Dr Oliver Mathematics
AQA Further Maths Level 2
June 2023 Paper 2
1 hour 45 minutes

The total number of marks available is 80.
You must write down all the stages in your working.
You are permitted to use a scientific or graphical calculator in this paper.

1. Solve (3)

$$\frac{8d-3}{3d-7} = \frac{5}{2}.$$

Solution

Cross-multiply:

$$\begin{aligned}\frac{8d-3}{3d-7} = \frac{5}{2} &\Rightarrow 2(8d-3) = 5(3d-7)\\ &\Rightarrow 16d - 6 = 15d - 35\\ &\Rightarrow \underline{\underline{d- = -29}}.\end{aligned}$$

2. (a) The first four terms of a linear sequence are (2)

$$15 \quad 18.5 \quad 22 \quad 25.5.$$

Work out an expression for the nth term.

Solution

Let the

$$n\text{th term} = an + b.$$

$$\begin{array}{ccccccc} \color{red}{15} & & 18.5 & & 22 & & 25.5 \\ & \color{blue}{3.5} & & 3.5 & & 3.5 & \end{array}$$

$$\begin{array}{ccccccc} \color{red}{a+b} & & 2a+b & & 3a+b & & 4a+b \\ & \color{blue}{a} & & a & & a & \end{array}$$

We compare terms:

$$a = 3.5$$

and

$$\begin{aligned} a + b = 15 &\Rightarrow 3.5 + b = 15 \\ &\Rightarrow b = 11.5. \end{aligned}$$

Hence,

$$n\text{th term} = \underline{\underline{3.5n + 11.5}}.$$

(b) A different linear sequence has nth term (2)

$$318 - 9n.$$

Work out the value of the first **negative** term in the sequence.

Solution

$$\begin{aligned} 318 - 9n < 0 &\Rightarrow 318 < 9n \\ &\Rightarrow n > 35\tfrac{1}{3}; \end{aligned}$$

so, $\underline{\underline{\text{36th term}}}$.

3. (2)

$$\begin{pmatrix} 3 & 5 \\ u & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} t \\ 6 \end{pmatrix}.$$

Work out the values of t and u.

Solution

Well,

$$\begin{pmatrix} 3 & 5 \\ u & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} t \\ 6 \end{pmatrix} \Rightarrow \begin{pmatrix} 23 \\ u + 8 \end{pmatrix} = \begin{pmatrix} t \\ 6 \end{pmatrix}.$$

So, $\underline{\underline{t = 23}}$ and

$$u + 8 = 6 \Rightarrow \underline{\underline{u = -2}}.$$

4. A line passes through $P(1, k)$ and $Q(r, 6)$, where k and r are constants. (4)

- The midpoint of PQ has x-coordinate 5.
- The gradient of the line is 2.

Work out the value of k.

Solution

Well, the midpoint of PQ has x-coordinate 5 so

$$\begin{aligned}\frac{1+r}{2} = 5 &\Rightarrow 1 + r = 10 \\ &\Rightarrow r = 9.\end{aligned}$$

Now,

$$\begin{aligned}\frac{6-k}{9-1} = 2 &\Rightarrow \frac{6-k}{8} = 2 \\ &\Rightarrow 6 - k = 16 \\ &\Rightarrow \underline{\underline{k = -10}}.\end{aligned}$$

5. (3)

$$y = 0.5x^4.$$

Work out the value of x for which the rate of change of y with respect to x is 6.75.

Solution

$$y = 0.5x^4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2x^3$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} = 6.75 &\Rightarrow 2x^3 = 6.75 \\ &\Rightarrow x^3 = 3.375 \\ &\Rightarrow \underline{\underline{x = 1.5}}.\end{aligned}$$

6. The equation of a circle is (2)

$$(x+7)^2 + (y-4)^2 = 36.$$

Complete these statements.

The coordinates of the centre of the circle are (. . . . . . , ...).

The radius of the circle is ....

Solution

The coordinates of the centre of the circle are $\underline{\underline{(-7, 4)}}$.

