Dr Oliver Mathematics
Applied Mathematics: Mechanics or Statistics
Section B
2013 Paper
1 hour

The total number of marks available is 32.
You must write down all the stages in your working.

1. Given that (2)

$$y = \sin(\mathrm{e}^{5x}),$$

find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$y = \sin(\mathrm{e}^{5x}) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \cos(\mathrm{e}^{5x}) \cdot 5\mathrm{e}^{5x}$$
$$\Rightarrow \underline{\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = 5\mathrm{e}^{5x}\cos(\mathrm{e}^{5x})}}.$$

2. Matrices are given as

$$\mathbf{A} = \begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} y & 3 \\ -1 & 2 \end{pmatrix}.$$

(a) Write (2)

$$\mathbf{A}^2 - 3\mathbf{B}$$

as a single matrix.

Solution

$$\mathbf{A}^2 - 3\mathbf{B} = \begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & x \\ 0 & 2 \end{pmatrix} - 3\begin{pmatrix} 5 & 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 16 & 6x \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 15 & 3 \\ 0 & 3 \end{pmatrix}$$
$$= \underline{\underline{\begin{pmatrix} 1 & 6x-3 \\ 0 & 1 \end{pmatrix}}}.$$

(b) (i) Given that $\mathbf{C}$ is non-singular, find $\mathbf{C}^{-1}$, the inverse of $\mathbf{C}$. (2)

Solution

$$\begin{aligned}\det \mathbf{C} &= 2y - (-3)\\ &= 2y + 3\end{aligned}$$

and so

$$\mathbf{C}^{-1} = \underline{\underline{\tfrac{1}{2y+3}\begin{pmatrix} 2 & -3 \\ 1 & y \end{pmatrix}}}.$$

(ii) For what value of y would matrix $\mathbf{C}$ be singular? (1)

Solution

$$\begin{aligned}\det \mathbf{C} = 0 &\Rightarrow 2y + 3 = 0\\ &\Rightarrow 2y = -3\\ &\Rightarrow \underline{\underline{y = -1\tfrac{1}{2}}}.\end{aligned}$$

3. Use integration by parts to obtain (4)

$$\int \frac{\ln x}{x^3}\, \mathrm{d}x,$$

where $x > 0$.

Solution

$$u = \ln x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = x^{-1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = x^{-3} \Rightarrow v = -\tfrac{1}{2}x^{-2}$$

Now,

$$\begin{aligned}\int \frac{\ln x}{x^3}\, \mathrm{d}x &= \int \left(\ln x \cdot \frac{1}{x^3}\right) \mathrm{d}x\\ &= -\tfrac{1}{2}x^{-2}\ln x - \int (x^{-1})(-\tfrac{1}{2}x^{-2})\, \mathrm{d}x\\ &= -\tfrac{1}{2}x^{-2}\ln x + \tfrac{1}{2}\int x^{-3}\, \mathrm{d}x\\ &= \underline{\underline{-\tfrac{1}{2}x^{-2}\ln x - \tfrac{1}{4}x^{-2} + c}}.\end{aligned}$$

4. (a) State (2)

$$\sum_{r=1}^{n} r \text{ and } \sum_{r=1}^{n} r^3$$

in terms of n.

Solution

$$\sum_{r=1}^{n} r = \underline{\underline{\tfrac{1}{2}n(n+1)}} \text{ and } \sum_{r=1}^{n} r^3 = \underline{\underline{\tfrac{1}{4}n^2(n+1)^2}}.$$

(b) Hence show that (2)

$$\sum_{r=1}^{n} (r^3 - 3r) = \frac{n(n+1)(n-2)(n+3)}{4}.$$

Solution

$$\begin{aligned}\sum_{r=1}^{n} (r^3 - 3r) &= \sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r \\ &= \tfrac{1}{4}n^2(n+1)^2 - \tfrac{3}{2}n(n+1) \\ &= \tfrac{1}{4}n(n+1)\left[n(n+1) - 6\right] \\ &= \tfrac{1}{4}n(n+1)\left[n^2 + n - 6\right]\end{aligned}$$

$$\left.\begin{array}{ll}\text{add to:} & +1 \\ \text{multiply to:} & -6\end{array}\right\} -2,\ +3$$

