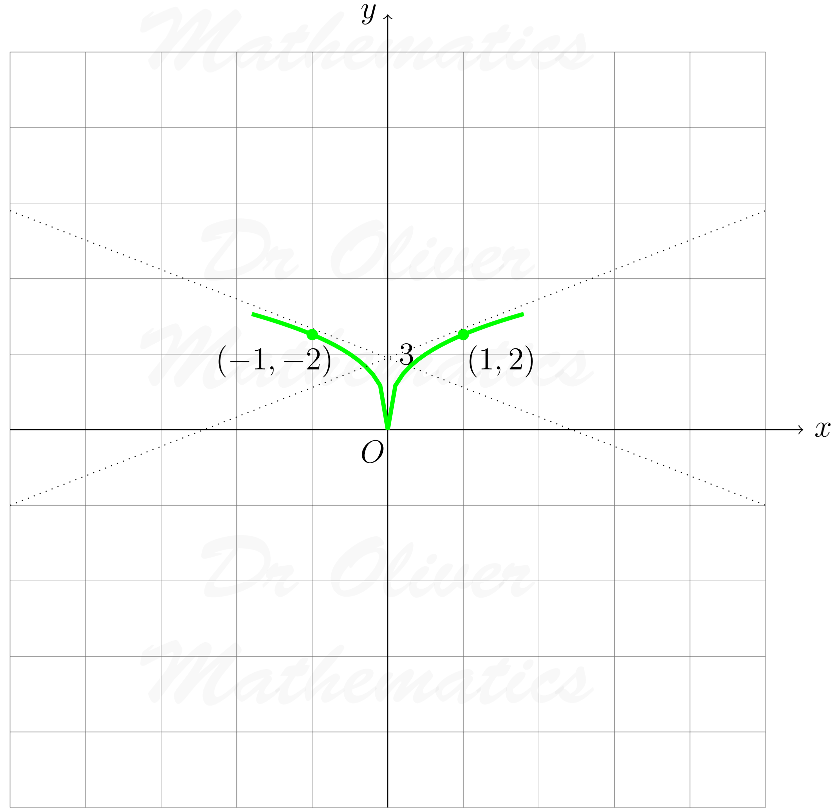
Solution



$g(x) = |\,f(x)|$.

(b) On a separate diagram, sketch $g(x)$. (2)
Include known asymptotes and points.

Solution



(c) State the range of values of $f'(x)$ given that $f'(0) = 2$. (1)

Solution
Because $f'(0) = 2$, the line has gradient of 2 and that is the steepest part of the line (why?). Now, $f'(x) > \frac{1}{2}$ because of the asymptotes. So, $\underline{\underline{\frac{1}{2} < f'(x) \leqslant 2}}$.

13. Let n be an integer. (4)
Using proof by contrapositive, show that if n^2 is even, then n is even.

Solution
The contrapositive is: "if n is odd, then n^2 is odd." So, suppose n is odd. That

mean $n = 2m + 1$ for some integer m. Now,

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2 \times \text{some integer} + 1 \end{aligned}$$

and, clearly, n^2 is odd. The contrapositive is true which means that, if n^2 is even, then n is even.

14. Find the particular solution of the differential equation (10)

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 8\sin x + 19\cos x,$$

given that $y = 7$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$ when $x = 0$.

Solution

Complementary function:

$$m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3 \text{ (repeated)}$$

and hence the complementary function is

$$y = (Ax + B)\mathrm{e}^{3x}.$$

Particular integral: try

$$\begin{aligned} y = C\cos x + D\sin x &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -C\sin x + D\cos x \\ &\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -C\cos x - D\sin x. \end{aligned}$$

Now,

$$\begin{aligned} & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 8\sin x + 19\cos x \\ \Rightarrow \quad & (-C\cos x - D\sin x) - 6(-C\sin x + D\cos x) + 9(C\cos x + D\sin x) \\ & \qquad = 8\sin x + 19\cos x \end{aligned}$$

and

$$-D + 6C + 9D = 8 \Rightarrow 6C + 8D = 8 \quad (1)$$
$$-C - 6D + 9C = 19 \Rightarrow 8C - 6D = 19 \quad (2).$$