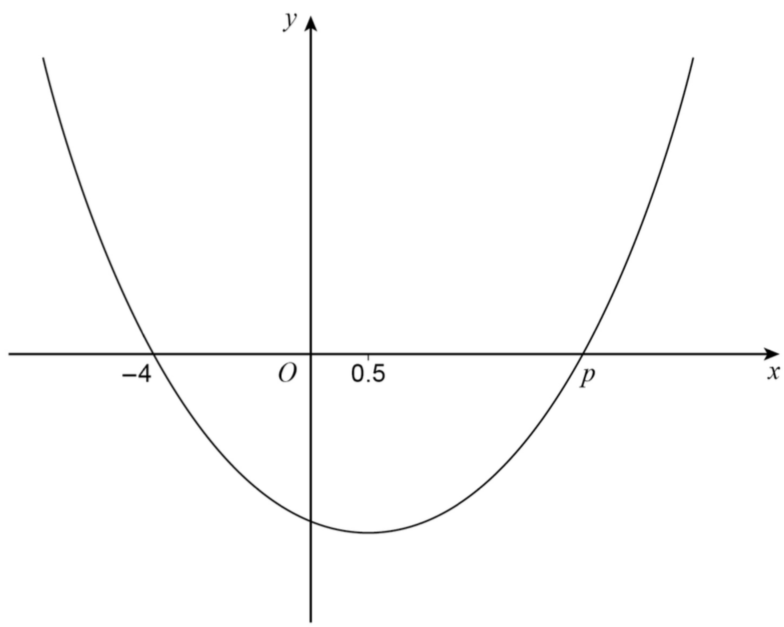
The radius of the circle is $\sqrt{36} = \underline{\underline{6}}$.

7. Here is a sketch of the curve

$$y = ax^2 + bx + c,$$

where a, b, and c are constants.

- The curve intersects the x-axis at $(-4, 0)$ and $(p, 0)$.
- The turning point has x-coordinate 0.5.



(a) Work out the value of p. (1)

Solution

So,

$$p = 0.5 + [0.5 - (-4)] = \underline{\underline{5}}.$$

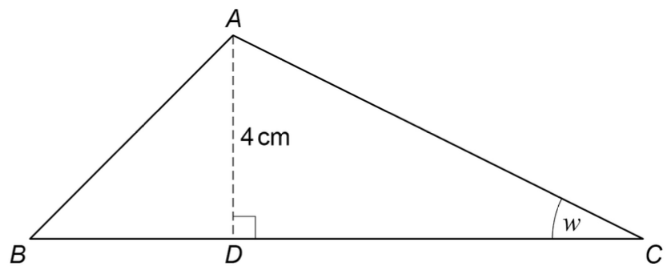
(b) Solve (2)

$$ax^2 + bx + c > 0.$$

Solution

$$\underline{\underline{x < -4 \text{ or } x > 5}}.$$

8. ABC is a triangle with perpendicular height AD. (4)



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- Area of $ABC = 25$ cm^2.
- $BD : DC = 2 : 3$.

Work out the size of angle w.

Solution

Well,

$$\begin{aligned} \text{area} = 25 &\Rightarrow \tfrac{1}{2} \times BC \times AD = 25 \\ &\Rightarrow \tfrac{1}{2} \times BC \times 4 = 25 \\ &\Rightarrow BC = 12.5. \end{aligned}$$

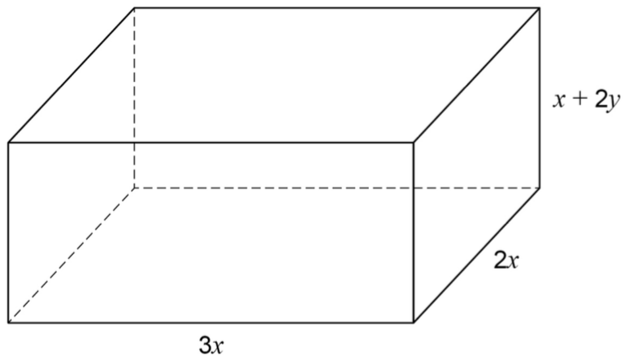
Now,

$$BD : DC = 2 : 3 = 5 : 7.5$$

and

$$\begin{aligned} \tan = \frac{\text{opp}}{\text{adj}} &\Rightarrow \tan w = \frac{4}{7.5} \\ &\Rightarrow w = 28.072\,486\,94 \text{ (FCD)} \\ &\Rightarrow \underline{\underline{w = 28.1^\circ \text{ (3 sf)}}}. \end{aligned}$$

9. The dimensions of the cuboid are given in centimetres.



The total length of all 12 edges is 300 cm.

(a) Show that (2)

$$y = \frac{75 - 6x}{2}.$$

Solution

The total length of all 12 edges is 300 cm so

$$\begin{aligned} 4[3x + 2x + (x + 2y)] = 300 &\Rightarrow 6x + 2y = 75 \\ &\Rightarrow 2y = 75 - 6x \\ &\Rightarrow \underline{\underline{y = \frac{75 - 6x}{2}}}, \end{aligned}$$

as required.