$$\begin{aligned}&= \tfrac{1}{4}n(n+1)(n-2)(n+3) \\ &= \underline{\underline{\frac{n(n+1)(n-2)(n+3)}{4}}},\end{aligned}$$

as required,

(c) Use the above result to evaluate (2)

$$\sum_{r=5}^{15} (r^3 - 3r).$$

Solution

$$\begin{aligned}\sum_{r=5}^{15}(r^3 - 3r) &= \sum_{r=1}^{15}(r^3 - 3r) - \sum_{r=1}^{4}(r^3 - 3r) \\ &= \tfrac{1}{4}(15)(16)(13)(18) - \tfrac{1}{4}(4)(5)(2)(7) \\ &= 14\,040 - 70 \\ &= \underline{\underline{13\,970}}.\end{aligned}$$

5. Find the general solution of the differential equation (6)

$$\frac{1}{x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6,\ x \neq 0.$$

Solution

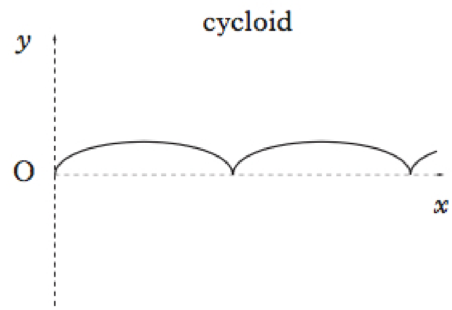
$$\frac{1}{x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 6x$$

$$\begin{aligned}\text{IF} &= \mathrm{e}^{\int 2x\,\mathrm{d}x} \\ &= \mathrm{e}^{x^2}\end{aligned}$$

$$\begin{aligned}&\Rightarrow \mathrm{e}^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\mathrm{e}^{x^2}y = 6x\,\mathrm{e}^{x^2} \\ &\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x^2}y) = 6x\,\mathrm{e}^{x^2} \\ &\Rightarrow \mathrm{e}^{x^2}y = \int 6x\,\mathrm{e}^{x^2}\,\mathrm{d}x \\ &\Rightarrow \mathrm{e}^{x^2}y = 3\,\mathrm{e}^{x^2} + c \\ &\Rightarrow \underline{\underline{y = 3 + c\,\mathrm{e}^{-x^2}}}.\end{aligned}$$

6. The cycloid curve below is defined by the parametric equations

$$x = t - \sin t,\ y = 1 - \cos t.$$



(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. (2)

Solution

$$x = t - \sin t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \cos t$$
$$y = 1 - \cos t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \sin t$$

Now,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \underline{\underline{\frac{\sin t}{1 - \cos t}}}.\end{aligned}$$

(b) Show that the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is always negative, in the case where $0 < t < 2\pi$. (5)

Solution

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin t}{1 - \cos t}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sin t}{1 - \cos t}\right) \times \frac{\mathrm{d}t}{\mathrm{d}x}\end{aligned}$$

$$u = \sin t \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = \cos t$$
$$v = 1 - \cos t \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} = \sin t$$

$$\begin{aligned}
&= \frac{(1-\cos t)(\cos t) - (\sin t)(\sin t)}{(1-\cos t)^2} \times \frac{1}{1-\cos t} \\
&= \frac{\cos t - \cos^2 - \sin^2 t}{(1-\cos t)^3} \\
&= \frac{\cos t - 1}{(1-\cos t)^3} \\
&= \frac{-(1-\cos t)}{(1-\cos t)^3} \\
&= -\frac{1}{(1-\cos t)^2}
\end{aligned}$$

and this is negative except when

$$\cos t = 1 \Rightarrow t = 0,\ 2\pi.$$

A particle follows the path of the cycloid where t is the time elapsed since the particle's motion commenced.

(c) Calculate the speed of the particle when $t = \frac{1}{3}\pi$. (2)

Solution

$$\begin{aligned}
\text{Speed} &= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \\
&= \sqrt{\left(1 - \cos \tfrac{1}{3}\pi\right)^2 + \left(\sin \tfrac{1}{3}\pi\right)^2} \\
&= \sqrt{\left(1 - \cos \tfrac{1}{3}\pi\right)^2 + \left(\sin \tfrac{1}{3}\pi\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} \\
&= \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} \\
&= \underline{\underline{1}}.
\end{aligned}$$