Next,

$$4 \times (1) : 24C + 32D = 32 \quad (3)$$
$$3 \times (2) : 24C - 18D = 57 \quad (4)$$

and (3) − (4):

$$\begin{aligned} 50D = -25 &\Rightarrow D = -\tfrac{1}{2} \\ &\Rightarrow 6C - 4 = 8 \\ &\Rightarrow 6C = 12 \\ &\Rightarrow C = 2. \end{aligned}$$

The particular integral is $y = 2\cos x - \frac{1}{2}\sin x$.

The general solution is

$$y = (Ax + B)e^{3x} + 2\cos x - \tfrac{1}{2}\sin x.$$

Now,

$$\begin{aligned} x = 0, y = 7 &\Rightarrow 7 = B + 2 \\ &\Rightarrow B = 5. \end{aligned}$$

Next,

$$\begin{aligned} & y = (Ax + B)e^{3x} + 2\cos x - \tfrac{1}{2}\sin x \\ \Rightarrow \quad & y = Axe^{3x} + Be^{3x} + 2\cos x - \tfrac{1}{2}\sin x \\ \Rightarrow \quad & \frac{dy}{dx} = Ae^{3x} + 3Axe^{3x} + 3Be^{3x} - 2\sin x - \tfrac{1}{2}\cos x. \end{aligned}$$

and

$$\begin{aligned} x = 0, \frac{dy}{dx} = \tfrac{1}{2} &\Rightarrow \tfrac{1}{2} = A + 15 - \tfrac{1}{2} \\ &\Rightarrow A = -14. \end{aligned}$$

Hence, the general solution is

$$\underline{\underline{y = (-14x + 5)e^{3x} + 2\cos x - \tfrac{1}{2}\sin x}}.$$

15. A beam of light passes through the points $B(7, 8, 1)$ and $T(-3, -22, 6)$.

(a) Obtain parametric equations of the line representing the beam of light. (2)

Solution

$$\overrightarrow{BT} = -10\mathbf{i} - 30\mathbf{j} + 5\mathbf{k} = -5(2\mathbf{i} + 6\mathbf{j} - \mathbf{k})$$

and the parametric equations of the line are

$$\underline{\underline{x = 2\lambda + 7,\ y = 6\lambda + 8,\ z = -\lambda + 1}}.$$

A sheet of metal is represented by a plane containing the points $P(2, 1, 9)$, $Q(1, 2, 7)$, and $R(-3, 7, 1)$.

(b) Find the Cartesian equation of the plane. (4)

Solution

$$\begin{aligned}\overrightarrow{PQ} &= -\mathbf{i} + \mathbf{j} - 2\mathbf{k}\\ \overrightarrow{PR} &= -5\mathbf{i} + 6\mathbf{j} - 8\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ -5 & 6 & -8 \end{vmatrix}\\ &= (-8 + 12)\mathbf{i} - (8 - 10)\mathbf{j} + (-6 + 5)\mathbf{k}\\ &= 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}.\end{aligned}$$

Finally, the Cartesian equation of the plane is

$$4x + 2y - z = 8 + 2 - 9 \Rightarrow \underline{\underline{4x + 2y - z = 1}}.$$

The beam of light passes through a hole in the metal at point H.

(c) Find the coordinates of H. (3)

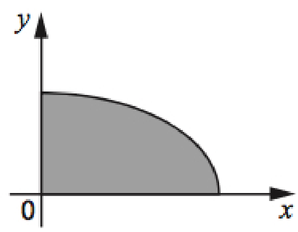
Solution

$$
\begin{aligned}
4(2\lambda+7)+2(6\lambda+8)-(-\lambda+1)=1 &\Rightarrow 21\lambda=-42\\
&\Rightarrow \lambda=-2\\
&\Rightarrow \underline{\underline{H(3,-4,3)}}.
\end{aligned}
$$

16. On a suitable domain, a curve is defined by the equation (5)

$$4x^2+9y^2=36.$$

A section of the curve in the first quadrant, illustrated in the diagram below, is rotated 360° about the y-axis.