The volume of the cuboid is V cm^3.

(b) Show that (2)

$$V = 450x^2 - 30x^3.$$

Solution

The vertical height is

$$x + 2y = x + (75 - 6x) = 75 - 5x$$

and

$$\begin{aligned} V &= 3x \times 2x \times (75 - 5x) \\ &= 6x^2(75 - 5x) \\ &= \underline{\underline{450x^2 - 30x^3}}, \end{aligned}$$

as required.

(c) Use calculus to work out the maximum value of V as x varies. (3)

Solution

Now,

$$V = 450x^2 - 30x^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}x} = 900x - 90x^2$$

and

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} = 0 &\Rightarrow 900x - 90x^2 = 0 \\ &\Rightarrow 90x(10 - x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 10. \end{aligned}$$

But $x \neq 0$ (think about it!) so $x = 10$. So

$$x = 10 \Rightarrow V_{\max} = \underline{\underline{15\,000}}.$$

10. Line K has equation (3)

$$4x - 5y = 17.$$

Line L passes through the points $(3, 6)$ and $(-5, 16)$.

Tick (✓) the correct statement about lines K and L.

The lines are parallel.	☐
The lines are perpendicular.	☐
The lines are neither parallel nor perpendicular.	☐

Show working to support your answer.

Solution

Line K:

$$4x - 5y = 17 \Rightarrow -5y = -4x + 17$$
$$\Rightarrow y = \tfrac{4}{5}x - \tfrac{17}{5}.$$

Line L:

$$\begin{aligned} m &= \frac{16 - 6}{-5 - 3} \\ &= \frac{10}{-8} \\ &= -\tfrac{5}{4}. \end{aligned}$$

Well, the product of their gradients is -1. As they are not horizontal or vertical lines, the lines are perpendicular.

11. Expand and simplify fully (4)

$$(2x^3 - 9)(3x^2 + 4) + x(x - 4)^2.$$

Solution

Well,

$\times$	$2x^3$	-9
$3x^2$	$6x^5$	$-27x^2$
$+4$	$+8x^3$	-36

and

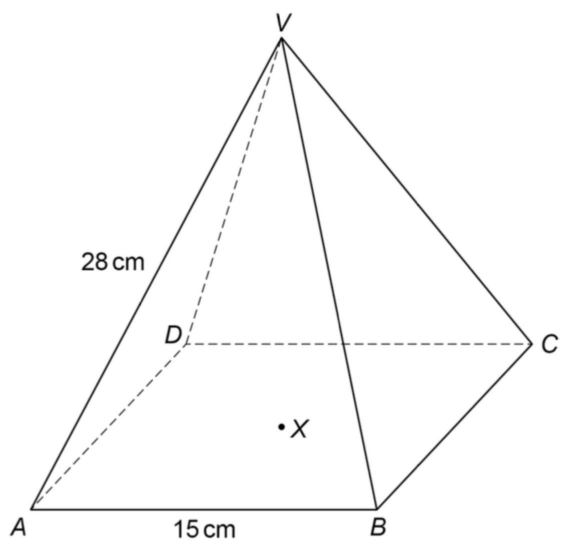
$\times$	x	-4
x	x^2	$-4x$
-4	$-4x$	$+16$

Now,

$$\begin{aligned} & (2x^3-9)(3x^2+4)+x(x-4)^2 \\ = \;& (6x^5+8x^3-27x^2-36)+x(x^2-8x+16) \\ = \;& 6x^5+8x^3-27x^2-36+x^3-8x^2+16x \\ = \;& \underline{\underline{6x^5+9x^3-35x^2+16x-36}}. \end{aligned}$$

12. $VABCD$ is a pyramid. (3)

- The square horizontal base, $ABCD$, has side length 15 cm.
- V is directly above the centre, X, of the base.
- $VA = 28$ cm.



Work out the size of the angle that VA makes with $ABCD$.

Solution

Well, let $x = \angle VAX$. Then

$$\begin{aligned} AC^2 = AB^2 + BC^2 &\Rightarrow AC^2 = 15^2 + 15^2 \\ &\Rightarrow AC^2 = 450 \\ &\Rightarrow AC = 15\sqrt{2} \\ &\Rightarrow AX = 7.5\sqrt{2} \end{aligned}$$

and

$$\cos = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos x = \frac{7.5\sqrt{2}}{28}$$
$$\Rightarrow x = 67.740\,182\,3 \text{ (FCD)}$$
$$\Rightarrow \underline{\underline{x = 67.7^\circ \text{ (3 sf)}}}.$$

13. (a) Circle the expression equivalent to $3x^{-7}$. (1)

$$-\frac{3}{x^7} \quad -\frac{1}{3x^7} \quad \frac{1}{3x^7} \quad \frac{3}{x^7}$$

Solution

$$-\frac{3}{x^7} \quad -\frac{1}{3x^7} \quad \frac{1}{3x^7} \quad \underline{\underline{\frac{3}{x^7}}}$$

(b) Simplify fully (2)

$$\frac{12w^8}{(4w^3)^2}.$$

Solution

$$\frac{12w^8}{(4w^3)^2} = \frac{12w^8}{16w^6}$$
$$= \underline{\underline{\tfrac{3}{4}w^2}}.$$

(c) (3)

$$\sqrt{y} \times \sqrt[3]{y} = \sqrt[c]{y^d},$$

where c and d are positive integers.

Work out the least possible values of c and d.

Solution

$$\begin{aligned}\sqrt{y} \times \sqrt[3]{y} &= y^{\frac{1}{2}} \times y^{\frac{1}{3}} \\ &= y^{\frac{3}{6}} \times y^{\frac{2}{6}} \\ &= y^{\frac{5}{6}} \\ &= \underline{\underline{\sqrt[6]{y^5}}};\end{aligned}$$

so, $\underline{\underline{c = 6}}$ and $\underline{\underline{d = 5}}$ is one possible pair.

14. Simplify fully (4)

$$\frac{15a^2}{a^2 + 6a - 16} \times \frac{8 - 4a}{3a}.$$

Solution

Well,

$$\left.\begin{array}{ll}\text{add to:} & +6 \\ \text{multiply to:} & -16\end{array}\right\} + 8,\ -2$$

so

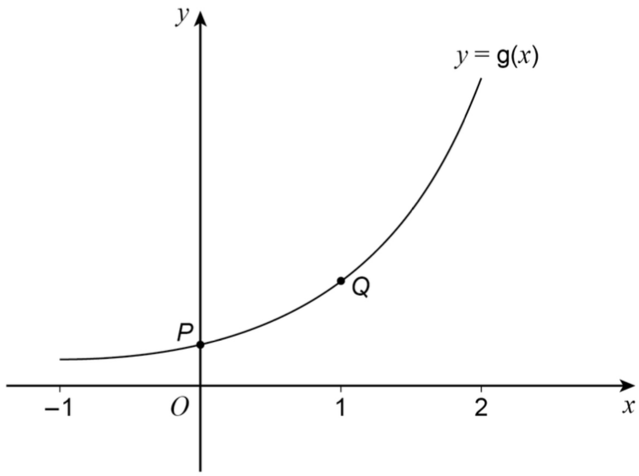
$$\begin{aligned}\frac{15a^2}{a^2 + 6a - 16} \times \frac{8 - 4a}{3a} &= \frac{5a}{(a+8)(a-2)} \times \frac{-4(a-2)}{1} \\ &= \frac{5a}{a+8} \times -4 \\ &= \underline{\underline{-\frac{20a}{a+8}}}.\end{aligned}$$

15. The function g is given by (4)

$$\mathrm{g}(x) = a \times b^x,$$

where a and b are constants.

- The domain of the function is $-1 \leqslant x \leqslant 2$,
- $P(0, \frac{1}{2})$ and $Q(1, \frac{3}{2})$ are points on the graph $y = \mathrm{g}(x)$.



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Work out the range of the function.

Solution

Well,

$$\begin{aligned} x = 0, y = \tfrac{1}{2} &\Rightarrow a = \tfrac{1}{2} \\ x = 1, y = \tfrac{3}{2} &\Rightarrow \tfrac{1}{2}b = \tfrac{3}{2} \\ &\Rightarrow b = 3, \end{aligned}$$

so the function is

$$\mathrm{g}(x) = \tfrac{1}{2} \times 3^x.$$

Now,

$$\begin{aligned} x = -1 &\Rightarrow y = \tfrac{1}{2} \times 3^{-1} = \tfrac{1}{6} \\ x = 2 &\Rightarrow y = \tfrac{1}{2} \times 3^2 = 4\tfrac{1}{2}; \end{aligned}$$

so the range of the function is

$$\underline{\underline{\tfrac{1}{6} \leqslant \mathrm{g}(x) \leqslant 4\tfrac{1}{2}}}.$$

16. $(2x - 3)$ is a factor of (4)

$$6x^3 - 25x^2 + 28x - 6.$$

Solve

$$6x^3 - 25x^2 + 28x - 6 = 0.$$

Give all solutions as **exact** values.