Calculate the exact value of the volume generated.

Solution

$$
\begin{aligned}
x=0 &\Rightarrow 9y^2=36\\
&\Rightarrow y^2=4\\
&\Rightarrow y=\pm 2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
4x^2+9y^2=36 &\Rightarrow 4x^2=36-9y^2\\
&\Rightarrow x^2=9-\tfrac{9}{4}y^2
\end{aligned}
$$

and the exact value of the volume generated is

$$
\begin{aligned}
\int_0^2 \pi(9-\tfrac{9}{4}y^2)\,\mathrm{d}y &= \pi\left[9y-\tfrac{3}{4}y^3\right]_{y=0}^{2}\\
&= \pi\left[(18-6)-(0-0)\right]\\
&= \underline{\underline{12\pi}}.
\end{aligned}
$$

17. The complex number $z = 2 + \mathrm{i}$ is a root of the polynomial equation

$$z^4 - 6z^3 + 16z^2 - 22z + q = 0,$$

where $q \in \mathbb{R}$.

(a) State a second root of the equation. (1)

Solution

$\underline{\underline{z = 2 - \mathrm{i}}}$.

(b) Find the value of q and the remaining roots. (6)

Solution

$\times$	z	-2	$+\mathrm{i}$
z	z^2	$-2z$	$+z\mathrm{i}$
-2	$-2z$	$+4$	$-2\mathrm{i}$
$-\mathrm{i}$	$-z\mathrm{i}$	$+2\mathrm{i}$	$+1$

$$[z - (2 - \mathrm{i})][z - (2 + \mathrm{i})] = z^2 - 4z + 5.$$

Let the remaining quadratic be

$$x^2 + ax + b$$

where $a,\, b \in \mathbb{R}$.

$\times$	z^2	$-4z$	$+5$
z^2	z^4	$-4z^3$	$+5z^2$
$+az$	$+az^3$	$-4az^2$	$+5az$
$+b$	$+bz^2$	$-4bz$	$+5b$

Now,

$$z^4 - 6z^3 + 16z^2 - 22z + q = z^4 + (-4 + a)z^3 + (5 - 4a + b)z^2 + (5a - 4b)z + 5b.$$

$$\begin{aligned} \text{Coefficient of } z^3: &\quad -4 + a = -6 \Rightarrow a = -2. \\ \text{Coefficient of } z^2: &\quad 5 + 8 + b = 16 \Rightarrow b = 3. \\ \text{Check in } z: &\quad 5a - 4b = 22 \quad \checkmark \end{aligned}$$