Solution

We use synthetic division:

$$\begin{array}{c|cccc} \frac{3}{2} & 6 & -25 & 28 & -6 \\ & \downarrow & 9 & -24 & 6 \\ \hline & 6 & -16 & 4 & 0 \end{array}$$

so

$$\begin{aligned} 6x^3 - 25x^2 + 28x - 6 &= (x - \tfrac{3}{2})(6x^2 - 16x + 4) \\ &= (2x-3)(3x^2 - 8x + 2). \end{aligned}$$

Now, the quadratic does not factorise nicely so $a = 3$, $b = -8$, and $c = 2$:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{8 \pm \sqrt{8^2 - 4 \times 3 \times 2}}{2 \times 3} \\ &= \frac{8 \pm \sqrt{40}}{6} \\ &= \frac{4 \pm \sqrt{10}}{3}. \end{aligned}$$

Hence, the solutions are

$$\underline{\underline{x = \tfrac{3}{2} \text{ or } \frac{4 \pm \sqrt{10}}{3}}}.$$

17. The function h is given by (4)

$$\mathrm{h}(x) = ax(3x^2 - 2) + 5x,$$

where a is a positive constant.

h is an **increasing** function for all values of x.

Work out the possible values of a.
Give your answer as an inequality.

Solution

Well,

$$\begin{aligned}
\mathrm{h}(x) = ax(3x^2-2)+5x &\Rightarrow \mathrm{h}(x) = 3ax^3 - 2ax + 5x \\
&\Rightarrow \mathrm{h}(x) = 3ax^3 + (5-2a)x \\
&\Rightarrow \frac{\mathrm{d}h}{\mathrm{d}x}(x) = 9ax^2 + (5-2a).
\end{aligned}$$

Now, since $9ax^2 \geqslant 0$,

$$\begin{aligned}
\frac{\mathrm{d}h}{\mathrm{d}x} > 0 &\Rightarrow 9ax^2 + (5-2a) > 0 \\
&\Rightarrow 5 - 2a > 0 \\
&\Rightarrow 2a < 5 \\
&\Rightarrow a < \tfrac{5}{2}.
\end{aligned}$$

Hence, the possible values of a is

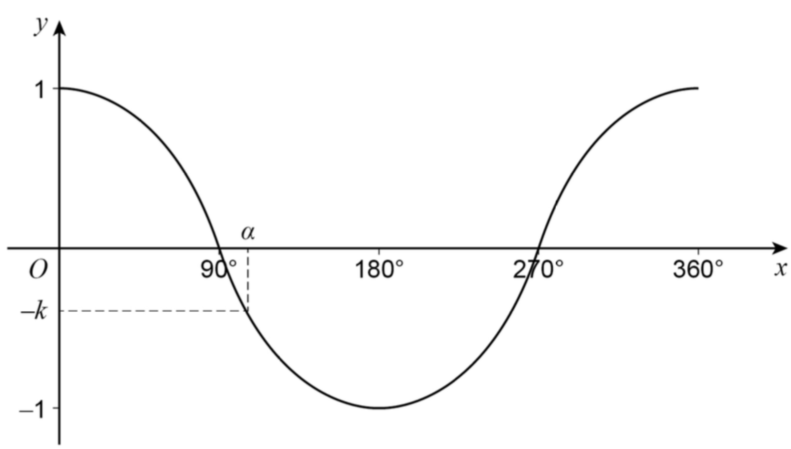
$$\underline{\underline{0 < a < \tfrac{5}{2}}}.$$

18. Here is a sketch of

$$y = \cos x$$

for values of x from 0° to 360°.

α is an obtuse angle measured in degrees.
$\cos\alpha = -k$, where k is a positive constant.



(a) Tick (✓) **two** boxes that show expressions for x where $\cos x = -k$. (2)

☐	$180° - \alpha$	☐	$180° + \alpha$
☐	$270° - \alpha$	☐	$270° + \alpha$
☐	$360° - \alpha$	☐	$360° + \alpha$

Solution
Tick $\underline{\underline{360^\circ - \alpha}}$ and $\underline{\underline{360^\circ + \alpha}}$.

(b) Circle the expression for x where $\sin x = -k$. (1)

$$\alpha \quad 90^\circ + \alpha \quad 180^\circ - \alpha \quad 180^\circ + \alpha$$

Solution

$$\alpha \quad \underline{\underline{90^\circ + \alpha}} \quad 180^\circ - \alpha \quad 180^\circ + \alpha$$

19. In these simultaneous equations, k is a positive constant. (3)

$$\begin{aligned} 3x + 4y &= k \\ y &= 2kx. \end{aligned}$$

Solve the simultaneous equations.
Give the answers in their simplest form in terms of k.

Solution

$$
\begin{aligned}
3x + 4y = k &\Rightarrow 3x + 4(2kx) = k \\
&\Rightarrow 3x + 8kx = k \\
&\Rightarrow x(3 + 8k) = k \\
&\Rightarrow x = \frac{k}{3 + 8k} \\
&\Rightarrow y = \frac{2k^2}{3 + 8k};
\end{aligned}
$$

hence,

$$\underline{\underline{x = \frac{k}{3 + 8k},\ y = \frac{2k^2}{3 + 8k}}}.$$

20. Show that (3)

$$2\sin^3 x + 2\sin x \cos^2 x + 5\tan x \cos x$$

simplifies to

$$p \sin x,$$

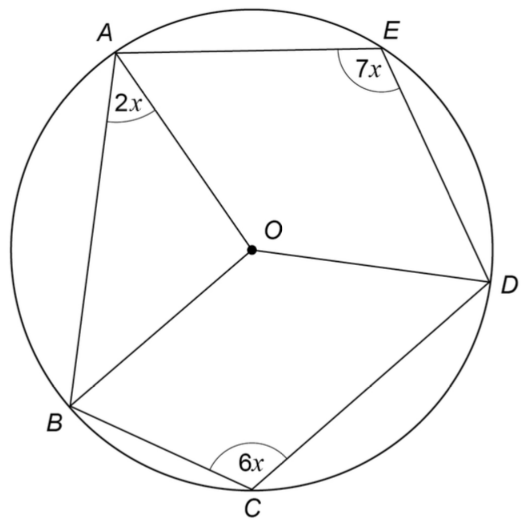
where p is a constant.

Solution

$$
\begin{aligned}
& 2\sin^3 x + 2\sin x \cos^2 x + 5\tan x \cos x \\
\equiv\ & 2\sin x \sin^2 x + 2\sin x \cos^2 x + 5\left(\frac{\sin x}{\cos x}\right)\cos x \\
\equiv\ & 2\sin x(\sin^2 x + \cos^2 x) + 5\sin x \\
\equiv\ & 2\sin x + 5\sin x \\
\equiv\ & \underline{\underline{7\sin x}};
\end{aligned}
$$

hence, $\underline{\underline{p = 7}}$.

21. A, B, C, D, and E are points on a circle, centre O. (4)



Work out the value of x.

Solution

Major $\angle AOD = 2 \times 7x = 14x$.
Major $\angle BOD = 2 \times 6x = 12x$.
$\angle ABO = 2x$ (base angles)
$\angle AOB = 180 - 2 \times 2x = 180 - 4x$ (completing the triangle).
Major $\angle AOB = 360 - (180 - 4x) = 180 + 4x$.

Now, because we have twice the angle that we need,

$$\begin{aligned} 14x + 12x + (180 + 4x) = 2 \times 360 &\Rightarrow 30x + 180 = 720 \\ &\Rightarrow 30x = 540 \\ &\Rightarrow \underline{\underline{x = 18}}. \end{aligned}$$

22. Five-digit integers are made using (3)

1 2 7 8 9.

For each integer, all the digits are used exactly once.

The integers are greater than 40 000 **and** odd.

How many different integers can be made?
You **must** show your working.

Solution

3 choices for the last number: 7, 8, and 9.

7: There is one choice for the first number (7, obviously!), two choices for the final digit (1 and 3), and $3! = 6$ ways for the middle digits to be arranged. So,

$$1 \times 6 \times 2 = 12 \text{ ways.}$$

8: There is one choice for the first number (8), three choices for the final digit (1 and 3), and $3! = 6$ ways for the middle digits to be arranged. So,

$$1 \times 6 \times 3 = 18 \text{ ways.}$$

9: There is one choice for the first number (9), two choices for the final digit (1 and 3), and $3! = 6$ ways for the middle digits to be arranged. So,

$$1 \times 6 \times 2 = 12 \text{ ways.}$$

Hence,

$$12 + 18 + 12 = \underline{\underline{42 \text{ ways}}}.$$