Hence, the value of q is

$$5 \times 3 = \underline{\underline{15}}$$

and the remaining roots are

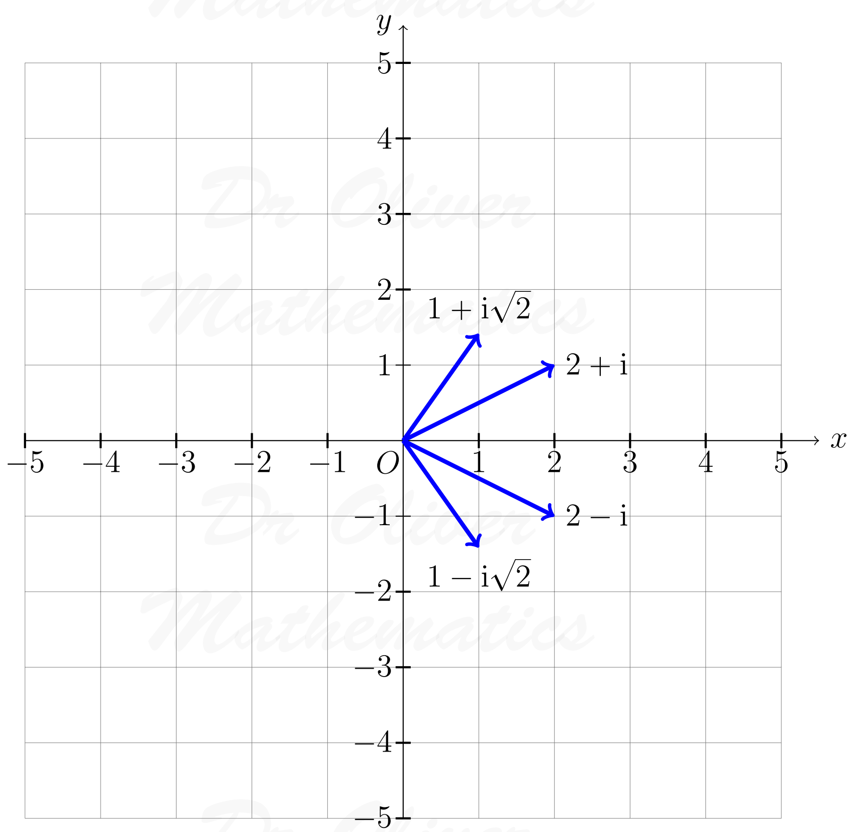
$$\begin{aligned} z^2 - 2z + 3 = 0 &\Rightarrow z^2 - 2z + 1 = -2 \\ &\Rightarrow (z-1)^2 = -2 \\ &\Rightarrow z - 1 = \pm \mathrm{i}\sqrt{2} \\ &\Rightarrow \underline{\underline{z = 1 \pm \mathrm{i}\sqrt{2}}}. \end{aligned}$$

(c) Show the solutions to (1)

$$z^4 - 6z^3 + 16z^2 - 22z + q = 0$$

on an Argand diagram.

Solution



18. The position of a particle at time t is given by the parametric equations

$$x = t\cos t,\ y = t\sin t,\ t \geqslant 0.$$

(a) Find an expression for the instantaneous speed of the particle. (5)

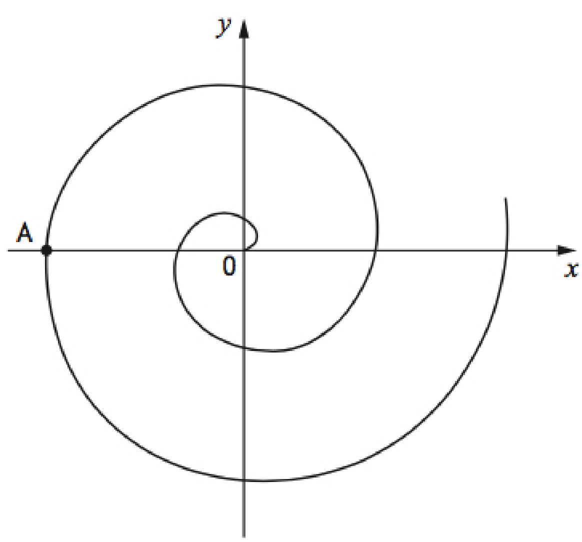
Solution

$$x = t\cos t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \cos t - t\sin t$$
$$y = t\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t + t\cos t.$$

Now,

$$\begin{aligned}
\text{speed} &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \\
&= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} \\
&= \sqrt{\left(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t\right) + \left(\sin^2 t - 2t\sin t\cos t + t^2\cos^2 t\right)} \\
&= \sqrt{\sin^2 + \cos^2 t + t^2(\sin^2 t + \cos^2 t)} \\
&= \underline{\underline{\sqrt{1 + t^2}}}.
\end{aligned}$$

The diagram below shows the path that the particle takes.



(b) Calculate the instantaneous speed of the particle at point A. (2)

Solution

$t = 3\pi$ (it has gone $1\frac{1}{2}$ times around) and

$$\text{speed} = \underline{\underline{\sqrt{1 + (3t)^2}}}.$$